Journal of Convex Analysis
Volume 24 (2017), No. 1, 41–53

A Best Proximity Point Theorem Using Discontinuous Functions

Binayak S. Choudhury
*Department of Mathematics,
Indian Institute of Engineering Science and Technology,
Shibpur, P.O. - B. Garden, Howrah, 711103, West Bengal, India
binayak12@yahoo.co.in*

Pranati Maity*
*Department of Mathematics,
Indian Institute of Engineering Science and Technology,
Shibpur, P.O. - B. Garden, Howrah, 711103, West Bengal, India
pranati.math@gmail.com*

Kishin Sadarangani†
*Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria,
Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
ksadaran@dma.ulpgc.es*

Received: November 13, 2013
Revised manuscript received: March 8, 2015
Accepted: March 19, 2015

In this paper, we prove a best proximity point theorem for generalized weak contractions with discontinuous control functions in the context of metric spaces and partially ordered metric spaces. The main argument in our proofs is based on transforming a question of best proximity point to one about a fixed point. Moreover, we will use a weaker condition than the P-property and we present two examples to illustrate our results.

Keywords: Non-self map, weak P-property, best proximity point, control function, weak contraction

1991 Mathematics Subject Classification: 54H25, 47H10

1. Introduction

Banach's contraction principle is one of the pivotal result of analysis. Its importance lies in its vast applicability in numerous branches of mathematics. This principle states that if (X, d) is a complete metric spaces and $T\colon X \to X$ is a k-contraction (this means that there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq$

*Corresponding author.
†The third author has been partial supported by the project MTM2013-44357-P.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

$k\,d(x,y)$ for any $x, y \in X$) then T has a unique fixed point. A huge number of generalizations and extended versions of this principle appear in the literature.

In this paper we are interested in nonself mappings and this question is closely related with the best proximity point theory. We present a brief discussion about best proximity points.

Let A be a nonempty subset of a metric space (X, d) and let $T\colon A \to X$ be a mapping. The solutions of the equation $Tx = x$ are fixed point of T. Consequently, $T(A) \bigcap A \neq \emptyset$ is a necessary condition for the existence of a fixed point for the operator T. If this necessary condition does not hold then $d(x, Tx) > 0$ for any $x \in A$ and the mapping $T\colon A \to X$ does not have any fixed point. In this setting, our aim is to find an element $x \in A$ such that $d(x, Tx)$ is minimum.

In the context of nonself mappings, we consider two nonempty subsets A and B of a complete metric space (X, d) and a mapping $T\colon A \to B$. A natural question is whether one can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx)\colon x \in A\}$ for any $x \in A$ (recall that $d(A, B) = \inf\{d(x, y)\colon x \in A$ and $y \in B\}$), the optimal solution to this problem will be the one for which the value $d(A, B)$ is attained by the real valued function $\varphi\colon A \to \mathbb{R}$ given by $\varphi(x) = d(x, Tx)$. These optimal solutions are called best proximity points of the mappings T. Some results about best proximity points can be found in [1, 2, 3, 4, 6, 8, 10, 11, 13, 14, 16, 17, 18, 19, 20].

Next, we introduce some notations and definitions which be used later.

Let A and B be two nonempty subsets of a metric space (X, d).

By A_0 and B_0 we denote the following sets:

$$\begin{aligned} A_0 &= \{x \in A\colon d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 &= \{y \in B\colon d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned}$$

Some sufficient conditions which guarantee the nonemptiness of A_0 and B_0 are discussed in [13].

The following definition was introduced in [19].

Definition 1.1. Let A and B be two nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

It is easily seen that for any nonempty subset A of a metric space (X, d), the pair (A, A) has the P-property.

In this paper, we present a best proximity point theorem where the contractive condition appears in terms of a discontinuous function in the context of metric spaces and partially ordered metric spaces.

