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On Legendre and Weierstrass Conditions in One-Dimensional Variational Problems*

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In this paper we show that classical conditions of Legendre and Weierstrass characterize lower semicontinuity of the correspondent integral functional in appropriate classes of functions. A strengthened version of these conditions characterize the property of the functional of convergence in energy.

Keywords: Integral functionals, Legendre condition, Weierstrass condition, convexity at a point, strict convexity at a point, lower semicontinuity, convergence in energy, Young measures

1. Introduction and statements of main results

Let $L(x,u,v) : R\times R^n\times R^n \to R$ be continuous.

The classical theory of the Calculus of Variations studies weak and strong local minimizers of the integral functional

$$J(u)=\int_a^b L(x,u(x),\dot u(x))dx, \quad -\infty<a<b<\infty, \tag{1}$$

see e.g. [1].

Definition 1.1. A function $u\in C^1([a,b];R^n)$ is called weak local minimizer of the functional (1) provided there exists $\epsilon>0$ such that for each $w\in C^1([a,b];R^n)$ with $w(a)=u(a)$, $w(b)=u(b)$, $||u-w||_{C^1}\le\epsilon$, we have $J(w)\ge J(u)$.

Definition 1.2. A function $u\in C^1([a,b];R^n)$ is called strong local minimizer of the functional (1) provided there exists $\epsilon>0$ such that for each

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$w \in C^1([a,b];R^n)$ with $w(a) = u(a)$, $w(b) = u(b)$, $||u-w||_C \leq \epsilon$, we have $J(w) \geq J(u)$.

Weak local minimizers satisfy the Legendre condition.

Theorem 1.3. *Let u be a weak local minimizer and let $L \in C^2(R^{2n+1})$. Then for all $x \in [a,b]$ we have*

$$\sum_{i,j=1}^{n} L_{v_i v_j}(x, u(x), \dot{u}(x))\xi_i \xi_j \geq 0, \quad \forall \xi \in R^n. \tag{2}$$

Strong local minimizers satisfy the Weierstrass condition.

Theorem 1.4. *Let u be a strong local minimizer and let $L \in C^1(R^{2n+1})$. Set*

$$\begin{aligned} &E(x, u(x), \dot{u}(x), w) \\ := \; &L(x, u(x), w) - L(x, u(x), \dot{u}(x)) - \langle L_v(x, u(x), \dot{u}(x)), w - \dot{u}(x)\rangle. \end{aligned}$$

Then for all $x \in [a,b]$ we have

$$E(x, u(x), \dot{u}(x), w) \geq 0, \quad \forall w \in R^n. \tag{3}$$

It turns out that Theorems 1.3, 1.4 are corollaries of more general assertions.

Definition 1.5. Let $G : U \to R$ be continious, where U is a convex subset of R^n. Let v_0 be a point of the relative interior $\operatorname{reint} U$ of the set U. Then we say that G is convex (strictly convex) at v_0 with respect U if for every $c_i > 0$, $v_i \in U$, $v_i \neq v_0$, $i = 1, \ldots, m$, such that $\sum c_i = 1$, $\sum c_i v_i = v_0$, we have (strict) Jensen inequality

$$\sum_{i=1}^{m} c_i G(v_i) \geq G(v_0) \quad \left(\sum c_i G(v_i) > G(v_0)\right). \tag{4}$$

We say that G is concave (strictly concave) at v_0 with respect to U if $-G$ is convex (strictly convex) at this point with respect to U.

It seems that convexity at a point was first introduced in [10]. Strict convexity at a point was introduced by the author in [7] and was already exploited in [6].

The following lemma indicates characterizations of the property of convexity at a point.

