Journal of Convex Analysis
Volume 24 (2017), No. 1, 75–92

Convexity with Respect to Beckenbach Families*

Mihály Bessenyei
Institute of Mathematics, University of Debrecen, 4002 Debrecen, Pf. 400, Hungary
besse@science.unideb.hu

Ágnes Konkoly
Institute of Mathematics, University of Debrecen, 4002 Debrecen, Pf. 400, Hungary
konkoly.agnes@freemail.hu

Bella Popovics
Institute of Mathematics, University of Debrecen, 4002 Debrecen, Pf. 400, Hungary
bellapopovics@gmail.com

Received: February 28, 2014
Revised manuscript received: March 10, 2015
Accepted: March 12, 2015

Beckenbach families are sets of functions possessing the two characteristic properties of Euclidean lines: their members are continuous and each distinct pairs of points of the plane can be interpolated by a unique member of the family. Applying Beckenbach families, the notion of (planar) convexity can be extended. Moreover, generalized convex functions can also be studied in this framework. The aim of this note is to prove the analogue of the Radon, Helly, Carathéodory and Minkowski Theorems in this generalized setting. The most important properties of generalized convex functions are presented, as well. As applications, some separation results are given.

Keywords: Beckenbach families, convex sets and functions, separation theorems

2010 Mathematics Subject Classification: Primary 52A10; Secondary 26A51, 39B62, 52A40

*The personal grant of this research was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 'National Excellence Program'. The logistic and infrastructural background of the research was supported by the Hungarian Scientific Research Fund (OTKA) Grants NK–81402.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

1. Introduction

Sandwich and separation theorems play an important role in many fields of mathematics, especially in Convex Analysis (see [19] and [27]). These kinds of results, roughly speaking, give sufficient and/or necessary conditions for a given pair of objects that can be "distinguished" in a certain sense with objects of particular importance. Let us recall here two results of this kind.

Assume that I is a real interval and we are given functions $f, g\colon I \to \mathbb{R}$ possessing a convex separator, that is, a convex function $\varphi\colon I \to \mathbb{R}$ fulfilling the inequalities $f \le \varphi \le g$. Then, as simple calculations show, for all $x, y \in I$ and $\lambda \in [0,1]$, we have

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda g(x) + (1-\lambda)g(y). \tag{1}$$

Due to a result of Baron, Matkowski, and Nikodem [2], this property is *characteristic*: If (1) holds for all $x, y \in I$ and $\lambda \in [0,1]$, then f and g can be separated by a convex function. The proof is based on the celebrated Carathéodory Theorem.

Similarly, assume that the functions $f, g\colon I \to \mathbb{R}$ have an affine separator $\varphi\colon I \to \mathbb{R}$. Then, as it can be checked directly, for all $x, y \in I$ and $\lambda \in [0,1]$, we have

$$\begin{aligned} f\big(\lambda x + (1-\lambda)y\big) &\le \lambda g(x) + (1-\lambda)g(y), \\ g\big(\lambda x + (1-\lambda)y\big) &\ge \lambda f(x) + (1-\lambda)f(y). \end{aligned} \tag{2}$$

According to a result of Nikodem and Wąsowicz [24], this is not only *necessary* but also a *sufficient* condition of the existence of an affine separator: If (2) holds for all $x, y \in I$ and $\lambda \in [0,1]$, then there is an affine function separating f and g. The statement can be verified via the Helly Theorem.

Notable, that in both cases the pure analytical statements are connected with some key tools of Convex Geometry. This connection reflects not only to the surprising beauty of the essential feature of the problem, but also to the fruitful marriage of Classical Analysis and Convex Geometry.

Separation problems of this spirit were studied intensively by several authors and in several contexts. The characterization of those pairs of functions that can be separated by a polynomial of given degree is due to Wąsowicz [30] and to Balaj and Wąsowicz [1]. The case of so-called two parameter Beckenbach families was solved by Nikodem and Páles [23]. The separation problem with respect to a given Chebyshev system (that is, to a given linear interpolation family) was presented by Bessenyei and Páles [8], while the case of convex Beckenbach families (that is, convex interpolation families), was presented by Bessenyei and Szokol [10].

All of the above-mentioned results belong to the problem of (generalized) affine separation. However, most of the problems of (generalized) convex separation are still open. In this direction, the only results are due to Nikodem and Páles

[23] and, in a special setting but in an alternative approach, to Bessenyei and Szokol [9].

These developments use the Carathéodory and the Helly Theorems directly, or in some modified form. Motivated by this phenomenon, the aim of the paper is to find the correspondence of some theorems of Convex Geometry and the adequate versions of affine and convex separation results in the contexts of two-parameter Beckenbach families.

Beckenbach families are sets of functions possessing the two characteristic properties of Euclidean lines: their members are continuous and each points of the plane (with distinct first coordinates) can be interpolated by a unique member of the family. Applying Beckenbach families, the notion of convex functions can be extended and studied [3], and many results turn out to be similar to the classical ones [4]. Krzyszkowski generalized the notion of (planar) convexity with the help of these families [17], and obtained a version of Carathéodory Theorem [16]. The results of [23] are partly rely on the investigations of Krzyszkowski.

Our paper is a continuation of the work initiated by Beckenbach and Krzyszkowski and is organized as follows. After the definition of generalized lines, we give their most important geometric and analytic properties. One of these facts is that uniform convergence on compact subintervals and convergence in two points are equivalent among sequences of Beckenbach lines.

For the generalized convexity of planar sets, we use a different approach then that of Krzyszkowski. Of course, our definition turns out to be equivalent to that of Krzyszkowski. The most important developments of this part are the correspondence of the Carathéodory, Radon, and Helly Theorems. These are formally similar to the classical ones. However, in the lack of linear structure, the usual rank-argument cannot be applied. Hence, an alternative approach has to be developed that based on some geometric methods and ideas.

We use the definition of generalized convex functions as Beckenbach introduced it, and present a variety of their characteristic properties. After clarifying the connection of generalized convex sets and generalized convex function, we prove the Minkowski Theorem in this setting. The paper concludes with some applications for convex and affine separation problems.

Part of the results are known in even more general settings. For the convenience of the Reader and for the sake of completeness, we always give an independent and self-contained proof in such cases.

2. Beckenbach families

Throughout this paper, we shall denote the set of *natural* and *real* numbers by the usual conventions $\mathbb{N}$ and $\mathbb{R}$. Under an *interval* I we mean a connected nonempty subset of $\mathbb{R}$ with at least two elements. We consider $\mathbb{R}^2$ equipped with the (unique) induced topology, or in case of $I \times \mathbb{R}$, with the subspace topology corresponding to the induced topology.