2. Main result

Our starting point in this section is to present the following classes of functions Ψ and Θ which will be used later,

$$\Psi = \{\psi\colon [0,\infty) \to [0,\ \infty)\colon \psi \text{ is continuous, non-decreasing and } \psi(t) = 0 \text{ if and only if } t = 0\}\,,$$

and

$$\Theta = \{\alpha\colon [0,\ \infty) \to [0,\ \infty)\colon \alpha \text{ is bounded on any bounded interval in } [0,\infty), \alpha \text{ is continuous at } 0 \text{ and } \alpha(0) = 0\}\,.$$

The functions belonging to the class Ψ are called altering distance function. The altering distance function was used in fixed point theory by Khan et al. [12] for the first time and later, in many works like [5, 9, 15].

These two classes of functions have been used in a recent paper to prove a result about coupled coincidence point [7].

In order to present the main result of [7], we need some definitions.

Definition 2.1. Let (X, d) be a metric space and $G, F\colon X \to X$ two mappings. We say that G and F are compatible if

$$\lim_{n\to\infty} d(GFx_n, FGx_n) = 0\,,$$

for any $(x_n) \subset X$ such that $\lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Fx_n = x$ for some $x \in X$.

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set and $F, G\colon X \to X$. We say that F is G-nondecreasing if for any $x, y \in X$ with $Gx \preceq Gy$ then $Fx \preceq Fy$.

The main result of [7] is the following.

Theorem 2.3 ([7]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) be a complete metric space. Let $T, G\colon X \to X$ two mappings such that G is continuous and nondecreasing, $T(X) \subset G(X)$, T is G-nondecreasing and the pair (G, T) is compatible. Suppose that there exist $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ such that*

a) $\psi(x) \preceq \varphi(y) \Rightarrow x \preceq y$.

b) *For any sequence $(x_n) \in [0,\infty)$ with $x_n \to t > 0$,*

$$\psi(t) - \overline{\lim}\,\varphi(x_n) + \underline{\lim}\,\theta(x_n) > 0\,.$$

c) *For any $x, y \in X$ with $Gx \preceq Gy$*

$$\psi(d(Tx, Ty)) \le \varphi(d(Gx, Gy)) - \theta(d(Gx, Gy))\,.$$

Moreover, suppose that

i) T *is continuous or* (X,d) *satisfies the following condition*

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x$$
$$\text{then } x_n \preceq x \text{ for any } n \in \mathbb{N}\,,$$

ii) *There exists* $x_0 \in X$ *such that* $Gx_0 \preceq Tx_0$.

Then G *and* T *have a coincidence point in* X*, that is, there exists* $x \in X$ *such that* $Gx = Tx$.

If in Theorem 2.3, we put $G = I_X$ the identity mapping on X we have the following corollary.

Corollary 2.4. *Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* (X,d) *be a complete metric space. Let* $T\colon X \to X$ *be a continuous and nondecreasing mapping, and there exist* $\psi \in \Psi$ *and* $\varphi, \theta \in \Theta$ *such that*

a) $\psi(x) \preceq \varphi(y) \Rightarrow x \preceq y$.

b) *For any sequence* $(x_n) \in [0,\infty)$ *with* $x_n \to t > 0$,

$$\psi(t) - \overline{\lim}\,\varphi(x_n) + \underline{\lim}\,\theta(x_n) > 0\,.$$

c) *For any* $x, y \in X$

$$\psi(d(Tx,Ty)) \le \varphi(d(x,y)) - \theta(d(x,y))\,.$$

Moreover, suppose that

i) T *is continuous or* (X,d) *satisfies the following condition*

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x$$
$$\text{then } x_n \preceq x \text{ for any } n \in \mathbb{N}\,,$$

ii) *There exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$.

Then T *has a fixed point.*

In [7], the authors present a sufficient condition in order to keep the paper selfcontained. They obtain a better result, that is, uniqueness of the common fixed point. This result is the following theorem.

Theorem 2.5 ([7]). *In addition to the hypotheses of Theorem 2.3, we suppose that for every* $x, y \in X$ *there exists* $u \in X$ *such that* Tu *is comparable to* Tx *and* Ty *Then* G *and* T *have a unique common fixed point, i.e., there exists a unique* $x \in X$ *such that* $Gx = Tx = x$.

When $G = I_X$ the identity mapping on X, we have the following corollary.