Lemma 1.6. *Let $G : U \to R$ be continuous, where U is a convex subset of R^n. Let v_0 be a point of the relative interior of the set U. Then the following statements are equivalent:*

1) the function G is convex at v_0 with respect to U;

2) *there exists $f \in R^n$ such that for all $v \in U$ we have*

$$G(v) - G(v_0) \geq \langle f, v - v_0 \rangle;$$

3) *for each probability measure ν with the support in U and with the centre of mass at v_0 we have*

$$\int_U G d\nu \geq G(v_0).$$

The following theorem is valid

Theorem 1.7. *Let $L \in C(R^{2n+1})$. Let also $\epsilon > 0$, $u_0 \in C^1([a, b]; R^n)$. Then*

$$\liminf_{k \to \infty} J(u_k) \geq J(u_0)$$

for all $u_k \in C^1$ such that $||u_k - u_0||_{C^1} \leq \epsilon$, $k \in N$, $||u_k - u_0||_C \to 0$, $k \to \infty$, if and only if for each $x \in [a, b]$ the function $L(x, u_0(x), \cdot) : \bar{B}(\dot{u}_0(x), \epsilon) \to R$ is convex at $\dot{u}_0(x)$.

We can state two corollaries of Theorem 1.7 which strengthen Theorems 1.3, 1.4.

Corollary 1.8. *Let $L \in C(R^{2n+1})$, $u \in C^1$. Assume that there exists $\epsilon > 0$ such that*

$$\liminf_{k \to \infty} J(u_k) \geq J(u)$$

for all $u_k \in C^1$ such that $||u_k - u||_{C^1} \leq \epsilon$, $k \in N$, $||u_k - u||_C \to 0$, $k \to \infty$. Assume also that for each $x \in [a, b]$ the function $L(x, u(x), \cdot)$ is twicely differentiable at $\dot{u}(x)$. Then the Legendre condition (2) *holds.*

Corollary 1.9. *Let $L \in C(R^{2n+1})$, $u \in C^1$ and let*

$$\liminf_{k \to \infty} J(u_k) \geq J(u)$$

for each sequence u_k bounded in C^1-norm and such that $||u_k - u||_C \to 0$, $k \to \infty$. Then for each $x \in [a, b]$ the function $L(x, u(x), \cdot)$ is Fenchel subdiffirentiable at $\dot{u}(x)$, i.e. the Weierstrass condition

$$L(x, u(x), v) - L(x, u(x), \dot{u}(x)) - \langle l, v - \dot{u}(x) \rangle \geq 0, \quad \forall v \in R^n,$$

holds, where l - any element of the subgradient.

What is interesting, strict condition at a point also characterizes certain property of integral functionals, more precisely – the property of convergence in energy, see [4], [5].

We first introduce convenient characterizations of the property of strict convexity at a point.

Lemma 1.10. *Let $G : U \to R^n$ be continuous, where U is a convex subset of R^n with dimension of relative interior equal to l, and let v_0 be a point of relative interior of U. Then the following assertions are equivalent:*

1) *G is strictly convex at v_0 with respect to U;*

2) *there exists $f \in R^n$ such that $G(v) - G(v_0) \geq \langle f, v - v_0 \rangle$ for all $v \in U$; moreover if H is a plane of dimension l which contains U then there exists a basis $e_1, \dots, e_l$ of the plane $H - \{v_0\}$ such that if $v \in U$, $v = v_0 + \sum_1^l c_i e_i$, where $c_i = 0$ for $i \in \{1, \dots, j-1\}$, $c_j > 0$,then $G(v) - G(v_0) > \langle f, v - v_0 \rangle$;*

3) *for each nontrivial (nonequal to a Dirac mass) probability measure ν with the support in U and with the centre of mass at v_0 we have the strict Jensen inequality*

$$\int_U L d\nu > L(v_0).$$

The following theorem holds

Theorem 1.11. *Let $L \in C(R^{2n+1})$. Let $\epsilon > 0$, $u \in C^1([a, b]; R^n)$. Then the convergences $||u_k - u||_C \to 0$, $J(u_k) \to J(u)$ imply the convergence $||u_k - u||_{W^{1,1}} \to 0$ for all $u_k \in C^1$ such that $||u_k - u||_{C^1} \leq \epsilon$, $k \in N$, if and only if either $L(x, u(x), \cdot)$ is strictly convex at $\dot{u}(x)$ with respect to ball $\bar{B}(\dot{u}(x), \epsilon)$ for a.e. $x \in [a, b]$ or this holds for $-L$.*

2. Auxiliary facts from Young measure theory

In this section we state certain facts from Young measure theory which we will use in this paper.