One of our most important notions is the concept of generalized lines due to Beckenbach [3]. These lines can be considered as members of a continuous two-parameter interpolation family:

Definition 2.1. A set $\mathcal{F}$ of real valued functions is called a *Beckenbach family* over the real interval I if its members are continuous functions defined on I such that, for all points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ of $I \times \mathbb{R}$ with $x_1 \neq x_2$, there exists exactly one member φ of $\mathcal{F}$ interpolating p_1 and p_2, that is fulfilling

$$\varphi(x_1) = y_1, \qquad \varphi(x_2) = y_2.$$

If we would like to emphasize that φ interpolates p_1 and p_2, we shall use the convention $\varphi = \varphi_{p_1,p_2}$. The members of $\mathcal{F}$ are termed *Beckenbach lines.* Let $\mathcal{F}$ be a given Beckenbach family. Under a *generalized line* we mean either an element of $\mathcal{F}$ or a vertical line $x = x_0$ where $x_0 \in I$.

A typical example for a Beckenbach family is the set of the lines of the Moulton plane [21]. An other important class is the linear hull generated by the components of a *regular pair.* A pair of continuous functions $(\omega_1, \omega_2)\colon I \to \mathbb{R}^2$ is termed regular, if

$$\begin{vmatrix} \omega_1(x_1) & \omega_1(x_2) \\ \omega_2(x_1) & \omega_2(x_2) \end{vmatrix} \neq 0$$

holds whenever x_1 and x_2 are distinct elements of I. Obviously, this determinant property is equivalent to the unique interpolation property of the linear family $\mathcal{F} = \mathcal{L}(\omega_1, \omega_2)$. The regular pair $(1, \mathrm{id})$ generates the standard Euclidean lines of the plane. Further important examples for regular pairs are the hyperbolic $(\cosh, \sinh)$ or the exponential $(1, \exp)$ ones (for details, see [5]).

Similarly to the Euclidean case, the mutual position of two generalized lines cannot be arbitrary. It turns out, that generalized lines cannot graze. For the sake of simplicity, we prove this fact only for Beckenbach lines. However, we shall often use this property in its full generality.

Lemma 2.2. *Let $\mathcal{F}$ be a Beckenbach family over a real interval I and let x_1, x_2 be distinct elements of I. For all φ, ψ distinct elements of $\mathcal{F}$, we have the following properties:*

(*i*) $\operatorname{sgn}(\varphi - \psi)$ *is either kept or is changed exactly once on* $[x_1, x_2]$*;*

(*ii*) *if* $\varphi(x_1) < \psi(x_1)$ *and* $\varphi(x_2) < \psi(x_2)$*, then* $\varphi < \psi$ *on* $[x_1, x_2]$*;*

(*iii*) *if* $\varphi(t_0) = \psi(t_0)$ *for some* $t_0 \in]x_1, x_2[$*, and* $\chi \in \mathcal{F}$ *coincides with* φ *(resp., with* ψ*) at* t_0 *such that*

$$\min\{\varphi(t), \psi(t)\} \leq \chi(t) \leq \max\{\varphi(t), \psi(t)\}$$

for some $t \in [x_1, x_2]$*, then these inequalities hold on the interval* $[x_1, x_2]$*.*

Proof. We shall prove only assertion (i) since the others are straightforward consequences of this. First, note, that the unique interpolation property allows at most one common point in which φ and ψ coincide. Assume that there exists some $t_0 \in]x_1, x_2[$ such that $\varphi(t_0) = \psi(t_0)$ and, for simplicity, $\varphi < \psi$. Consider the Beckenbach line χ determined by the properties

$$\chi(x_1) = \varphi(x_1), \qquad \chi(x_2) = \psi(x_2),$$

furthermore the function $f\colon [x_1, x_2] \to \mathbb{R}$ given by $f|_{[x_1,t_0]} = \varphi|_{[x_1,t_0]}$ and $f|_{[t_0,x_2]} = \psi|_{[t_0,x_2]}$. Then, f is a well-defined continuous function and, by Bolzano's Theorem, there exists some $s_0 \in]x_1, x_2[$ such that $\chi(s_0) = f(s_0)$. If $s_0 \in]x_1, t_0]$, then the points $\big(s_0, \psi(s_0)\big)$ and $\big(x_2, \psi(x_2)\big)$ can be interpolated by two different Beckenbach lines ψ and χ simultaneously, which is a contradiction. The other possibility $s_0 \in [t_0, x_2[$ leads to a contradiction in a similar way. □

The geometrical properties of Beckenbach lines have strong analytical consequences: if a sequence of Beckenbach lines converges at two points, then it converges uniformly on compact subsets of the domain. An analogue property remains true in a more general setting and is due to Tornheim [28]. Applying the Dini Theorem we present here an independent and direct proof.

Lemma 2.3. *If $\mathcal{F}$ is a Beckenbach family over a real interval I, x_1, x_2 are distinct elements of I, and (φ_n) is sequence in $\mathcal{F}$, then the following statements are equivalent:*

(i) *(φ_n) converges pointwise on $\{x_1, x_2\}$;*
(ii) *(φ_n) converges pointwise on $[x_1, x_2]$;*
(iii) *(φ_n) converges uniformly on $[x_1, x_2]$.*

Moreover, in each cases above, the limit function φ of the sequence (φ_n) belongs to the family $\mathcal{F}$ and is determined by the properties

$$\varphi(x_1) = \lim_{n\to\infty} \varphi_n(x_1), \qquad \varphi(x_2) = \lim_{n\to\infty} \varphi_n(x_2).$$

Proof. Fix the real numbers y_1, y_2 and consider first the special sequence (φ_n) which is given by the interpolation properties

$$\varphi_n(x_1) = y_1 + \frac{1}{n}, \qquad \varphi_n(x_2) = y_2 + \frac{1}{n}.$$

Denote the Beckenbach line that is determined by the points (x_1, y_1) and (x_2, y_2) by φ. Clearly, with these settings, the first assertion of the lemma is satisfied. Moreover, by the second statement of Lemma 2.2, the sequence $(\varphi_n(t))$ is monotone decreasing for all $t \in [x_1, x_2]$. On the other hand, due to the second statement of Lemma 2.2 again, $\varphi_n(t) \geq \varphi(t)$. Hence $(\varphi_n(t))$ is convergent, and we have that

$$c := \lim_{n\to\infty} \varphi_n(t) \geq \varphi(t).$$

For the implication $(i) \Rightarrow (ii)$, assume to the contrary that $c > \varphi(t)$. Let $\alpha \in \mathbb{R}$ be chosen such that $c > \alpha > \varphi(t)$, and define the Beckenbach lines ψ_1, ψ_2 by the properties

$$\psi_1(x_1) = \varphi(x_1), \quad \psi_1(t) = \alpha; \qquad \psi_2(x_2) = \varphi(x_2), \quad \psi_2(t) = \alpha.$$

Take $\psi \in \mathcal{F}$ such that $\psi(t) = \alpha$ and $\psi_1(x_1) < \psi(x_1) < \psi_2(x_1)$ be valid. Then, according to the third statement of Lemma 2.2, we have that $\psi > \varphi$ on the interval $[x_1, x_2]$. Since (φ_n) converges to φ on the set $\{x_1, x_2\}$, there exists some $m \in \mathbb{N}$ such that

$$\varphi(x_1) < \varphi_m(x_1) < \psi(x_1), \qquad \varphi(x_2) < \varphi_m(x_2) < \psi(x_2).$$

Therefore, by (ii) of Lemma 2.2, $\varphi_m < \psi$ on $[x_1, x_2]$; in particular, $\varphi_m(t) < \psi(t) < \alpha$, which is a contradiction. That is, $c = \varphi(t)$.