Corollary 2.6. *In addition to the hypotheses of Corollary* 2.4*, we suppose that for every $x, y \in X$ there exists $u \in X$ which is comparable to x and y, we obtain uniqueness of the fixed point.*

Although the result appearing in [7] are presented in the context of partially ordered metric spaces, following a similar argument Corollaries 2.4 and 2.6 have a counterpart in the context for complete metric spaces. However, in order to that the paper is selfcontained, we present the proof.

Theorem 2.7. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying*

$$\psi(d(Tx,Ty)) \leq \varphi(d(x,y)) - \theta(d(x,y)), \quad \textit{for all } x, y \in X\,,$$

where $\psi \in \Psi$, $\varphi, \theta \in \Theta$ and such that

i) $\psi(x) \leq \varphi(y) \Rightarrow x \leq y$.

ii) *For any sequence $(t_n) \in [0, \infty)$ with $t_n \to t_0 > 0$,*

$$\psi(t_0) - \overline{\lim}\, \varphi(t_n) + \underline{\lim}\, \theta(t_n) > 0\,.$$

Then T has a unique fixed point.

Proof. We take $x_0 \in X$ and put $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \cdots$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then $x_{n_0+1} = Tx_{n_0} = x_{n_0}$ and the part of existence of fixed point is proved.

Now suppose that $x_{n+1} \neq x_n$ for any $n \in \mathbb{N}$. Then, using the contractive condition, we get

$$\begin{aligned} \psi(d(x_{n+1}, x_n)) &= \psi(d(Tx_n, Tx_{n-1})) \\ &\leq \varphi(d(x_n, x_{n-1})) - \theta(d(x_n, x_{n-1})) \\ &\leq \varphi(d(x_n, x_{n-1}))\,. \end{aligned} \tag{1}$$

From assumption $i)$, we infer that

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) \quad \text{for any } n \in \mathbb{N}\,.$$

Consequently, $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of nonnegative real numbers and, therefore, $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$ for certain $r \geq 0$.

Next, we will prove that $r = 0$.

In fact, taking $n \to \infty$ in (1) and using the continuity of Ψ and the fact that $\varphi, \theta \in \Theta$, we have

$$\begin{aligned} \psi(r) &\leq \overline{\lim}\, \varphi(d(x_n, x_{n-1})) + \overline{\lim}\, (-\theta(d(x_n, x_{n-1}))) \\ &= \overline{\lim}\, \varphi(d(x_n, x_{n-1})) - \underline{\lim}\, \theta(d(x_n, x_{n-1}))\,, \end{aligned}$$

or, equivalently,

$$\psi(r) - \overline{\lim}\,\varphi(d(x_n, x_{n-1})) + \underline{\lim}\,\theta(d(x_n, x_{n-1})) \leq 0\,,$$

where $\lim_{n\to\infty} d(x_n, x_{n-1}) = r$. Using assumption ii), we deduce that $r = 0$.

In what follows, we will prove that $\{x_n\}$ is a Cauchy sequence.

In contrary case, by using standard arguments, we can find $\epsilon > 0$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ satisfying:

$$\begin{aligned} &\text{a)} \quad d(x_{m(k)}, x_{n(k)}) \geq \epsilon\,, \qquad d(x_{m(k)-1}, x_{n(k)}) < \epsilon\,, \\ &\text{b)} \quad \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon\,. \end{aligned} \tag{2}$$

Taking into account the contractive condition and that ψ is nondecreasing, we get

$$\begin{aligned} 0 < \psi(\epsilon) \leq \psi(d(x_{m(k)}, x_{n(k)})) &= \psi(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\leq \varphi(d(x_{m(k)-1}, x_{n(k)-1})) - \theta(d(x_{m(k)-1}, x_{n(k)-1}))\,. \end{aligned}$$

Taking upper limit, we have that

$$\begin{aligned} 0 < \psi(\epsilon) &\leq \overline{\lim}\,\varphi(d(x_{m(k)-1}, x_{n(k)-1})) + \overline{\lim}\,\big(-\theta(d(x_{m(k)-1}, x_{n(k)-1}))\big) \\ &= \overline{\lim}\,\varphi(d(x_{m(k)-1}, x_{n(k)-1})) - \underline{\lim}\,\theta(d(x_{m(k)-1}, x_{n(k)-1}))\ , \end{aligned}$$

and, therefore,

$$\psi(\epsilon) - \overline{\lim}\,\varphi(d(x_{m(k)-1}, x_{n(k)-1})) + \underline{\lim}\,\theta(d(x_{m(k)-1}, x_{n(k)-1})) \leq 0\,,$$

where $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon$ (by (2)).