Theorem 2.1. *Let $\xi_k : [a, b] \to R^n$, $k \in N$, be a sequence of measurable functions such that $||\xi_k||_{L^\infty} \leq M < \infty$, $k \in N$. Then there exists a subsequence ξ_k, $k \in N$, (not relabeled) and a family of probability measures $(\nu_x)_{x \in [a,b]}$, where $\operatorname{supp} \nu_x \subset \bar{B}(0, M)$ for a.e. $x \in [a, b]$, which is generated by the subsequence ξ_k as a Young measure that means that for each continuous function $L(x, v) : [a, b] \times R^n \to R$ we have*

$$L(\cdot, \xi_k(\cdot)) \rightharpoonup^* \int_{R^n} L(\cdot, v) d\nu_{(\cdot)} \quad in\ L^\infty \tag{5}$$

(here and everywhere further " $\rightharpoonup$ " denotes the weak convergence). In particular $\xi_k \rightharpoonup^* \xi$ in L^∞, where $\xi(x)$ is the centre of mass of ν_x, $x \in [a, b]$. Moweover $||\xi_k - \xi||_{L^1} \to 0$, $k \to \infty$, if and only if $\nu_x = \delta_{\xi(x)}$ for a.e. $x \in [a, b]$.

Theorem 2.2. *Let $(\nu_x)_{x \in [a,b]}$ be a sequence of probability measures such that $\operatorname{supp} \nu_x \subset \bar{B}(0, M)$ for a.e. $x \in [a, b]$. Let also for each continuous function $L(x, v) : [a, b] \times R^n \to R$ the function $x \to \int_{R^n} L(x, v) d\nu_x$ be measurable. Then there exists a sequence of measurable functions $\xi_k : [a, b] \to B(0, M)$, $k \in N$, which generate $(\nu_x)_{x \in [a,b]}$ as a Young measure.*

Theorems 2.1, 2.2 are the standard facts of Young measure theory. Their proofs could be found in [9], [8]. We will also use the following corollary of these theorems.

Theorem 2.3. *Let $u \in C^1([a,b];R^n)$, $(\nu_x)_{x\in[a,b]}$ be a Young measure such that for a.e. $x \in [a,b]$ the centre of mass of the measure ν_x is $\dot{u}(x)$ and $\operatorname{supp}\nu_x \subset \bar{B}(\dot{u}(x),M)$. Then there exists a sequence of functions $u_k \in C^1([a,b],R^n)$, $k \in N$, such that $||u_k - u||_{C^1} < M$, $u_k(a) = u(a)$, $u_k(b) = u(b)$, $k \in N$, $||u_k - u||_C \to 0$, $k \to \infty$, and the sequence $\dot{u}_k$, $k \in N$, generate Young measure $(\nu_x)_{x\in[a,b]}$.*

Proof. Due to Theorem 2.2 there exists a sequence $\xi_k \in L^\infty$, $k \in N$, such that $||\xi_k - \dot{u}||_{L^\infty} < M$ and such that ξ_k generate Young measure $(\nu_x)_{x\in[a,b]}$. There exists also a sequence $\eta_k \in C([a,b];R^n)$, $k \in N$, such that $||\eta_k - \dot{u}||_{L^\infty} < M$, $k \in N$, $||\eta_k - \xi_k||_{L^1} \to 0$, $k \to \infty$. Then η_k, $k \in N$, also generate Young measure $(\nu_x)_{x\in[a,b]}$.