For the implication $(ii) \Rightarrow (iii)$, observe that the functions φ_n and φ fulfill the assumptions of the Dini Theorem on $[x_1, x_2]$. Therefore (φ_n) converges uniformly on $[x_1, x_2]$. The last implication $(iii) \Rightarrow (i)$ is trivial.

Assume now that φ_n is an arbitrary sequence that fulfills assertion (i). We shall prove that, in this case, the convergence on $[x_1, x_2]$ is uniform. Fix $\varepsilon > 0$. The previous argument guarantees the existence of Beckenbach lines ψ_1, ψ_2 with the properties $\psi_1 < \varphi < \psi_2$ and $\|\psi_1 - \psi_2\| < \varepsilon$. If $\varphi_n \to \varphi$ on the set $\{x_1, x_2\}$, then $\psi_1(x) < \varphi_n(x) < \psi_2(x)$ if $x \in \{x_1, x_2\}$ for sufficiently large indices; that is, $\psi_1(x) < \varphi_n(x) < \psi_2(x)$ for all $[x_1, x_2]$ in view of Lemma 2.2. Therefore,

$$\|\varphi_n - \varphi\| \le \|\psi_1 - \psi_2\| < \varepsilon. \qquad \square$$

The final result of this section is a set-theoretical lemma. This lemma turns out to be essential in representing generalized convex hulls (see next section), and its relatives play an important role in the theory of abstract convexity structures [29].

Lemma 2.4. *Assume that $\Phi\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a monotone, idempotent mapping of which singletons are fixed points. Then, for all $H \subset X$,*

$$\Phi(H) = \bigcap \big\{K \subset X \,|\, H \subset K, K = \Phi(K)\big\}.$$

Proof. Let $\mathcal{K} := \big\{K \subset X \,|\, H \subset K, K = \Phi(K)\big\}$. If $K \in \mathcal{K}$, then $H \subset K$ holds; hence, by the monotonicity, $\Phi(H) \subset \Phi(K) = K$ follows. That is, $\Phi(H) \subset \bigcap \mathcal{K}$. For the converse inclusion, take $h \in H$. Then the fixed point property guarantees $\{h\} = \Phi(\{h\}) \subset \Phi(H)$, yielding $H \subset \Phi(H)$. Therefore, due to the idempotency, $\Phi(H) \in \mathcal{K}$, and hence we get $\bigcap \mathcal{K} \subset \Phi(H)$. $\square$

3. Generalized convex sets

Applying Beckenbach families, the concept of convex sets can be extended in a quite natural way. This extension is due to Krzyszkowski (consult [17] and [16]). In the present section, we use an alternative approach via the inter representation of convexity: first we introduce generalized convex polytopes, that is, the $\mathcal{F}$-convex hulls of finite sets of points. Then, the $\mathcal{F}$-convex hull of an arbitrary set can be introduced as the set of all generalized polytopes with vertices in the set.

Definition 3.1. Let $\mathcal{F}$ be a Beckenbach family over the real interval I and let $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$ be given points of $I \times \mathbb{R}$. Under the *generalized segment* spanned by p_0 and p_1 we mean the set $[p_0, p_1]$ given by (distinguishing the cases $x_0 = x_1$ and $x_0 \neq x_1$)

$$[p_0, p_1] := \{(x, y) \in I \times \mathbb{R} \mid x_0 = x = x_1,\ \min\{y_0, y_1\} \leq y \leq \max\{y_0, y_1\}\};$$
$$[p_0, p_1] := \{(x, y) \in I \times \mathbb{R} \mid \min\{x_0, x_1\} \leq x \leq \max\{x_0, x_1\},\ y = \varphi_{p_0,p_1}(x)\}.$$

Assume that the set $[p_0, \ldots, p_n]$ has already been defined and let $p_{n+1} \in I \times \mathbb{R}$ be a given point. Then, $[p_0, \ldots, p_{n+1}] := \{[p, p_{n+1}] \mid p \in [p_0, \ldots, p_n]\}$. For a subset H of $I \times \mathbb{R}$, the *$\mathcal{F}$-convex hull* $\operatorname{conv}_{\mathcal{F}}(H)$ is given by

$$\operatorname{conv}_{\mathcal{F}}(H) := \bigcup_{n \in \mathbb{N}} \{[p_0, \ldots, p_n] \mid p_k \in H,\ k = 0, \ldots, n\}.$$

We say that $K \subset I \times \mathbb{R}$ is *$\mathcal{F}$-convex*, if it coincides with its $\mathcal{F}$-convex hull, that is, if $K = \operatorname{conv}_{\mathcal{F}}(K)$ holds.

The next lemma subsumes the most important properties of the convexity notion above. It turns out that the $\mathcal{F}$-convex hull of a finite set does not depend on the order of its points. The second statement shows that our convexity notion is equivalent to that of Krzyszkowski. The last statement, similarly to the classical case, gives an external representation for the $\mathcal{F}$-convex hull of a set.

Lemma 3.2. *Let $\mathcal{F}$ be a Beckenbach family over a real interval I. Then,*

(i) *the $\mathcal{F}$-convex hull of a finite set is invariant under the permutations of its elements;*
(ii) *K is $\mathcal{F}$-convex if and only if $[p_1, p_2] \subset K$ holds for all $p_1, p_2 \in K$;*
(iii) *the mapping* $\operatorname{conv}_{\mathcal{F}} \colon \mathcal{P}(I \times \mathbb{R}) \to \mathcal{P}(I \times \mathbb{R})$ *is monotone and idempotent;*
(iv) *the intersection of $\mathcal{F}$-convex sets is $\mathcal{F}$-convex;*
(v) *the union of nested $\mathcal{F}$-convex sets is $\mathcal{F}$-convex;*
(vi) *for all subset H of $I \times \mathbb{R}$,*

$$\operatorname{conv}_{\mathcal{F}}(H) = \bigcap \{K \subset I \times \mathbb{R} \mid H \subset K,\ \operatorname{conv}_{\mathcal{F}}(K) = K\}.$$