From assumption ii), we deduce that $\epsilon = 0$ which is a contradiction.

Therefore, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Notice that our contractive condition implies the continuity of T, since, for any $x, y \in X$,

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \theta(d(x, y)) \leq \varphi(d(x, y))\,,$$

and, by assumption i),

$$d(Tx, Ty) \leq d(x, y)\,.$$

Therefore, from $\lim_{n\to\infty} x_n = x$ and the fact that T is continuous, we infer that

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x\,.$$

This proves that x is a fixed point of T.

Next, we will prove the uniqueness of the fixed point.

Suppose that $y \in X$ is another fixed point of T, i.e., $Ty = y$, then

$$\psi(d(x,y)) = \psi(d(Tx,Ty)) \leq \varphi(d(x,y)) - \theta(d(x,y))\,,$$

and, this gives us

$$\psi(d(x,y)) = \psi(d(Tx,Ty)) \leq \varphi(d(x,y)) + \theta(d(x,y)) \leq 0\,,$$

where we consider the constant sequence $\{d(x,y)\}$ in assumption ii).

Therefore, $d(x,y) = 0$ and, consequently, $x = y$.

This completes the proof. □

Before proceeding the main result of the paper, we need some definitions.

Definition 2.8 ([20]). Let (A,B) be a pair of nonempty subsets of a metric space X with $A_0 \neq \emptyset$ ($A_0 = \{x \in A : d(x,y) = \operatorname{dist}(A,B)$ for some $y \in B\}$ and $B_0 = \{y \in B : d(x,y) = \operatorname{dist}(A,B)$ for some $x \in A\}$). Then the pair (A,B) is said to have the weak P-property if and only if

$$\left.\begin{array}{l} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{array}\right\} \Rightarrow d(x_1,x_2) \leq d(y_1,y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

In the literature, the definition of the P-property appears as

$$\left.\begin{array}{l} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{array}\right\} \Rightarrow d(x_1,x_2) = d(y_1,y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

The following example proves that the weak P-property does not imply the P-property.

Example 2.9. Let $X = R^2$ be the usual plane with the euclidean metric and let A and B be the subsets of X given by $A = \{(x, x+2) : -1 \leq x \leq 0\} \cup \{(x, -x+2) : 0 \leq x \leq 1\}$ and $B = \{(-2^n, 0), (2^n, 0)\}$ for certain $n \in N$.

A picture of these sets is Figure 2.1.

Notice that $d(A,B) = d((-1,1),(-2^n,0)) = \sqrt{(2^n-1)^2+1}$. Moreover, $A_0 = \{(-1,1),(1,1)\}$ and $B_0 = B$. Notice that $d((-1,1),(-2^n,0) = d(A,B)$, $d((1,1),(2^n,0) = d(A,B)$ and $d((-1,1),(1,1)) = 2$, $d((-2^n,0),(2^n,0)) = 2^{n+1}$. Therefore, (A,B) satisfies the weak P-property and dose not satisfy the P-property.

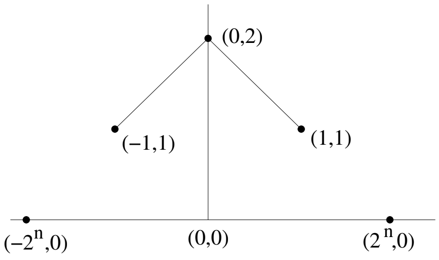

Figure 2.1:

Definition 2.10 ([17]). A mapping $T : A \to B$ is said to be *proximally increasing* if it satisfies the condition that

$$\left.\begin{array}{l} y_1 \preceq y_2 \\ d(x_1, Ty_1) = d(A,B) \\ d(x_2, Ty_2) = d(A,B) \end{array}\right\} \Rightarrow x_1 \preceq x_2\,,$$

where $x_1, x_2, y_1, y_2 \in A$.