Since $\eta_k \rightharpoonup^* \dot{u}$ L^∞ there exists another sequence $\phi_k \in C([a,b],R^n)$ such that $||\phi_k - \eta_k||_C \to 0$, $k \to \infty$, $||\phi_k - \dot{u}||_C < M$, $\int_a^b \phi_k dx = \int_a^b \dot{u}dx$, $k \in N$. We can take the sequence ϕ_k, $k \in N$, satisfying an additional condition: if $u_k(y) = u(a) + \int_a^y \phi_k(x)dx$, $y \in [a,b]$, $k \in N$, then $||u_k - u||_{C^1} < M$, $k \in N$. The sequence u_k, $k \in N$, satisfies all the requirements of the theorem. □

3. Convexity and strict convexity at a point

In this section we prove Lemmas 1.6, 1.10.

Proof of Lemma 1.6. *1) => 2)*. First assume that additionally U is a compact set. Then we define out of U the function L as $+\infty$. Consider the convexification $L^{**} : U \to R$ of the function L, i.e.

$$L^{**} := \inf\Big\{\sum c_i L(v_i) : c_i > 0, v_i \in U,$$
$$i = 1,\dots,m, m \in N, \sum c_i = 1, \sum c_i v_i = v\Big\}.$$

Then $L^{**} : U \to R$ is a convex function and by Hahn-Banach theorem there exists $f \in R^n$ such that

$$L^{**}(v) - L^{**}(v_0) \geq \langle f, v - v_0\rangle, \quad v \in U.$$

Since $L^{**}(v_0) = L(v_0)$, $L \geq L^{**}$ everywhere we derive that

$$L(v) - L(v_0) \geq \langle f, v - v_0\rangle, \quad v \in U.$$

In the case of arbitrary convex set we consider an extending sequence of compact convex sets U_k such that $v_0 \in U_k$, $k \in N$, $\cup_{k=1}^\infty U_k = \operatorname{reint} U$. Then there exists a bounded sequence f_k, $k \in N$, such that

$$L(v) - L(v_0) \geq \langle f_k, v - v_0\rangle, \quad \forall v \in U_k, \ k \in N.$$

For a cluster point f of the sequence f_k we have

$$L(v) - L(v_0) \geq \langle f, v - v_0 \rangle, \ \forall v \in U.$$

2) => 3). In case ν is a probability measure with the support in U and with the centre of mass at v_0 we have

$$\int_{R^n} L(v)d\nu - L(v_0) = \int_{R^n} (L(v) - L(v_0))d\nu \geq \int_{R^n} \langle f, v - v_0 \rangle d\nu = 0.$$

3) => 1). Since the Jensen inequality

$$\int_U L(v)d\nu \geq L(v_0)$$

is valid for any probability measure with the support in U and with the centre of mass at v_0 this holds also for the case $\nu = \sum_{i=1}^m c_i\delta_{v_i}$. Then

$$\sum_{i=1}^m c_i L(v_i) = \int_U L(v)\nu \geq L(v_0),$$

which means convexity at the point.

Lemma 1.6 is proved. □

Proof of Lemma 1.10. *1) => 2)*. Let $f \in R^n$ be such that

$$L(v) - L(v_0) \geq \langle f, v - v_0 \rangle, \quad v \in U,$$

see Lemma 1.6.

Consider the set

$$V := \{v \in U : L(v) - L(v_0) - \langle f, v - v_0 \rangle = 0\}.$$

Since L is strictly convex at v_0 with respect to U the convex hull $\tilde{V}$ of the set $V \setminus \{v_0\}$ does not contain v_0. Then due to Hahn-Banach theorem there exists $e_1 \in R^n$, $v_0 + e_1 \in U$, such that $L(v) - L(v_0) > \langle f, v - v_0 \rangle$ for $v \in U$ with $\langle e_1, v - v_0 \rangle > 0$. Consider the set $U_1 := U \cap \{v : \langle e_1, v - v_0 \rangle = 0\}$. Since the function L is strictly convex at v_0 with respect to U_1 we obtain analogously that there exists $e_2 \in R^n$, $v_0 + e_2 \in U_1$, such that $L(v) - L(v_0) > \langle f, v - v_0 \rangle$ for $v \in U_1$ with $\langle v - v_0, e_2 \rangle > 0$.