Proof. For sets of two elements, assertion (i) is clear by definition. Let p_0, p_1, p_2 be pairwise distinct points of $I \times \mathbb{R}$ and $p \in [p_0, p_1, p_2]$. Then, there exists some $q \in [p_0, p_1]$ such that $p \in [q, p_2]$. In view of Lemma 2.2 and of the Bolzano Theorem, there exists some $r \in [p_0, p_2]$ such that $r \in \varphi_{p_1,p}(t)$. This means that $p \in [r, p_1]$, yielding

$$[p_0, p_1, p_2] \subset [p_0, p_2, p_1].$$

The reversed inclusion can be proved similarly. Assume that the $\mathcal{F}$-convex hulls of $(n+1)$-element sets are invariant under permutations. Take pairwise distinct points $p_0, \dots, p_{n+1}$ of $I \times \mathbb{R}$. We may assume that $p_{n+1} \notin [p_0, \dots, p_{n-1}, p_n]$ and $p_n \notin [p_0, \dots, p_{n-1}, p_{n+1}]$ hold. Let p be an arbitrary point of the set $[p_0, \dots, p_{n+1}]$. Then, there exists some $r \in [p_0, \dots, p_n]$ such that $p \in [r, p_n]$; similarly, there exists some $q_0 \in [p_0, \dots, p_{n-1}]$ such that $r \in [q_0, p_{n+1}]$. That is, $p \in [q_0, q_1, q_2]$ where we use the notations $q_1 := p_n$ and $q_2 := p_{n+1}$. The previous part ensures that $p \in [q_0, q_2, q_1]$ holds. On the other hand, applying the argument of the present part backward, we have

$$[q_0, q_2, q_1] \subset [p_0, \dots, p_{n-1}, p_{n+1}, p_n].$$

In other words, $[p_0, \dots, p_{n+1}] \subset [p_0, \dots, p_{n-1}, p_{n+1}, p_n]$, and the reversed inclusion can be verified via a similar way.

The sufficiency of the statement in assertion (ii) is clear. For the necessity, take arbitrary points p_0, p_1, p_2 of K. Then, $[p_0, p_1] \subset K$ and hence, for all $p \in [p_0, p_1]$, we have that $[p, p_2] \subset K$ holds. That is, $[p_0, p_1, p_2] \subset K$. In more general, applying induction, $[p_0, \dots, p_n] \subset K$ follows for all elements $p_0, \dots, p_n$ of K. Therefore, taking into consideration the definition of $\mathcal{F}$-convex hulls, we arrive at $\operatorname{conv}_{\mathcal{F}}(K) \subset K$, which completes the proof of (ii).

The monotonicity of the hull operator is trivial. For the idempotency, first we restrict the proof on the special case $H = [p_0, \dots, p_n]$. The cases $n = 0$ (when H is a singleton) and $n = 1$ are obvious. Assume $n = 2$, that is, $H = [p_0, p_1, p_2]$ and take points q_1, q_2 of H. Then, there exist points r_1, r_2 of $[p_1, p_2]$ such that $q_1 \in [r_1, p_0]$ and $q_2 \in [r_2, p_0]$ hold. If q is an arbitrary point of the segment $[q_1, q_2]$, then Lemma 2.2 provides a point r of the segment $[r_1, r_2]$ which also belongs to the generalized line interpolating the points p_0 and q. That is, $q \in [p_0, r_1, r_2]$. On the other hand, $[p_0, r_1, r_2] \subset [p_0, p_1, p_2]$ and q is an arbitrary point belonging to $[q_1, q_2]$. This shows that $[p_0, p_1, p_2]$ contains line segments determined by its arbitrary two points; in other words, due to the previous assertion, $[p_0, p_1, p_2]$ is $\mathcal{F}$-convex. Combining this idea by induction, one can prove that the convex hull of a finite set is a fixed point of the hull operator, that is, for all points $p_0, \dots, p_n$ of $I \times \mathbb{R}$,

$$\operatorname{conv}_{\mathcal{F}}[p_0, \dots, p_n] = [p_0, \dots, p_n].$$

Assume now that $H \subset I \times \mathbb{R}$ is arbitrary and take points $a, b \in \operatorname{conv}_{\mathcal{F}}(H)$. Then, there exist some finite subsets A and B of H such that $a \in \operatorname{conv}_{\mathcal{F}}(A)$

and $b \in \mathrm{conv}_{\mathcal{F}}(B)$. Then, $a, b \in \mathrm{conv}_{\mathcal{F}}(A \cup B)$. The set $A \cup B$ is finite; therefore, applying the previous observation, we arrive at

$$[a, b] \subset \mathrm{conv}_{\mathcal{F}}\big(\mathrm{conv}_{\mathcal{F}}(A \cup B)\big) = \mathrm{conv}_{\mathcal{F}}(A \cup B) \subset \mathrm{conv}_{\mathcal{F}}(H).$$

In view of the previous assertion again, this means the $\mathcal{F}$-convexity $\mathrm{conv}_{\mathcal{F}}(H)$, or equivalently, the idempotency of the hull operator. The statements of assumptions (*iv*) and (*v*) are trivial, while the last statement is an immediate consequence of Lemma 2.4 and the properties of (*iii*). □

In the rest of this section we give the $\mathcal{F}$-convex correspondence of the Radon Theorem [25], the Helly Theorem [13] (see also [15]), and the Carathéodory Theorem [11]. For a comprehensive view (and also for interesting historical data) of these results, we refer to [12] and [18].

Theorem 3.3. *If $\mathcal{F}$ is a Beckenbach family over a real interval I and p_0, p_1, p_2, p_3 are pairwise distinct points of $I \times \mathbb{R}$, then there exists a partition $\{A, B\}$ of the set $\{p_0, p_1, p_2, p_3\}$ such that*

$$\mathrm{conv}_{\mathcal{F}}(A) \cap \mathrm{conv}_{\mathcal{F}}(B) \neq \emptyset.$$

Proof. Consider the lexicographical order $\preceq$ on $I \times \mathbb{R}$; that is, $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 < x_2$, or $x_1 = x_2$ and $y_1 \leq y_2$. Without hurting generality, assume that the labeling of the points corresponds to this ordering; that is,

$$p_0 \preceq p_1 \preceq p_2 \preceq p_3.$$

If the points are collinear, then $A = \{p_0, p_3\}$ and $B = \{p_1, p_2\}$ is a suitable partition. Assume that there are three collinear points. Denote these points by q_0, q_1, q_2, where the indices reflect the lexicographical ordering again. In this case, $A = \{q_0, q_2\}$ and $B = \{p_0, p_1, p_2, p_3\} \setminus A$ are proper choices. Finally assume that there is no three element collinear subset of the points $\{p_0, p_1, p_2, p_3\}$. Then, the generalized lines determined by the points $p_0; p_k$ are pairwise distinct, and hence some of them separates those two points that do not belong to this line. Denote this line by φ and assume that it is determined by $q_0 := p_0$ and some point $q_1 \in \{p_1, p_2, p_3\}$. Consider now the generalized line ψ that interpolates the other two points, denoted by q_2 and q_3. If ψ separates q_0 and q_1, then let $A = \{q_0, q_1\}$ and $B = \{q_2, q_3\}$; else $A = \{q_0, q_2, q_3\}$ and $B = \{q_1\}$ represent the desired partition. □