One can see that, for a self-mapping, the notion of proximally increasing mapping reduces to that of increasing mapping.

The main results are the following theorems.

Theorem 2.11. *Let (A, B) be a pair of non empty closed subsets of a complete metric space $(X,\ d)$ such that $A_0 \neq \emptyset$. Let $T : A \longrightarrow B$ be a mapping satisfying $T(A_0) \subseteq B_0$ and such that*

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \theta(d(x, y)), \quad \textit{for all } x, y \in A, \ \textit{where } \psi \in \Psi, \theta \in \Theta$$

i) $\psi(x) \leq \varphi(y) \Longrightarrow \ x \leq y,$
ii) *for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$,*

$$\psi(t) - \overline{\lim}\,\varphi(x_n) + \underline{\lim}\,\theta(x_n) > 0.$$

Suppose that the pair (A, B) has the weak P-property, then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A, B)$.

Proof. Notice that the contractive condition implies that T is continuous. In fact, for any $x, y \in A$, we have

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \theta(d(x, y)) \leq \varphi(d(x, y)),$$

and, by $i)$, $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in A$
and this proves the continuity of T. For better readability, we break the proof in several steps.

Step 1. B_0 is closed. In fact, suppose that $\{y_n\} \subseteq B_0$ and $y_n \to y \in B$, we have to prove that $y \in B_0$. Since $\{y_n\} \subseteq B_0$, we can find $\{x_n\} \subseteq A_0$ such that

$$d(x_n, y_n) = d(A, B) \quad \text{for all } n \in N. \tag{3}$$

Since the pair (A, B) has the weak P-property, $d(x_n, x_m) \leq d(y_n, y_m)$ for any $n, m \in N$. Since y_n is convergent, it is Cauchy sequence and, therefore, from the last inequality, we infer that x_n is a Cauchy sequence. Since A is closed, $x_n \to x$ for certain $x \in A$. By using the continuity of the distance d, taking $n \to \infty$ in (3), we have $d(x, y) = d(A, B)$ and this proves that $y \in B_0$.

Step 2. $T(\overline{A_0}) \subseteq B_0$, where $\overline{A_0}$ is the closure of A_0. Taking into account our assumption $T(\overline{A_0}) \subseteq B_0$, it is sufficient to prove that $T(\overline{A_0} - A_0) \subseteq B_0$. In fact, we have $x \in \overline{A_0} - A_0$. Then, we can find a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. By using the continuity of T, the fact that $T(A_0) \subseteq B_0$ and that B_0 is closed subset (by step 1), we have $Tx_n \to Tx \in B_0$ and this proves our claim.

Step 3. For each $y \in T(\overline{A_0})$, there exists a unique $x \in A_0$ such that $d(x, y) = d(A, B)$. In fact, since $d(x, y) = d(A, B)$ (by step 1), we can find $x \in A_0$ such that $d(x, y) = d(A, B)$ and this proves the part of existence. For the uniqueness, suppose that $x_1, x_2 \in A_0$ and such that

$$\begin{aligned} d(x_1, y) &= d(A, B), \\ d(x_2, y) &= d(A, B), \end{aligned}$$

by the weak P-property, $d(x_1, x_2) \leq d(y, y) = 0$ and, therefore, $x_1 = x_2$. Now, we define the mapping $L : T(\overline{A_0}) \to A_0$ by $Ly = x$, where x satisfies $d(x, y) = d(A, B)$. By step 3, the mapping L is well defined. Now, we consider the mapping $LT : \overline{A_0} \to \overline{A_0}$. Then, for any $x_1, x_2 \in \overline{A_0}$ we have

$$d(LTx_1, Tx_1) = d(A, B), d(LTx_2, Tx_2) = d(A, B)$$

and by the weak P-property, $d(LTx_1, LTx_2) \leq d(Tx_1, Tx_2)$ and since ψ is nondecreasing

$$\psi(d(LTx_1, LTx_2)) \leq \psi(d(Tx_1, Tx_2)) \leq \varphi(d(x_1, x_2)) - \theta(d(x_1, x_2)).$$

