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Archimedean Cones in Vector Spaces

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In the case of an ordered vector space (briefly, OVS) with an order unit, the Archimedeanization method was recently developed by Paulsen and Tomforde [4]. We present a general version of the Archimedeanization which covers arbitrary OVS. Also we show that an OVS (V, V_+) is Archimedean if and only if $\inf_{\tau \in \{\tau\},\, y \in L}(x_\tau - y) = 0$ for any bounded below decreasing net $\{x_\tau\}_\tau$ in V, where L is the collection of all lower bounds of $\{x_\tau\}_\tau$, and give characterization of the almost Archimedean property of V_+ in terms of existence of a linear extension of an additive mapping $T : U_+ \to V_+$.

Keywords: Ordered vector space, pre-ordered vector space, Archimedean, Archimedean element, almost Archimedean, Archimedeanization, linear extension

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1. Introduction

In Section 3, the Archimedeanization method, due to Paulsen and Tomforde [4] for OVS with an order unit, is extended to arbitrary ordered vector space. Namely, the following result is proved.

Theorem 1.1. *Any ordered vector space possesses a unique, up to an order isomorphism, Archimedeanization.*

The following characterization of the Archimedean property is well known in the case of vector lattices (see, for example, [3, Thm. 22.5]). In the general setting of OVS, it has been published recently in [2, Prop. 2] as an auxiliary result with a gap in the proof of the implication $(ii) \Rightarrow (i)$. A corrected proof is given in Section 3 below.

Theorem 1.2. *For an OVS (V, V_+), the following conditions are equivalent:*

(i) *V_+ is Archimedean.*

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(*ii*) *For any bounded below decreasing net* $\{x_\tau\}_\tau$ *in* V,

$$\inf_{\tau\in\{\tau\}, y\in L}(x_\tau - y) = 0\,,$$

where $L = \{y \in V : (\forall\tau \in \{\tau\})[y \leq x_\tau]\}$ *is the collection of all lower bounds of* $\{x_\tau\}_\tau$.

We also characterize the almost Archimedean property of V_+ in terms of existence of an extension of an additive mapping $T : U_+ \to V_+$ of the positive cone U_+ of an OVS U into V_+ to a linear operator from U into V.

Theorem 1.3. *For an OVS* (V, V_+), *the following conditions are equivalent:*

(*i*) V_+ *is almost Archimedean.*
(*ii*) *For any OVS* (U, U_+) *and any additive map* $T : U_+ \to V_+$, *there is an extension of* T *to a linear operator from* U *into* V.
(*iii*) *For any additive map* $T : \mathbb{R}_+ \to V_+$, *there is an extension of* T *to a linear map from* $\mathbb{R}$ *into* V.

Note that existence of such an extension is well known in the Archimedean setting (see, for example, [1, Lem. 1.26], [6, Lem. 83.1]). Theorem 1.3 is proved in Section 3.

2. Preliminaries

In the present paper, we always exclude zero from the natural numbers (i.e. $\mathbb{N} := \{1, 2, ...\}$) and consider vector spaces only over reals.

A subset W of a vector space V is called a *wedge* if it satisfies

$$W + W \subseteq W$$

and

$$rW \subseteq W \quad (\forall\, r \geq 0). \tag{1}$$

A wedge K is called a *cone* if

$$K \cap (-K) = \{0\}.$$

A cone K is said to be *generating* if

$$K - K = V. \tag{2}$$

Given a cone (resp., a wedge) V_+ in V, we say that (V, V_+) is an *ordered vector space* (briefly an OVS) (resp., a *pre-ordered vector space* or a POVS). The partial ordering (resp., partial pre-ordering) $\leq$ on V is defined by

$$x \leq y \;\text{ whenever }\; y - x \in V_+\,.$$

(V, V_+) is also denoted by $(V, \leq)$ or simply by V, if the partial ordering $\leq$ is well understood. We say that $e \in V$ is an *order unit* if for each $x \in V$ there is some $n \in \mathbb{N}$ such that $x \leq ne$. Clearly, V_+ is generating if V possesses an order unit.

For every x, y in a POVS $(V, \leq)$, the (possibly empty, if $x \not\leq y$) set

$$[x, y] := \{z : x \leq z \leq y\}$$

is called an *order interval.* A subset $A \subseteq V$ is said to be *order convex* if $[a, b] \subseteq A$ for all $a, b \in A$. Any order convex vector subspace of V is called an *order ideal.* If Y is an order ideal in a POVS V, the quotient space V/Y is partially ordered by:

$$[0] \leq [f] \quad \text{if } \exists q \in Y \text{ with } 0 \leq f + q\,.$$

The order convexity of Y is needed for the property

$$[f] \leq [g] \leq [f] \Rightarrow [f] = [g]\,,$$

which guarantees that $[V_+]_Y := V_+ + Y$ is a cone in V/Y.