Theorem 3.4. *If $\mathcal{F}$ is a Beckenbach family over a real interval I and $\mathcal{K}$ is a finite collection of $\mathcal{F}$-convex subsets of $I \times \mathbb{R}$ such that, for all members K_0, K_1, K_2 of $\mathcal{K}$ we have $K_0 \cap K_1 \cap K_2 \neq \emptyset$ then $\bigcap \mathcal{K} \neq \emptyset$.*

Proof. Assume first that $\mathcal{K} = \{K_0, K_1, K_2, K_3\}$. Then, for each index $k \in \{0, 1, 2, 3\}$, there exists some element p_k belonging to $\bigcap\{K_j \mid j \neq k\}$. By Theorem 3.3, there exists a partition $\{A, B\}$ of the points p_0, p_1, p_2, p_3 such that

$\mathrm{conv}_{\mathcal{F}}(A) \cap \mathrm{conv}_{\mathcal{F}}(B)$ is nonempty. Let p be an element of this intersection; we claim that p belongs to $\bigcap \mathcal{K}$. Indeed, let k be fixed; we may assume that $p_k \notin K_k$. If $p_k \in A$, then $p_k \notin B$, and hence $p_k \notin K_k$ guarantees $B \subset K_k$. Applying *(iii)* of Lemma 3.2 and the convexity of K_k, we have that $\mathrm{conv}_{\mathcal{F}}(B) \subset K_k$, implying $p \in K_k$. If $p_k \notin A$, then $p_k \in B$, and the same argument results $p \in K_k$.

Assume that the statement of the theorem remains true for any n-element collection of sets and consider a family $\{K_0, \dots, K_n\}$ satisfying the requirements of the theorem. Then the previous part ensures that the $(n-1)$-member family $\{K_0 \cap K_1, K_2 \dots, K_n\}$ has a nonempty intersection, and the proof is completed. □

According to the Riesz Lemma, if a centered family of closed sets contains a compact member, then the entire intersection of the family is nonempty. This argument, combined with the above version of the Helly Theorem, leads to the next result. The precise details of the proof is omitted.

Theorem 3.5. *If $\mathcal{F}$ is a Beckenbach family over a real interval I and $\mathcal{K}$ is a collection of $\mathcal{F}$-convex, closed sets of $I \times \mathbb{R}$ such that, for all members K_0, K_1, K_2 of $\mathcal{K}$ we have $K_0 \cap K_1 \cap K_2 \neq \emptyset$, and $\mathcal{K}$ contains a compact set, then $\bigcap \mathcal{K} \neq \emptyset$.*

The classical Carathéodory Theorem guarantees that the convex hull of a set in Euclidean n-spaces can be reconstructed using $(n+1)$-simplices. In $\mathcal{F}$-convex structures, we have an analogue result: Now the underlying space is two dimensional, and we need generalized triangles for building $\mathcal{F}$-convex hulls.

Theorem 3.6. *If $\mathcal{F}$ is a Beckenbach family over a real interval I and $H \subset I \times \mathbb{R}$, then, for all element p of $\mathrm{conv}_{\mathcal{F}}(H)$, there exists a subset $\{p_0, p_1, p_2\}$ of H such that $p \in [p_0, p_1, p_2]$.*

Proof. If p belongs to $\mathrm{conv}_{\mathcal{F}}(H)$, then $p \in \mathrm{conv}_{\mathcal{F}}(A)$ with some finite subset A of H. For simplicity, assume that A is maximal in sense $p \notin \mathrm{conv}_{\mathcal{F}}\big(A \setminus \{p\}\big)$ for each element p of A. Let $p_0, p_1 \in A$ be chosen such that one of the half planes determined by φ_{p_0,p_1} contain all points of A. Then, there exists some $p_2 \in A$ such that φ_{p_0,p_2} separates the sets $\{p_1\}$ and $F \setminus \{p_1\}$. Using this algorithm, one can label the points of A by $p_0, \dots, p_n$ such that, for every index k, the generalized line φ_{p_0,p_k} separates the sets $\{p_1, \dots, p_{k-1}\}$ and $\{p_{k+1}, \dots, p_n\}$, respectively. Consider now the family

$$\mathcal{A} := \{[p_0, p_k, p_{k+1}] \mid k = 1, \dots, n-1\}.$$

The labeling ensures that, for all $T_1, T_2 \in \mathcal{A}$, the intersection $T_1 \cap T_2$ is either the singleton $\{p_0\}$ or some segment $[p_0, p_k]$. Moreover, we have the representation $\mathrm{conv}_{\mathcal{F}}(A) = \bigcup \mathcal{A}$. Hence, $p \in \mathrm{conv}_{\mathcal{F}}(A)$ implies that there exists some index k such that $p \in [p_0, p_k, p_{k+1}]$ holds, and the proof is completed. □

4. Generalized convex functions

The particular importance of Beckenbach families is that, beside the convexity of planar sets, the convexity of function can also be interpreted within this framework: A function f is generalized convex with respect to a given Beckenbach family, if the generalized segments joining the points of the graph belong to the epigraph. More precisely, we have the following:

Definition 4.1. Let $\mathcal{F}$ be a Beckenbach family over the real interval I. A function $f\colon I \to \mathbb{R}$ is called an *$\mathcal{F}$-convex function*, if, for all $x_1 < x_2$ elements of I and $x \in [x_1, x_2]$, we have the inequality $f(x) \le \varphi(x)$ where $\varphi \in \mathcal{F}$ satisfies $\varphi(x_1) = f(x_1)$ and $\varphi(x_2) = f(x_2)$.

The next theorem gives various characterizations for a function to be generalized convex. It turns out that the role of segments in the definition above can be replaced by the property that Beckenbach lines intersect the graph alternately. The third assertion guarantees the existence of generalized supports of generalized convex functions at interior points of the domain, while the last one shows a desired connection between the $\mathcal{F}$-convexity of sets and functions.

These properties are well-known in the theory of convex functions (consult the books [22] or [26]). Moreover, if the underlying Beckenbach family is induced by a regular pair, analogous results can be obtained (for details, see [7] and [6]; or, in a comprehensive form, [5]). However, in the lack of linear structure, other techniques have to be applied.