This way we can find a basis $e_1, \ldots, e_l$ of the space $H - \{v_0\}$ with the required properties.

2) => 3). Let ν be a probability measure with the support in U and with the centre of mass at v_0. There exists $f \in R^n$ such that

$$L(v) - L(v_0) \geq \langle f, v - v_0 \rangle, \quad v \in U,$$

and *2*) holds. Then $\int_U L d\nu \geq L(v_0)$. Assume $\int_U L d\nu = L(v_0)$. Then

$$\operatorname{supp}\nu \cap \{v \in U : \langle e_1, v - v_0\rangle > 0\} = \emptyset,$$

i.e.

$$\operatorname{supp}\nu \subset \{v \in U : \langle e_1, v - v_0\rangle = 0\}.$$

Analogously we can establish that

$$\operatorname{supp}\nu \subset \{v \in U : \langle e_i, v - v_0\rangle = 0\}, \quad i = 2, \dots, l.$$

Therefore $\nu = \delta_{v_0}$. This way we proved that $\int_U L d\nu > L(v_0)$ for the nontrivial measure ν.

3) => *1*). This statement is valid since a convex combination of Dirac masses $\sum c_i \delta_{v_i}$ is a special case of a probability measure ν with the support in U and with centre of mass at v_0.

Lemma 1.10 is proved. □

Further we will use one more lemma.

Lemma 3.1. *Let U be a convex subset of R^n of dimension l, let v_0 be a point of the relative interior of U. Assume also that $G : U \to R$ is a continuous function which is convex but not strictly convex at the point v_0 with respect to the set U. Then there exist $c_i > 0$, $v_i \in U$, $v_i \neq v_0$, $i = 1, \dots, l+1$, with the properties*

$$\sum_{i=1}^{l+1} c_i = 1, \qquad \sum_{i=1}^{l+1} c_i v_i = v_0, \qquad \sum_{i=1}^{l+1} c_i G(v_i) = G(v_0).$$

Proof. Since G is convex at v_0 with respect to U we can apply Lemma 6 to find $f \in R^n$ such that

$$G(v) - G(v_0) - \langle f, v - v_0\rangle \geq 0, \quad \forall v \in U. \tag{6}$$

Let V be a set of $v \in U$, $v \neq v_0$, for which we have equality in (6). By conditions of lemma the convex hull of V contains v_0. Since dimension of the set V is not larger than l, we obtain by Carathéodory theorem, see e.g. [3], that the convex hull consists of convex combinations of $l+1$ elements of the set V, that proves the lemma. □

4. Theorem 1.7

Proof of Theorem 1.7. Let $\dot{u} \in C^1([a,b]; R^n)$, $L(x,u,v) \in C([a,b] \times R^n \times R^n)$ and assume that for all $x \in [a,b]$ function $L(x,u(x),\cdot)$ is convex at $\dot{u}(x)$ with respect to the ball $\bar{B}(\dot{u}(x), \epsilon)$. Let $u_k \in C^1([a,b]; R^n)$ be a sequence such that $||u_k - u||_{C^1} \leq \epsilon$, $k \in N$, $||u_k - u||_C \to 0$, $k \to \infty$. Replacing this sequence by a subsequence we can assume that $J(u_k) \to I$, $k \to \infty$. We have to show that $I \geq J(u)$.

Due to Theorem 2.1 an appropriate subsequence of the sequence $\dot{u}_k$, $k \in N$, generates a Young measure $(\nu_x)_{x\in[a,b]}$. In this case the centre of mass of ν_x is $\dot{u}(x)$ and, moreover, $\operatorname{supp}\nu_x \subset \bar{B}(\dot{u}(x), \epsilon)$ for a.e. $x \in [a, b]$. In view of Theorem 2.1 we also have

$$J(u_k) \to \int_a^b \int_{R^n} L(x, u(x), v)d\nu_x dx,$$

since $(u_k, \dot{u}_k)$ generates Young measure $(\delta_{u(x)} \otimes \nu_x)_{x\in[a,b]}$.