Therefore, since $\overline{A_0}$ is a complete metric space and $LT : \overline{A_0} \to \overline{A_0}$ satisfies the contractive condition appearing in Theorem 2.7, by this result, we can find a unique $x_0 \in \overline{A_0}$ such that $(LT)(x_0) = L(Tx_0) = x_0$.

Taking into account the definition of the mapping L, this means that $d(x_0, Tx_0) = d(A, B)$. This proves the part of existence of our result. For the uniqueness, suppose that there exists another element $x_1 \in A_0$ such that $d(x_1, Tx_1) = d(A, B)$. This means that $x_1 \in A_0$ and $L(Tx_1) = x_1$ and, consequently, x_1 is a fixed point of LT. Since LT has a unique fixed point, $x_1 = x_0$.

This completes the prove. □

In what follows, we prove a version of Theorem 2.11 in the context of partially ordered metric spaces.

Theorem 2.12. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let (A, B) be a pair of non empty closed subsets of X such that $A_0 \neq \emptyset$ and the pair (A, B) satisfies the weak P-property. Let $T : A \longrightarrow B$ be a nondecreasing and continuous mapping satisfying $T(A_0) \subseteq B_0$ and such that*

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \theta(d(x, y)), \quad \text{for all } x, y \in A, \text{ with } x \succeq y,$$

where $\psi \in \Psi$, $\theta \in \Theta$

i) $\psi(x) \leq \varphi(y) \Longrightarrow x \leq y$,
ii) *for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \longrightarrow t > 0$,*

$$\psi(t) - \overline{\lim}\, \varphi(x_n) + \underline{\lim}\, \theta(x_n) > 0.$$

Suppose that T is proximally increasing and there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \leq x_1$, then there exists a x^* in A such that $d(x^*, Tx^*) = d(A, B)$.

Proof. Following the same steps as in the proof of Theorem 2.11, we have that B_0 is closed and $T(\overline{A_0}) \subseteq B_0$. On the other hand, for $x, y \in A_0$ with $x \succeq y$, since by Theorem 2.11

$$d(LTx, Tx) = d(A, B),$$
$$d(LTy, Ty) = d(A, B),$$

the weak P-property gives us $d(LTx, LTy) \leq d(Tx, Ty)$ and, since ψ is nondecreasing

$$\psi(d(LTx, LTy)) \leq \psi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \theta(d(x, y)).$$

Therefore, the contractive condition appearing in Corollary 2.4 is satisfied. Finally, from $x_0 \leq x_1$,

$$d(x_1, Tx_0) = d(A, B),$$
$$d(LTx_1, Tx_1) = d(A, B),$$

and, by using that T is proximally increasing, we infer that $x_1 \leq LTx_1$.

Applying Corollary 2.4, LT has a fixed point, that is, $x^* \in A_0$ such that $LTx^* = x^*$. Notice that we can apply Corollary 2.4 because since $\overline{A_0}$ is closed, $\overline{A_0}$ is a complete metric space. But, as $d(LTx^*, Tx^*) = d(A, B)$, we get $d(x^*, Tx^*) = d(A, B)$.

This completes the prove. □

For uniqueness of the best proximity point, we need the following condition:

$$\text{for } x, y \in \overline{A_0} \text{ there exists } z \in \overline{A_0} \text{ which is comparable to } x \text{ and } y. \qquad (A)$$

Theorem 2.13. *Adding condition* (A) *to the assumptions of Theorem* 2.12 *we obtain uniqueness of the best proximity point.*

Proof. Let $x_1^*, x_2^* \in \overline{A_0}$ be such that $d(x_1^*, Tx_1^*) = d(x_2^*, Tx_2^*) = d(A, B)$.