Theorem 4.2. *If $\mathcal{F}$ is a Beckenbach family over an open interval I and $f\colon I \to \mathbb{R}$ is a given function, then the following statements are equivalent:*

(*i*) *f is $\mathcal{F}$-convex;*

(*ii*) *for all elements $x_0 \le x_1 < x_2 \le x_3$ of I, we have the inequalities*

$$f|_{[x_0,x_1]} \ge \varphi|_{[x_0,x_1]}, \qquad f|_{[x_1,x_2]} \le \varphi|_{[x_1,x_2]}, \qquad f|_{[x_2,x_3]} \ge \varphi|_{[x_2,x_3]},$$

where φ denotes the Beckenbach line determined by $\varphi(x_1) = f(x_1)$ and $\varphi(x_2) = f(x_2)$;

(*iii*) *for all element $p \in I$ there exists a Beckenbach line φ such that $\varphi \le f$ and $\varphi(p) = f(p)$;*

(*iv*) *for all element $p \in I$, we have the representation*

$$f(p) = \sup\{\varphi(p) \mid \varphi \in \mathcal{F},\ \varphi \le f\};$$

(*v*) $\operatorname{epi}(f)$ *is an $\mathcal{F}$-convex set.*

Proof. For the implication $(i) \Rightarrow (ii)$, fix elements $x_0 \le x_1 < x_2 \le x_3$ of I and denote the Beckenbach line fulfilling the properties $\varphi(x_1) = f(x_1)$ and $\varphi(x_2) = f(x_2)$ by φ. Assume indirectly that there exists some $t \in [x_0, x_1]$ such that $f(t) < \varphi(t)$. For simplicity we may assume that $t = x_0$. Take the

Beckenbach line ψ that interpolates the points $(x_0, f(x_0))$ and $(x_2, f(x_2))$. Since $\psi(x_0) = f(x_0) < \varphi(x_0)$ and $\psi(x_2) = f(x_2) = \varphi(x_2)$, we have

$$\psi|_{]x_0,x_2[} < \varphi|_{]x_0,x_2[}.$$

On the other hand,

$$\psi|_{]x_0,x_2[} \geq f|_{]x_0,x_2[}$$

by the $\mathcal{F}$-convexity of f. In particular, applying these inequalities on the choice $x_1 \in]x_0, x_2[$, we get the contradiction

$$f(x_1) \leq \psi(x_1) < \varphi(x_1) = f(x_1).$$

For the implication $(ii) \Rightarrow (iii)$, take a point $p \in I$. Fix $x_1, x_2 \in I$ such that $x_1 < p < x_2$ hold. Consider the Beckenbach lines φ and ψ satisfying the properties

$$\begin{aligned} \varphi(x_2) &= f(x_2), \qquad & \varphi(p) &= f(p); \\ \psi(x_1) &= f(x_1), \qquad & \psi(p) &= f(p). \end{aligned}$$

Take a sequence (p_n) in $]p, x_2[$ tending to p and the corresponding sequence (φ_n) of Beckenbach lines determined by

$$\varphi_n(p_n) = f(p_n), \qquad \varphi_n(p) = f(p).$$

The second inequality of (ii) and the definition of φ ensure that $f(p_n) \leq \varphi(p_n)$, and hence $\varphi_n(p_n) \leq \varphi(p_n)$. That is, $\varphi_n(t) \leq \varphi(t)$ for all $t \in [p, x_2]$. Similarly, $f(s) \leq \psi(s)$ if $s \in [x_1, p]$. On the other hand, the first inequality of (ii) guarantees $\varphi_n(s) \leq f(s)$; that is $\varphi_n(s) \leq \psi(s)$ for all $s \in [x_1, p]$. Therefore, $\varphi_n(t) \geq \psi(t)$ for all $t \in [p, x_2]$. More precisely, applying similar arguments, we have the next system of inequalities for the allocation of the lines φ_n, φ, ψ:

$$\begin{aligned} \varphi(s) \leq \varphi_n(s) \leq \psi(s), &\quad s \in [x_1, p]; \\ \varphi(t) \geq \varphi_n(t) \geq \psi(t), &\quad t \in [p, x_2]. \end{aligned}$$

According to the first row of inequalities, the sequence $(\varphi_n(x_1))$ is bounded. In view of the Bolzano–Weierstrass theorem we may assume that $\varphi_n(x_1) \to z_1$ with some $z_1 \in \mathbb{R}$ as $n \to \infty$. Consider the Beckenbach line χ that interpolates the points (x_1, z_1) and $(p, f(p))$. Our claim is that χ satisfies (iii). Indeed, if $s < p$, then $\varphi_n(s) \leq f(s)$. Therefore, $\chi(s) \leq f(s)$ by the second assertion of Lemma 2.3. If $p < t$, then $p_n < t$ for sufficiently large $n \in \mathbb{N}$. That is, $\varphi_n(t) < f(t)$ which implies the inequality $\chi(t) \leq f(t)$.

The implication $(iii) \Rightarrow (iv)$ is trivial.

For the implication $(iv) \Rightarrow (v)$, assume indirectly that the epigraph is not an $\mathcal{F}$-convex set. Then, by Lemma 3.2, there exist points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ of $\operatorname{epi}(f)$ such that $[p_1, p_2]$ does not belong to the epigraph. Clearly,

$x_1 \neq x_2$. Let ψ be the Beckenbach line that interpolates p_1 and p_2; then, the indirect assumptions yields $\psi(x) < f(x)$ with some $x \in]x_1, x_2[$. Therefore, due to the representation of (iv), there exists a Beckenbach line φ such that $\varphi \leq f$ and $\psi(x) < \varphi(x)$. On the other hand, p_1 and p_2 belongs to epi(f); therefore,

$$\varphi(x_1) \leq f(x_1) \leq \psi(x_1);$$
$$\varphi(x_2) \leq f(x_2) \leq \psi(x_2).$$

This means that $\varphi \leq \psi$ on the interval $[x_1, x_2]$ which contradicts to $\psi(x) < \varphi(x)$.

The final implication $(v) \Rightarrow (i)$ is straightforward due to Lemma 3.2. $\square$

The notion of $\mathfrak{F}$-concave functions can be introduced similarly to the $\mathfrak{F}$-convex ones, via the reversed inequality between the function and the generalized segment. Clearly, $(-f)$ is not necessarily $\mathfrak{F}$-convex if f is $\mathfrak{F}$-concave. However, applying an argument analogous to the previous one, the next result can be proved.

Theorem 4.3. *If $\mathfrak{F}$ is a Beckenbach family over an open interval I and $f\colon I \to \mathbb{R}$ is a given function, then the following statements are equivalent:*

(i) *f is $\mathfrak{F}$-concave;*

(ii) *for all elements $x_0 \leq x_1 < x_2 \leq x_3$ of I, we have the inequalities*

$$f|_{[x_0,x_1]} \leq \varphi|_{[x_0,x_1]}, \qquad f|_{[x_1,x_2]} \geq \varphi|_{[x_1,x_2]}, \qquad f|_{[x_2,x_3]} \leq \varphi|_{[x_2,x_3]},$$

where φ denotes the Beckenbach line determined by $\varphi(x_1) = f(x_1)$ and $\varphi(x_2) = f(x_2)$;

(iii) *for all element $p \in I$ there exists a Beckenbach line φ such that $\varphi \geq f$ and $\varphi(p) = f(p)$;*

(iv) *for all element $p \in I$, we have the representation*

$$f(p) = \inf\{\varphi(p) \mid \varphi \in \mathfrak{F},\, \varphi \geq f\};$$

(v) hyp(f) *is an $\mathfrak{F}$-convex set.*

One of the most important consequences of the third assertions of Theorem 4.2 and Theorem 4.3 is that generalized convex/concave functions are continuous at the interior points of their domains. The proof of this fact is left to the Reader; let us concentrate here to an other corollary of these theorems. This corollary gives a characterization of $\mathfrak{F}$-convex and compact sets in terms of epigraphs and hypographs of generalized convex and concave functions.