Let $x_n \in V$ and $u \in V_+$. The sequence $(x_n)_n$ is said to be *u-uniformly convergent* to a vector $x \in V$, in symbols $x_n \overset{(u)}{\to} x$, if

$$k_n(x_n - x) \in [-u, u]$$

for some sequence $k_n \uparrow \infty$ in $\mathbb{N}$. Clearly, $x_n \overset{(u)}{\to} x$ if and only if

$$x_n - x \in [-\varepsilon_n u, \varepsilon_n u]$$

for some sequence ε_n of reals such that $\varepsilon_n \downarrow 0$. A subset $A \subseteq V$ is said to be *uniformly closed* in V if it contains all limits of all u-uniformly convergent sequences $(x_n)_n$ in A for all $u \in V_+$.

Definition 2.1. A wedge K in a vector space V is said to be Archimedean (also we say that the POVS (V, K) is Archimedean), whenever

$$(\forall x, y \in V)\big[[(\forall n \in \mathbb{N})\ x - ny \in K] \Rightarrow [y \in -K]\big]\,. \tag{3}$$

Clearly, (3) in Definition 2.1 can be replaced by

$$(\forall x \in V_+, y \in V)\big[[(\forall n \in \mathbb{N})\ x - ny \in K] \Rightarrow [y \in -K]\big]\,. \tag{3'}$$

It is obvious that any subspace of an Archimedean POVS is Archimedean. It is also worth noticing that if V admits a linear topology τ for which a wedge $K \subseteq V$ is closed, then K is Archimedean (cf. [1, Lem. 2.3]). In the case when τ is Hausdorff and a wedge $K \subseteq V$ has nonempty τ-interior, K is Archimedean if and only if it is τ-closed (cf. [1, Lem. 2.4]). It is well known (see also Theorem 1.2, where it is included in a slightly more general form) that an OVS (V, V_+) is Archimedean if and only if $\inf_{n\geq 1} \frac{1}{n} y = 0$ for every $y \in V_+$. One may also define the Archimedean property by saying that, whenever $x \in V_+$ and $y \in V$ with $0 \leq x + ny$ for all $n \in \mathbb{N}$, then $0 \leq y$. This observation leads to the following definition, that is motivated by the definition of an *Archimedean order unit* in [4, Def. 2.7].

Definition 2.2. If (V, V_+) is a POVS and $x \in V_+$, we say that x is an Archimedean element if, whenever $y \in V$ with $x + ny \in V_+$ for all $n \in \mathbb{N}$, then $y \in V_+$.

Notice that $0 \in V$ is always Archimedean. Clearly, V_+ is Archimedean if and only if every $x \in V_+$ is Archimedean. Furthermore, if $0 \leq z \leq x$ and x is Archimedean, then z is Archimedean. In particular, in any POVS V with an Archimedean order unit, all elements of V_+ are Archimedean. Given a POVS V, denote

$$\mathrm{Arch}(V) = \{x \in V | x \ \text{ is Archimedean}\}.$$

Assertion 2.3. $\mathrm{Arch}(V)$ *is a wedge which is contained in* V_+. *In particular, if* V_+ *is a cone then* $\mathrm{Arch}(V)$ *is a cone.*

Proof. The inclusion $\mathrm{Arch}(V) \subseteq V_+$ follows directly from the definition. It is also clear that $\alpha x \in \mathrm{Arch}(V)$ for all $0 < \alpha \in \mathbb{R}$ and $x \in \mathrm{Arch}(V)$. To complete the proof it is enough to show that $\mathrm{Arch}(V) + \mathrm{Arch}(V) \subseteq \mathrm{Arch}(V)$. Let $x_1, x_2 \in \mathrm{Arch}(V)$. Suppose that

$$x_1 + x_2 + nv \in V_+ \quad (\forall n \in \mathbb{N}).$$

Take an $m \in \mathbb{N}$. Then

$$x_1 + nx_2 + nmv \in V_+ \quad (\forall n \in \mathbb{N})$$

and hence

$$x_1 + n(x_2 + mv) \in V_+ \quad (\forall n \in \mathbb{N}).$$

So, we obtain $x_2 + mv$ because x_1 is Archimedean. Since $m \in \mathbb{N}$ is arbitrary,

$$x_2 + mv \in V_+ \quad (\forall m \in \mathbb{N}).$$

As $x_2 \in \mathrm{Arch}(V)$, then $v \in V_+$, which means, by Definition 2.2, that $x_1 + x_2 \in \mathrm{Arch}(V)$. $\square$

Definition 2.4. A wedge W in a vector space V is said to be almost Archimedean (also we say that the POVS (V, W) is almost Archimedean) if

$$(\forall x, y \in V)\big[[(\forall n \in \mathbb{Z}) \ x - ny \in W] \Rightarrow [y = 0]\big]. \tag{4}$$

A standard example of an OVS which is not almost Archimedean is $(\mathbb{R}^2, \leq_{lex})$, the Euclidean plane with the lexicographic ordering. Clearly, any $x \in V$ satisfying $[(\forall n \in \mathbb{Z})\ x - ny \in W]$ in (4) is indeed an element of W.