Due to Lemma 1.6 we have

$$\int_{R^n} L(x, u(x), \cdot)d\nu_x \geq L(x, u(x), \dot{u}(x)), \quad x \in [a, b].$$

Then, indeed, $I \geq J(u)$.

Conversely, let $I \geq J(u)$. We have to show that for each $x \in [a, b]$ the function $L(x, u(x), \cdot)$ is convex at $\dot{u}(x)$ with respect to $\bar{B}(\dot{u}(x), \epsilon)$. Assume the contrary. Then for a $x_0 \in [a, b]$ we can isolate $c_i > 0$, $v_i \neq \dot{u}(x_0)$, $i = 1, \ldots, m$, such that $v_i \in \bar{B}(\dot{u}(x_0), \epsilon)$, $\sum c_i L(v_i) = \dot{u}(x_0)$, $\sum c_i = 1$,

$$\sum_{i=1}^{m} c_i L(x_0, u(x_0), v_i) < L(x_0), u(x_0), \dot{u}(x_0)).$$

Let $v_i(x) = v_i + \dot{u}(x) - \dot{u}(x_0)$, $i = 1, \ldots, m$. For all x sufficiently close to x_0 we have $\sum c_i v_i(x) = \dot{u}(x)$,

$$\sum_{i=1}^{m} c_i L(x, u(x), v_i(x)) < L(x, u(x), \dot{u}(x)).$$

For these x we assume $\nu_x = \sum c_i \delta_{v_i(x)}$, for other $x \in [a, b]$ we assume $\nu_x = \delta_{\dot{u}(x)}$. By Theorem 2.2 $(\nu_x)_{x\in[a,b]}$ is a Young measure which is generated due to Theorem 2.3 by a sequence $\dot{u}_k$, $k \in N$, where $u_k \in C^1([a, b]; R^n)$, $||u_k - u||_{C^1} < \epsilon$, $k \in N$, $||u_k - u||_C \to 0$, $k \to \infty$. By Theorem 2.1

$$J(u_k) \to \int_a^b \int_{R^n} L(x, u(x), \cdot)d\nu_x dx < J(u),$$

which is a contradiction. Therefore we established that for all $x \in [a, b]$ function $L(x, u(x), \cdot)$ is convex at $\dot{u}(x)$ with respect to $\bar{B}(\dot{u}(x), \epsilon)$.

Theorem 1.7 is proved. $\square$

5. Theorem 1.11

Proof of Theorem 1.11. Let $u_k \in C^1([a,b];R^n)$, $k \in N$, be a sequence such that $||u_k - u||_C \to 0$, $k \to \infty$, $||u_k - u||_{C^1} \leq \epsilon$, $k \in N$. Assume that $L \in C(R^{2n+1})$ and that function $L(x, u(x), \cdot)$ is strictly convex at $\dot{u}(x)$ with respect to $\bar{B}(\dot{u}(x), \epsilon)$ for a.e. $x \in [a,b]$. Assume that $J(u_k) \to J(u)$, $k \to \infty$, but $||u_k - u||_{W^{1,1}} \not\to 0$. Then due to Theorem 2.1 a subsequence $\dot{u}_k$, $k \in N$, generates a nontrivial (not equal to a Dirac mass in the set of positive measure) Young measure $(\nu_x)_{x\in[a,b]}$. By Theorem 2.1 we also have

$$J(u_k) \to \int_a^b \int_{R^n} L(x, u(x), \cdot) d\nu_x dx.$$

Because of convexity of $L(x, u(x), \cdot)$ at $\dot{u}(x)$ and due to Lemma 1.6 we obtain

$$\int_{R^n} L(x, u(x), \cdot) d\nu_x \geq L(x, u(x), \dot{u}(x)), \quad x \in [a,b].$$

Because of strict convexity of $L(x, u(x), \cdot)$ at $\dot{u}(x)$ for a.e. $x \in [a,b]$ we obtain due to Lemma 1.10 that in the set, where ν_x is nontrivial,

$$\int_{R^n} L(x, u(x), \cdot) d\nu_x > L(x, u(x), \dot{u}(x)).$$

Finally we obtain

$$\lim_{k\to\infty} J(u_k) = I > J(u),$$

which is a contradiction.