Since $d(LTx_1^*, Tx_1^*) = d(A, B)$, the weak P-property gives us

$$d(x_1^*, LTx_1^*) \leq d(Tx_1^*, Tx_1^*) = 0.$$

Therefore $d(x_1^*, LTx_1^*) = 0$. Similarly, we can prove that $d(x_2^*, LTx_2^*) = 0$. Therefore, $LTx_1^* = x_1^*$, $LTx_2^* = x_2^*$.

By Corollary 2.6, since LT has a unique fixed point, we deduce that $x_1^* = x_2^*$. □

3. Example

Example 3.1. Let X=R^2 is a metric space with $d((x_1, y_1), (x_2, y_2)) =\mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid$.

We consider the two closed subsets $A = \{(0, 2\lambda - 1) : 0 \leq \lambda \leq 1\}$ and $B = \{(3-2\lambda, 2) : 0 \leq \lambda \leq 1\} \cup \{(3, 4\lambda - 2) : 0 \leq \lambda \leq 1\} \cup \{(3-2\lambda, -2) : 0 \leq \lambda \leq 1\}$.

Here, $A_0 = \{(0, 1), (0, -1)\}$, $B_0 = \{(1, 2), (1, -2)\}$ and $d(A, B) = 2$.

Now $d((0, 1), (1, 2)) = d(A, B) = 2$, $d((0, -1), (1, -2)) = d(A, B) = 2$, $d((0, 1), (0, -1)) = 2 \neq d((1, 2), (1, -2)) = 4$.

So the pair (A, B) satisfies the weak P-property.

Let $T : A \longrightarrow B$ be a mapping defined as

$$T(0, 2\lambda - 1) = \left(1.5 - |2\lambda - 1| + \frac{1}{2}|2\lambda - 1|^2, 2\right), \quad \text{for } 0 \leq \lambda \leq 1.$$

Then T is continuous and $T(A_0) \subseteq B_0$.

Let $\psi : [0, \infty) \longrightarrow [0, \infty)$ be defined as $\psi(t) = t$, for all $t \in [0, \infty)$, and let $\varphi, \theta : [0, \infty) \longrightarrow [0, \infty)$ be defined respectively as

$$\varphi(t) = \begin{cases} \dfrac{1}{2}[t]^2, & \text{if } 2 < t < 3, \\ t - \frac{1}{2}t^2, & \text{otherwise,} \end{cases} \qquad \theta(t) = \begin{cases} \dfrac{1}{2}[t]^2, & \text{if } 2 < t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Then the required conditions of Theorem 2.11 are satisfied. Here $(0, 1)$ a unique best proximity point of the mapping T.

Example 3.2. Let $X = R^2$ is a euclidean metric space and A and B be two closed subsets of X given by $A = \{(2-2\alpha, 2\alpha-2) : 0 \leq \alpha \leq 1\}$ and $B = \{(-\alpha, 2-\alpha) : 0 \leq \alpha \leq 1\} \cup \{(\alpha, 2-\alpha) : 0 \leq \alpha \leq 1\}$.

We define a partial order '$\preceq$' on A as $x \preceq y$ if and only if $x \geq y$ for all $x, y \in A$.

Here, $A_0 = \{(0,0)\}$, $B_0 = \{(-1,1),(1,1)\}$ and $d(A,B) = \sqrt{2}$.

Now $d((0,0),(-1,1)) = d(A,B) = \sqrt{2}$, $d((0,0),(1,1)) = d(A,B) = \sqrt{2}$, $d((0,0),(0,0)) = 0 \neq d((-1,1),(1,1)) = 2$. So the pair (A,B) satisfies the weak P-property.

Let $T : A \longrightarrow B$ be a mapping defined as

$$\begin{aligned} &T(2-2\alpha, 2\alpha-2) \\ &= (1-2(1-\alpha)^2 + (1-\alpha), 1 + 2(\alpha-1)^2 + (\alpha-1)), \quad \text{for } 0 \leq \alpha \leq 1. \end{aligned}$$

Then T is continuous and $T(A_0) \subseteq B_0$.