Given a POVS (V, V_+), denote

$$N_V := \{x \in V | (\exists y \in V)(\forall n \in \mathbb{N})[-y \leq nx \leq y]\} \tag{5}$$

the set of all *infinitely small elements* of V. The wedge V_+ is almost Archimedean if and only if $N_V = \{0\}$. Since $V_+ \cap (-V_+) \subseteq N_V$, we see that every almost Archimedean POVS is an OVS. Although, the wedge V in a vector space $V \neq \{0\}$ is Archimedean but not almost Archimedean, any Archimedean OVS is clearly almost Archimedean. The converse is not true even if $\dim(V) = 2$ (cf. also Example 3.3).

Example 2.5. Let Γ be a set containing at least two elements, and V be the space of all bounded real functions on Γ, partially ordered by

$$f \leq g \ \text{ if either } f = g \text{ or } \inf_{t \in \Gamma}[g(t) - f(t)] > 0\,.$$

The OVS $(V, \leq)$ is almost Archimedean but not Archimedean. It can be shown that $\inf_{n\geq 1} \frac{1}{n} f$ does not exist for any $0 \neq f \in V_+$.

The following two assertions are immediate.

Assertion 2.6. *A wedge W in a vector space V is almost Archimedean if and only if it does not contain a straight line, say $x + ty$ with $x, y \in V$, $y \neq 0$. Furthermore, a POVS $(V, \leq) = (V, V_+)$ is almost Archimedean if and only if $\bigcap_{n\geq 1}[-\frac{1}{n}x, \frac{1}{n}x] = \{0\}$ for every $x \in V_+$.*

Assertion 2.7. *Any subcone of an almost Archimedean cone is almost Archimedean.*

It is worth noticing, and it follows directly from Definitions 2.1, 2.4, that the intersection of any nonempty family of Archimedean (resp., almost Archimedean) wedges in a vector space is an Archimedean (resp., almost Archimedean) wedge.

If A is an order ideal in a POVS (V, V_+) such that the POVS $(V/A, V_+ + A)$ is almost Archimedean, then A is uniformly closed (cf. the part (a) of Proposition 3.2). When $A = \{0\}$, the converse is true (e.g. by the part (b) of Proposition 3.2). Thus V is almost Archimedean if and only if $\{0\}$ is uniformly closed. In general, the question, whether or not V/A is Archimedean assuming an order ideal A to be uniformly closed, is rather nontrivial (it has a positive answer due to Veksler [5] in the vector lattice setting). An OVS V is said to be a *vector lattice* if every nonempty finite subset of V has a least upper bound. It is well known that any almost Archimedean vector lattice is Archimedean. In Example 2.5, $V = V/\{0\}$ is not Archimedean although $\{0\}$ is uniformly closed. Similarly to Definition 2.4, we introduce almost Archimedean elements.

Definition 2.8. If (V, V_+) is an POVS and $x \in V_+$, we say that x is an almost Archimedean element if, whenever $y \in V$ with $x + ny \in V_+$ for all $n \in \mathbb{Z}$, then $y = 0$.

Given a POVS V, denote

$$\text{a-Arch}(V) = \{v \in V | v \text{ is almost Archimedean}\}.$$

Clearly a-Arch$(V) \subseteq V_+$,

$$V_+ \text{ is a cone } \Leftrightarrow \ 0 \in \text{a-Arch}(V) \ \Leftrightarrow \ \text{a-Arch}(V) \neq \emptyset,$$

$$V_+ \text{ is almost Archimedean } \Leftrightarrow \ V_+ = \text{a-Arch}(V),$$

and

$$\alpha \cdot \text{a-Arch}(V) = \text{a-Arch}(V) \quad (\forall\ 0 < \alpha \in \mathbb{R}).$$

We shall also make of use the following definition.

