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On Fixed Point Properties of a Convex Combination of Metric Projections

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To illustrate the usability of a famous Browder's theorem, some fixed point properties of a convex combination of metric projections are provided.

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1. Introduction and preliminaries

The convex combinations of projections were intensively investigated in the recent past about the image recovery or the convex feasibility problems – see for example the basic papers [1] and [6]. In this paper we present some novel properties of a convex combination of projections in a Hilbert space, to illustrate once again the usability of a famous fixed point theorem due to Felix Browder.

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$.

To better understand what follows, we recall that an operator $T : H \to H$ is said to be nonexpansive(resp. firmly nonexpansive) if for all $x, y \in H$

$$\|Tx - Ty\| \leq \|x - y\| \quad (\text{resp. } \langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2\,).$$

If $T : H \to H$ is firmly nonexpansive, using the Schwartz inequality, we have

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle \leq \|Tx - Ty\| \cdot \|x - y\|\,,$$

therefore T is nonexpansive.

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Let P_K be the metric projection onto a closed convex nonempty subset K of H. The projection operator P_K is characterized by

$$P_K x \in K : \langle P_K x - x, y - P_K x\rangle \geq 0 \text{ for all } y \in K.$$

It follows from this characterization that for $x, y \in H$

$$\begin{aligned} & \langle P_K x - P_K y, x - y\rangle \\ = \; & \langle P_K x - P_K y, x - P_K x\rangle + \|P_K x - P_K y\|^2 + \langle P_K x - P_K y, P_K y - y\rangle \\ \geq \; & \|P_K x - P_K y\|^2 , \end{aligned}$$

so P_K is a firmly nonexpansive operator, thus P_K is nonexpansive. Also:

1. $\|x - P_K x\| = d(x; K) = \inf\{\|x - y\| \, / \, y \in K\}$,
2. $\mathrm{Fix}(P_K) = K$, where $\mathrm{Fix}(T)$ denote the set of fixed points of an operator $T : C \subseteq H \to H$.

Let A, B be two bounded closed convex nonempty subsets of H and P_A, P_B be the metric projections onto A, respectively B.

The aim of this paper is to investigate the fixed point properties of a convex combination of operators P_A and P_B.

2. Results

Consider the operator $P : H \to H$ defined by $Px = \alpha P_A x + \beta P_B x (\alpha, \beta > 0, \alpha + \beta = 1)$. P is nonexpansive as a convex combination of nonexpansive operators and, in order to establish results regarding its fixed points, we use the following famous theorem due to F. Browder [2]:

Theorem 2.1. *Let X be a uniformly convex Banach space, U a nonexpansive mapping of the bounded closed convex subset C of X into C. Then U has a fixed point in C.*

If $C = \bar{\mathrm{co}}(A \cup B)$, the closure of the convex hull of $A \cup B$, then P is a nonexpansive mapping of the bounded closed convex subset C of the Hilbert space H into C and, by Theorem 2.1, $\mathrm{Fix}(P) \neq \phi$.

In order to study $\mathrm{Fix}(P)$ it is necessary to prove that, generally, the operator P is not a metric projection. The example below justifies this last assertion.

Consider the real Hilbert space $H = \mathbb{R}^2$ with the Euclidean inner product. Let $A = [-2, -1] \times [0, 1]$ and $B = [1, 2] \times [0, 1]$. A and B are nonempty closed convex bounded subsets of H and $C = \bar{\mathrm{co}}(A \cup B) = [-2, 2] \times [0, 1]$.

For $\alpha = \beta = \frac{1}{2}$ the operator P becomes $P = \frac{1}{2}(P_A + P_B)$. Prove now that $\mathrm{Med}(A; B) = \{(0, x) : x \in [0, 1]\} \subset \mathrm{Fix}(P)$.

Indeed, for $u = (0, x) \in \mathrm{Med}(A; B)$ we obtain that $Pu = \frac{1}{2}[P_A(0, x) + P_B(0, x)] = \frac{1}{2}[(-1, x) + (1, x)] = (0, x) = u$ for all $x \in [0, 1]$, thus $u \in \mathrm{Fix}(P)$.

Now we are in position to affirm that, generally, the operator P is not a metric projection, because $\mathrm{Fix}(P)$ is not a singleton.