Theorem 4.4. *If $\mathfrak{F}$ is a Beckenbach family over an interval I and K is a compact subset of $I \times \mathbb{R}$, then the following statements are equivalent:*

(i) *K is an $\mathfrak{F}$-convex set;*

(*ii*) *there exists a compact subinterval J of I and there exists an $\mathcal{F}$-convex function $f\colon J \to \mathbb{R}$ and an $\mathcal{F}$-concave function $g\colon J \to \mathbb{R}$, such that $K = \mathrm{epi}(f) \cap \mathrm{hyp}(g)$.*

Proof. If K is a compact and $\mathcal{F}$-convex set, then it is arcwise connected. Therefore, its image under the first projection is a compact and connected subset of I, that is, a compact interval J. Define the functions $f, g\colon J \to \mathbb{R}$ by

$$f(x) = \inf\{y \in \mathbb{R} \mid (x, y) \in K\}, \qquad g(x) = \sup\{y \in \mathbb{R} \mid (x, y) \in K\}.$$

Clearly, $K = \mathrm{epi}(f) \cap \mathrm{hyp}(g)$; moreover, applying the $\mathcal{F}$-convexity property of K on the graph points of f and g, we get the $\mathcal{F}$-convexity of f and $\mathcal{F}$-concavity of g, respectively. The converse statement follows immediately from the last statements of Theorem 4.2 and Theorem 4.3, considering the second assertion of Lemma 3.2. □

The previous theorem enables us to prove the $\mathcal{F}$-convex version of the Minkowski Theorem [20]. As in the classical case, a point p is called an extremal point of the $\mathcal{F}$-convex set K, if the set $K \setminus \{p\}$ is also $\mathcal{F}$-convex.

Theorem 4.5. *If $\mathcal{F}$ is a Beckenbach family over an interval I and K is an $\mathcal{F}$-convex, compact subset of $I \times \mathbb{R}$, then*

$$K = \mathrm{conv}_{\mathcal{F}}\big(\mathrm{extr}(K)\big).$$

Proof. Observe first that Theorem 4.2, Theorem 4.3, and Theorem 4.4 guarantee that K has a supporting generalized line in its each boundary point. That is, for all $p \in \partial K$, there exists a generalized line φ passing trough $p = (x, y)$ such that, for all elements $q = (u, v)$ of K, one of the following properties holds:

$$x \le u; \qquad x \ge u; \qquad \varphi(u) \le v; \qquad \varphi(u) \ge v.$$

The compactness of K ensures that the intersection of K and φ is a (closed) segment; our claim is that the extreme points of this segment are also extreme points of K. Indeed, denote one of the endpoints of the segment by r and assume that $r \in]a, b[$ with some $a, b \in K$. Then, φ and the generalized line determined by a and b are distinct and grazing, which is impossible. Hence, any boundary point of K belongs to the convex hull of at most two extremal points; that is,

$$\partial K \subset \mathrm{conv}_{\mathcal{F}}\big(\mathrm{extr}(K)\big).$$

Assume that q is an interior point of K. Then, due the the compactness again, there exists a segment $[p_0, p_1]$ spanned by boundary points so that $q \in [p_0, p_1]$. The convexity of K implies $[p_0, p_1] \subset K$. The previous argument shows that both p_0 and p_1 belong to some (probably degenerating) segments spanned by extremal points. That is, q belongs to the $\mathcal{F}$-convex hull of some extremal points, resulting

$$K^\circ \subset \mathrm{conv}_{\mathcal{F}}\big(\mathrm{extr}(K)\big).$$

Since K can be obtained as the union of its boundary and interior points, we get that K is a subset of the $\mathcal{F}$-convex hull of the extreme points. The reversed inclusion is trivial, since the set of extreme points is a subset of K, and the $\mathcal{F}$-convex hull operator is monotone. □

5. Applications

Finally we present two applications concerning convex and affine separation problems. The first corollary is an extension of the Baron–Matkowski–Nikodem Theorem [2]. Their original idea turns out to be so effective that it works in case of $\mathcal{F}$-convexity, as well. There are only two parts of their proof that have to be adapted to our setting. To do this, we need the Carathéodory Theorem and the characterization of $\mathcal{F}$-convex functions via $\mathcal{F}$-convex sets.

Corollary 5.1. *If $\mathcal{F}$ is a Beckenbach family over a real interval I and $f, g\colon I \to \mathbb{R}$ are given functions, then the following statements are equivalent:*

(*i*) *there exists an $\mathcal{F}$-convex function $h\colon I \to \mathbb{R}$ such that $f \le h \le g$;*
(*ii*) *for all elements $x_0 \le x_1 \le x_2$ of I, we have the inequality $f(x_1) \le \varphi(x_1)$, where $\varphi \in \mathcal{F}$ is defined by the properties $\varphi(x_0) = g(x_0)$ and $\varphi(x_2) = g(x_2)$.*

Proof. The implication $(i) \Rightarrow (ii)$ is trivial. For the converse implication, the substitution $x_0 = x_1 = x_2$ gives that $f \le g$ holds on I. Denote the $\mathcal{F}$-convex hull of the epigraph of g by E; that is,

$$E := \operatorname{conv}_{\mathcal{F}}\big(\operatorname{epi}(g)\big).$$

Let p be a point of E fixed arbitrarily. Then, by Theorem 3.6, there exist points p_0, p_1, p_2 of $\operatorname{epi}(g)$ such that $p \in [p_0, p_1, p_2] =: S_p$. Assume that $p = (x, y)$ and define

$$z_0 := \inf\{z \in \mathbb{R} \mid (x, z) \in S_p\}.$$

Then, $z_0 \le y$ and (x, z_0) belongs to the boundary of S_p, for example, to the segment $[p_0, p_2]$. Then, the first coordinates of p_0 and p_2 satisfy $x_0 < x_2$; furthermore, $x_0 < x < x_2$. Define the points q_0 and q_2 by

$$q_0 := (x_0, g(x_0)), \qquad q_2 := (x_2, g(x_2)).$$

Denote the Beckenbach line that interpolates the points q_0 and q_2 by φ, and the Beckenbach line that interpolates the points p_0 and p_2 by ψ, respectively. Since p_0 and p_2 belong to the epigraph of g, their second coordinates, in turn, are not smaller than the corresponding second coordinates of q_0 and q_2. That is, for all $x \in [x_0, x_2]$, we have $\varphi(x) \le \psi(x)$ in view of Lemma 2.2. Suppose indirectly that $z_0 < f(x)$. Then, applying (ii) for $x_1 = x$, we get the contradiction

$$z_0 < f(x) \le \varphi(x) \le \psi(x) = z_0.$$

Define

$$h(x) := \inf\{z \in \mathbb{R} \mid (x, z) \in E\}.$$

The previous argument shows that $h\colon I \to \mathbb{R}$ is a function, indeed; moreover, $f \le h \le g$ also holds. To complete the proof, observe that $\operatorname{epi}(h) = E$ and hence $\operatorname{epi}(h)$ is an $\mathcal{F}$-convex set. In view of Theorem 4.2, this means that h is an $\mathcal{F}$-convex function. $\square$

The second corollary generalizes the Nikodem–Wąsowicz Theorem [24] and can be considered as a counterpart of the previous one. This corollary can also be found in [23], and we omit its proof.