Therefore $\nu_x = \delta_{\dot{u}(x)}$ for a.e. $x \in [a,b]$, that implies via Theorem 2.1 that $||u_k - u||_{W^{1,1}} \to 0$, $k \to \infty$.

Now we prove the converse statement. First note that for each $x \in [a,b]$ the function $L(x, u(x), \cdot)$ is either convex or concave at $\dot{u}(x)$ with respect to $\bar{B}(\dot{u}(x), \epsilon)$. Assume contrary. Then there exist

$$c_i^1 > 0,\ v_i^1 \in \bar{B}(\dot{u}(x), \epsilon),\ v_i^1 \neq \dot{u}(x),\ i = 1, \ldots, m_1,$$

$$c_i^2 > 0,\ v_i^2 \in \bar{B}(\dot{u}(x), \epsilon),\ v_i^2 \neq \dot{u}(x),\ i = 1, \ldots, m_2,$$

such that

$$\sum_{i=1}^{m_1} c_i^1 = 1, \quad \sum_{i=1}^{m_1} c_i^1 v_i^1 = \dot{u}(x), \quad \sum_{i=1}^{m_1} c_i^1 L(x, u(x), v_i^1) > L(x, u(x), \dot{u}(x)), \quad (7)$$

$$\sum_{i=1}^{m_2} c_i^2 = 1, \quad \sum_{i=1}^{m_2} c_i^2 v_i^2 = \dot{u}(x), \quad \sum_{i=1}^{m_2} c_i^2 L(x, u(x), v_i^2) < L(x, u(x), \dot{u}(x)). \quad (8)$$

Define $v_i^1(y) := v_i^1 - \dot{u}(x) + \dot{u}(y)$, $v_i^2(y) := v_i^2 - \dot{u}(x) + \dot{u}(y)$. Then (7), (8) is also valid with $v_i^1(y)$, $v_i^2(y)$ for all y sufficiently close to x. In particular we can isolate nonintersecting intervals I_1, I_2 such that in I_1 (I_2) the inequality (7) ((8)) holds. Assume $\nu_y = \sum_{i=1}^{m_1} c_i^1 \delta_{v_i^1(y)}$ for $y \in I_1$, $\nu_y = \sum_{i=1}^{m_2} c_i \delta_{v_i^2(y)}$ for $y \in I_2$. We can also select I_1, I_2 in such a way that

$$\int_{I_1 \cup I_2} \int_{R^n} L(y, u(y), \cdot) d\nu_y dy = \int_{I_1 \cup I_2} L(y, u(y), \dot{u}(y)) dy. \tag{9}$$

Assume also $\nu_y = \delta_{\dot{u}(y)}$ for $y \notin I_1 \cup I_2$.

Due to Theorem 2.3 there exists a sequence $u_k \in C^1([a,b]; R^n)$ with the properties $u_k(a) = u(a)$, $u_k(b) = u(b)$, $||u_k - u||_{C^1} < \epsilon$, $k \in N$, $||u_k - u||_C \to 0$, $k \to \infty$, and such that $\dot{u}_k$ generate a Young measure $(\nu_x)_{x \in [a,b]}$. Then

$$J(u_k) \to \int_a^b \int_{R^n} L(x, u(x), \cdot) d\nu_x dx = J(u),$$

but $||\dot{u}_k - \dot{u}||_{L^1} \not\to 0$ since Young measure $(\nu_x)_{x \in [a,b]}$ is nontrivial, cf. Theorem 2.1. This contradiction with the assumptions of the theorem shows that either $L(x, u(x), \cdot)$ is convex or concave at $\dot{u}(x)$ with respect to $\bar{B}(\dot{u}(x), \epsilon)$.