If $x_1 \preceq x_2$ and $d(y_1, Tx_1) = d(A,B) = \sqrt{2}$, $d(y_2, Tx_2) = d(A,B) = \sqrt{2}$ for some $y_1, y_2, x_1, x_2 \in A$, then we have $y_1 = y_2 = (0,0), x_1 = x_2 = (0,0)$. Then T is proximally increasing.

Let $\psi : [0,\infty) \longrightarrow [0,\infty)$ be defined as $\psi(t) = t^2$, for all $t \in [0,\infty)$,

and let $\varphi, \theta : [0,\infty) \longrightarrow [0,\infty)$ be defined respectively as

$$\varphi(t) = \begin{cases} \frac{1}{2}\, t^2, & \text{if } t > 5, \\ \frac{4}{5}\, t^2, & \text{if } t < 5, \end{cases} \qquad \theta(t) = \begin{cases} \frac{1}{3}\, t^2, & \text{if } t > 5, \\ 0, & \text{if } t < 5. \end{cases}$$

Then the required conditions of Theorem 2.13 are satisfied. Here $(0,0)$ a unique best proximity point of the mapping T.

References

[1] A. Abkar, M. Gabeleh: Best proximity points for cyclic mappings in ordered metric spaces, J. Optim. Theory Appl. 151 (2011) 418–424.

[2] M. A. Al-Thagafi, N. Shahzad: Convergence and existence results for best proximity points, Nonlinear Anal. 70 (2009) 3665–3671.

[3] A. Amini-Harandi: Best proximity points for proximal generalized contractions in metric spaces, Optim. Lett. 7 (2013) 913–921.

[4] J. Caballero, J. Harjani, K. Sadarangani: A best proximity point theorem for Geraghty-contractions, Fixed Point Theory Appl. (2012), Article ID 231, 9p.

[5] B. S. Choudhury, A. Kundu: (ψ, α, β)-weak contractions in partially ordered metric spaces, Appl. Math. Lett. 25(1) (2012) 6–10.

[6] B. S. Choudhury, P. Maity, P. Konar: A global optimality result using nonself mappings, Opsearch 51 (2014) 312–320.

[7] B. S. Choudhury, N. Metiya, M. Postolache: A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory Appl. (2013), Article ID 152, 21 p.

[8] C. Di Bari, T. Suzuki, C. Vetro: Best proximity points for cyclic Meir-Keeler contractions, Nonlinear Anal. 69 (2008) 3790–3794.

[9] D. Doric: Common fixed point for generalized ψ, ϕ-weak contractions, Appl. Math. Lett. 22 (2009) 1896–1900.

[10] A. A. Eldred, P. Veeramani: Existence and convergence of best proximity points, J. Math. Anal. Appl. 323(2) (2006) 1001–1006.

[11] S. Karpagam, S. Agrawal: Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps, Nonlinear Anal. 74 (2011) 1040–1046.

[12] M. S. Khan, M. Swaleh, S. Sessa: Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 30(1) (1984) 1–9.

[13] W. A. Kirk, S. Reich, P. Veeramani: Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24(7–8) (2003) 851–862.

[14] W. A. Kirk, P. S. Srinivasan, P. Veeramani: Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory 4(1) (2003) 79–89.

[15] O. Popescu: Fixed points for (ψ, ϕ)-weak contractions, Appl. Math. Lett. 24 (2011) 1–4.

[16] V. Pragadeeswarar, M. Marudai: Best proximity points: approximation and optimization in partially ordered metric spaces, Optim Lett. 7 (2013) 1883–1892.

[17] S. Sadiq Basha: Best proximity points: optimal solutions, J. Optim. Theory Appl. 151 (2011) 210–216.

[18] S. Sadiq Basha: Discrete optimization in partially ordered sets, J. Glob. Optim. 54 (2011) 511–517.

[19] V. Sankar Raj: A best proximity point theorem for weakly contractive non-self-mappings, Nonlinear Anal. 74(14) (2011) 4804–4808.

[20] J. Zhang, Y. Su, Q. Cheng: A note on 'A best proximity point theorem for Geraghty-contractions', Fixed Point Theory Appl. (2013), Arcticle ID 99, 4p.