Corollary 5.2. *If $\mathcal{F}$ is a Beckenbach family over a real interval I and $f, g\colon I \to \mathbb{R}$ are given functions, then the following statements are equivalent:*

(*i*) *there exists an $\mathcal{F}$-affine function $h\colon I \to \mathbb{R}$ such that $f \le h \le g$;*

(*ii*) *there exists an $\mathcal{F}$-convex function $h_1\colon I \to \mathbb{R}$ and an $\mathcal{F}$-concave function $h_2\colon I \to \mathbb{R}$ such that $f \le h_1 \le g$ and $f \le h_2 \le g$;*

(*iii*) *for all elements $x_0 \le x_1 \le x_2$ of I, we have $f(x_1) \le \varphi_1(x_1)$ and $g(x_1) \ge \varphi_2(x_1)$, where $\varphi_1, \varphi_2 \in \mathcal{F}$ are defined by the properties*

$$\varphi_1(x_0) = g(x_0), \quad \varphi_1(x_2) = g(x_2); \qquad \varphi_2(x_0) = f(x_0), \quad \varphi_2(x_2) = f(x_2).$$

These corollaries can be applied to obtain stability results of the Hyers–Ulam type [14]. Such results can be found in [23]. Note, that Krzyszkowski also investigated Ulam's stability problem [16] in Beckenbach's setting, but in a direct way. Let us also remark, that the Authors of [23] used an alternative approach to the corollaries above than ours. Their proofs rely on the $\mathcal{F}$-convex version of Kakutani Theorem, stating that two disjoint $\mathcal{F}$-convex sets of $I \times \mathbb{R}$ can be separated by complementary $\mathcal{F}$-convex sets.

The second corollary can be generalized for convex Beckenbach families of arbitrary parameter [10]. However, such kind of extension of the first corollary is not known even in that case when the underlying family (of arbitrary parameter) has a linear structure.

References

[1] M. Balaj, Sz. Wąsowicz: Haar spaces and polynomial selections, Math. Pannon. 14(1) (2003) 63–70.

[2] K. Baron, J. Matkowski, K. Nikodem: A sandwich with convexity, Math. Pannon. 5(1) (1994) 139–144.

[3] E. F. Beckenbach: Generalized convex functions, Bull. Amer. Math. Soc. 43(6) (1937) 363–371.

[4] E. F. Beckenbach, R. H. Bing: On generalized convex functions, Trans. Amer. Math. Soc. 58 (1945) 220–230.

[5] M. Bessenyei: Hermite-Hadamard-type inequalities for generalized convex functions, JIPAM. J. Inequal. Pure Appl. Math. 9(3) (2008) Article 63, 51 p.

[6] M. Bessenyei, Zs. Páles: Characterization of convexity via Hadamard's inequality, Math. Inequal. Appl. 9(1) (2006) 53–62.

[7] M. Bessenyei, Zs. Páles: Hadamard-type inequalities for generalized convex functions, Math. Inequal. Appl. 6(3) (2003) 379–392.

[8] M. Bessenyei, Zs. Páles: Separation by linear interpolation families, J. Nonlinear Convex Anal. 13(1) (2012) 49–56.

[9] M. Bessenyei, P. Szokol: Convex separation by regular pairs, J. Geom. 104(1) (2013) 45–56.

[10] M. Bessenyei, P. Szokol: Separation by convex interpolation families, J. Convex Analysis 20(4) (2013) 937–946.

[11] C. Carathéodory: Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen, Rend. Circ. Mat. Palermo 32 (1911) 193–217.

[12] L. Danzer, B. Grünbaum, V. Klee: Helly's theorem and its relatives, in: Convexity (Seattle, 1961), Proc. Sympos. Pure Math. 7, V. Klee (ed.), Amer. Math. Soc., Providence (1963) 101–180.

[13] E. Helly: Über Mengen konvexer Körper mit gemeinschaftlichen Punkten, Jahresber. Dtsch. Math.-Ver. 32 (1923) 175–176.

[14] D. H. Hyers, S. M. Ulam: Approximately convex functions, Proc. Amer. Math. Soc. 3 (1952) 821–828.

[15] D. Kőnig: Über konvexe Körper, Math. Zeitschr. 14 (1922) 208–220.

[16] J. Krzyszkowski: Approximately generalized convex functions, Math. Pannon. 12(1) (2001) 93–104.

[17] J. Krzyszkowski: Generalized convex sets, Rocz. Nauk.-Dydakt., Pr. Mat. 14 (1997) 59–68.

[18] S. R. Lay: Convex Sets and their Applications, Pure and Applied Mathematics, John Wiley & Sons, New York (1982).

[19] G. G. Magaril-Il'yaev, V. M. Tikhomirov: Convex Analysis: Theory and Applications, Translations of Mathematical Monographs 222, Amer. Math. Soc., Providence (2003).

[20] H. Minkowski: Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs, in: Gesammelte Abhandlungen II, D. Hilbert (ed.), Teubner, Leipzig (1911) 131–229.

[21] F. R. Moulton: A simple non-desarguesian geometry, Trans. Amer. Math. Soc. 14(2) (1902) 192–195.

[22] C. P. Niculescu, L.-E. Persson: Convex Functions and their Applications, CMS Books in Mathematics 23, Springer, New York (2006).

[23] K. Nikodem, Zs. Páles: Generalized convexity and separation theorems, J. Convex Analysis 14(2) (2007) 239–247.

[24] K. Nikodem, Sz. Wąsowicz: A sandwich theorem and Hyers-Ulam stability of affine functions, Aequationes Math. 49(1–2) (1995) 160–164.

[25] J. Radon: Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten, Math. Ann. 83 (1921) 113–115.

[26] A. W. Roberts, D. E. Varberg: Convex Functions, Pure and Applied Mathematics 57, Academic Press, New York (1973).

[27] R. T. Rockafellar: Convex Analysis, Princeton Mathematical Series 28, Princeton University Press, Princeton (1970).

[28] L. Tornheim: On n-parameter families of functions and associated convex functions, Trans. Amer. Math. Soc. 69 (1950) 457–467.

[29] M. L. J. van de Vel: Theory of Convex Structures, North-Holland Mathematical Library 50, North-Holland, Amsterdam (1993).

[30] Sz. Wąsowicz: Polynomial selections and separation by polynomials, Stud. Math. 120(1) (1996) 75–82.