Analogously we can show that for all $x \in [a,b]$ either $L(x, u(x), \cdot)$ is convex at $\dot{u}(x)$ with respect to $\bar{B}(\dot{u}(x), \epsilon)$ or this holds with replacement of convexity by concavity.

To prove the theorem we have to show only that for a.a. $x \in [a,b]$ convexity (concavity) has the strict character. This is enough to consider the case of convexity. Assume the contrary, i.e. assume that the set G, where the function $L(x, u(x), \cdot)$ is not strictly convex at $\dot{u}(x)$ with respect to $\bar{B}(\dot{u}(x), \epsilon)$, does not have zero measure. Due to Lemma 3.1 for sufficiently small $\delta > 0$ the subset G_δ of the set G such that for each $x \in G_\delta$ there exist $c_i > 0$, $v_i \in \bar{B}(\dot{u}(x), \epsilon)$, $|v_i - \dot{u}(x)| \geq \delta$, $i = 1, \ldots, n+1$, with the properties

$$\sum_{i=1}^{n+1} c_i = 1, \quad \sum_{i=1}^{n+1} c_i v_i = \dot{u}(x), \quad \sum_{i=1}^{n+1} c_i L(x, u(x), v_i) = L(x, u(x), \dot{u}(x)), \tag{10}$$

is not a set of zero measure. However G_δ is a closed set. In fact, if $x_k \in G_\delta$, $x_k \to x_0$, then there exists a subsequence x_k (not relabeled) such that $c_i^k \to c_i$, $v_i^k \to v_i$, $k \to \infty$, $i = 1, \ldots, n+1$, and (10) holds for each $k \in N$. Because of continuity of L, u, $\dot{u}$ we obtain that (10) holds also for the limit elements, i.e. we obtain that $x_0 \in G_\delta$. Then the measure of G_δ is positive. Further, the multivalued mapping V, associating to each $y \in G_\delta$ the set of all $(c_1, \ldots, c_{n+1}, v_1 \ldots, v_{n+1})$, for which (10) is valid, is upper semicontinuous, where upper semicontinuity means that if $u_k \in V(y_k)$, $k \in N$, $u_k \to u$, $y_k \to y$, $k \to \infty$, then $u \in V(y)$. By the theorem on measurable selector, see e.g. [2], there exists a measurable selector $u : G_\delta \to R^{2n+2}$ of the multivalued map, here

$u(\cdot) = (c_1(\cdot), \ldots, c_{n+1}(\cdot), v_1(\cdot), \ldots, v_{n+1}(\cdot))$. Assume $\nu_x = \sum_{i=1}^{n+1} c_i(x)\delta_{v_i(x)}$ for $x \in G_\delta$, $\nu_x = \delta_{\dot{u}(x)}$ for $x \in [a,b] \setminus G_\delta$. By Theorem 2.2 $(\nu_x)_{x\in[a,b]}$ is a Young measure. We have also

$$\int_a^b \int_{R^n} L(x, u(x), \cdot) d\nu_x dx = J(u). \tag{11}$$

By Theorem 2.3 there exists a sequence $u_k \in C^1([a,b]; R^n)$ such that $u_k(a) = u(a)$, $u_k(b) = u(b)$, $||u_k - u||_{C^1} < \epsilon$, $k \in N$, $||u_k - u||_C \to 0$, $k \to \infty$, and $\dot{u}_k$ generate Young measure $(\nu_x)_{x\in[a,b]}$. Then by Theorem 2.1

$$J(u_k) \to \int_a^b \int_{R^n} L(x, u(x), \cdot) d\nu_x dx = J(u),$$

where $||\dot{u}_k - \dot{u}||_{L^1} \not\to 0$ since Young measure $(\nu_x)_{x\in[a,b]}$ is nontrivial. This contradiction shows that the measure of the set G is equal to zero.

Theorem 1.11 is proved. □

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