To obtain a general characterization of $\mathrm{Fix}(P)$ we need (see [3, 4, 5]):

Theorem 2.2. *Let X be a bounded closed convex subset of H and $U_i : X \to X$, $i \in I$, be nonexpansive operators with a common fixed point, let $\omega_i > 0 (i \in I)$, $\sum_{i \in I} \omega_i = 1$ and let $U = \sum_{i \in I} \omega_i U_i$. Then $\mathrm{Fix}(U) = \bigcap_{i \in I} \mathrm{Fix}(U_i)$.*

So, applying Theorem 2.2 we have: if $A \cap B \neq \phi$, then $\mathrm{Fix}(P) = \mathrm{Fix}(P_A) \cap \mathrm{Fix}(P_B) = A \cap B$.

In order to study $\mathrm{Fix}(P)$ when $A \cap B = \phi$, we provide in this paper a new characterization of $\mathrm{Fix}(P)$.

If $x \in \mathrm{Fix}(P)$, then $Px = \alpha P_A x + \beta P_B x = x = (\alpha + \beta)x$. Thus

$$\alpha(P_A x - x) = \beta(x - P_B x). \tag{1}$$

Consequently $\alpha \|P_A x - x\| = \beta \|x - P_B x\|$ and it follows that $\alpha d(x; A) = \beta d(x; B)$.

We obtain that $\mathrm{Fix}(P) \subset \{x \in H : \alpha d(x; A) = \beta d(x; B)\}$ and this inclusion offers an useful characterization of $\mathrm{Fix}(P)$, when $A \cap B = \phi$.

Further we study the set of fixed points of the nonexpansive operator $-P$.

Theorem 2.3. *The operator $T = I + P : H \to H$ is a homeomorphism, where I is the identity of H.*

Proof. The operator T satisfies

$$\|Tx - Ty\| \leq \|x - y\| + \|Px - Py\| \leq 2\|x - y\| \tag{2}$$

and

$$\begin{aligned} & \langle Tx - Ty, x - y \rangle \\ = \ & \langle x - y, x - y \rangle + \alpha \langle P_A x - P_A y, x - y \rangle + \beta \langle P_B x - P_B y, x - y \rangle \\ \geq \ & \|x - y\|^2 + \alpha \|P_A x - P_A y\|^2 + \beta \|P_B x - P_B y\|^2 \geq \|x - y\|^2 \end{aligned} \tag{3}$$

for all $x, y \in H$.

For $f \in H$ and $\gamma > 0$ we consider now the operator $S_\gamma : H \longrightarrow H$ defined by

$$S_\gamma x = x - \gamma(Tx - f).$$

We have

$$\begin{aligned} ||S_\gamma x - S_\gamma y||^2 &= \langle x - \gamma Tx - y + \gamma Ty, x - \gamma Tx - y + \gamma Ty \rangle \\ &= \langle x - y - \gamma(Tx - Ty), x - y - \gamma(Tx - Ty) \rangle \\ &= \|x - y\|^2 - 2\gamma \langle Tx - Ty, x - y \rangle + \gamma^2 \|Tx - Ty\|^2 \\ &\leq \|x - y\|^2 - 2\gamma \|x - y\|^2 + 4\gamma^2 \|x - y\|^2, \end{aligned}$$

so

$$||S_\gamma x - S_\gamma y|| \leq \sqrt{4\gamma^2 - 2\gamma + 1}\,\|x - y\|\,, \quad \text{for all } x, y \in H.$$

If $\gamma \in (0, \frac{1}{2})$, then S_γ is a contraction, because $\sqrt{4\gamma^2 - 2\gamma + 1} < 1$.

Applying the Banach fixed point theorem it results that S_γ has a unique fixed point, that is there exists a unique element $u \in H$ such that $S_\gamma u = u$. We obtain that $u - \gamma(Tu - f) = u$, and consequently u is the unique solution of the equation $Tx = f$.

Thus $T : H \to H$ is a bijection.

Now, using (3), with the Schwartz inequality, we have

$$\|Tx - Ty\| \cdot \|x - y\| \geq \langle Tx - Ty, x - y\rangle \geq \|x - y\|^2 \quad \text{for all } x, y \in H$$

and, consequently,

$$\|x - y\| \leq \|Tx - Ty\| \leq 2\,\|x - y\| \quad \text{for all } x, y \in H.$$

The above double inequality justifies that T and T^{-1} are Lipschitz continuous. Therefore T is a homeomorphism and the proof of Theorem 2.3 is complete. □

As a consequence of Theorem 2.3 the operator $I + P$ has a unique zero, so we obtain

Theorem 2.4. *The operator $-P : H \to H$ is a nonexpansive operator having unique fixed point.*

Generally, the nonexpansive operators have multiple fixed points. We provided above an example of nonexpansive operator having a unique fixed point.

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