Definition 2.9. Let $V = (V, V_+)$ be an OVS. Consider the category $\mathcal{C}_{Arch}(V)$ whose objects are pairs $\langle U, \phi \rangle$, where $U = (U, U_+)$ is an Archimedean OVS and $\phi : V \to U$ is a positive linear map (that is: $\phi(V_+) \subseteq U_+$), and morphisms $\langle U_1, \phi_1 \rangle \to \langle U_2, \phi_2 \rangle$ are positive linear maps $q_{12} : U_1 \to U_2$ such that $q_{12} \circ \phi_1 = \phi_2$. If $\mathcal{C}_{Arch}(V)$ possesses an initial object $\langle U_0, \phi_0 \rangle$, then U_0 is said to be an *Archimedization* of V.

For further information on ordered vector spaces we refer to [1].

3. The Archimedeanization of an ordered vector space

Given a POVS (V, V_+), denote

$$D_V := \{x \in V | (\exists \xi \in V_+)(\forall n \in \mathbb{N})\ nx + \xi \in V_+\},$$

and let I_V be the intersection of all uniformly closed order ideals in (V, V_+).

It is straightforward to verify, that $D_V \cap (-D_V) = N_V$, where N_V was defined in (5), N_V is an order ideal in (V, V_+), D_V is a wedge, and $V_+ \subseteq D_V$. Consider the sets $V_+ + N_V = [V_+]_{N_V}$ and $D_V + N_V = [D_V]_{N_V}$ in the quotient space V/N_V. Since V_+, D_V, and N_V are wedges, then $V_+ + N_V$ and $D_V + N_V$ are wedges in V/N_V. However, in Example 2.5, D_V is not a cone. Clearly,

$$\begin{aligned}(D_V + N_V) \cap -(D_V + N_V) &= (D_V + N_V) \cap (-D_V + N_V) \\ &= D_V \cap (-D_V) = N_V.\end{aligned}$$

So, $D_V + N_V$ is a cone in V/N_V, and $V_+ + N_V \subseteq D_V + N_V$ is a cone as well. Let us collect some further elementary properties.

Proposition 3.1. *Given an POVS* (V, V_+)*. Then*

(a) *N_V is an order ideal in the POVS (V, D_V);*
(b) *$N_V \subseteq I_V$;*
(c) *if V possesses an order unit then $N_{V/A} = I_{V/A}$ for every ideal $A \subseteq V$;*
(d) *if (V, V_+) is a vector lattice then $(V/N_V, D_V + N_V)$ is a vector lattice.*

Proof. (a) Let $u, v \in N_V$, $w \in V$, with $u \le w \le v$ in (V, D_V). Then

$$-x \le nu \le x\,, \quad -y \le nv \le y \quad (\forall n \in \mathbb{N}) \tag{6}$$

for some $x, y \in V_+$, and

$$0 \le n(w-u) + \xi_1\,, \quad 0 \le n(v-w) + \xi_2 \quad (\forall n \in \mathbb{N}) \tag{7}$$

for some $\xi_1, \xi_2 \in V_+$. It follows from (6) and (7) that

$$0 \le nw + x + \xi_1\,, \quad 0 \le -nw + y + \xi_2 \quad (\forall n \in \mathbb{N}).$$

Then

$$-x + \xi_1 \le nw \le y + \xi_2 \quad (\forall n \in \mathbb{N})\,,$$

and therefore

$$-(x + y + \xi_1 + \xi_2) \le nw \le (x + y + \xi_1 + \xi_2) \quad (\forall n \in \mathbb{N})\,,$$

which means that $w \in N_V$.

(b) Take an $f \in N_V$, then for some $g \in V$

$$-g \le nf \le g \quad (\forall n \in \mathbb{N})$$

or equivalently

$$-\frac{1}{n} g \le 0 - f \le \frac{1}{n} g \quad (\forall n \in \mathbb{N}).$$

Let $A \subseteq V$ be a uniformly closed order ideal. The sequence $a_n := 0 \in A$ converges g-uniformly to f. Thus, $f \in A$. Since A is arbitrary, then $f \in N_V$.

(c) Denote by e an order unit in V. Firstly, assume that $A = \{0\}$. Given $f \in N_V$, then for some $g \in V$

$$-g \le nf \le g \quad (\forall n \in \mathbb{N})$$

and, since $g \le ce$ for some $c \in \mathbb{R}$, one gets

$$-e \le nf \le e \quad (\forall n \in \mathbb{N}).$$

We have to show that N_V is uniformly closed. Suppose $f_n \overset{(u)}{\to} z$, then $f_n \overset{(e)}{\to} z$. That is

$$-e \le k_n(f_n - z) \le e \quad (\forall k \in \mathbb{N})$$

for some $\mathbb{N} \ni k_n \uparrow \infty$. But also

$$-e \leq k_n f_n \leq e \quad (\forall k \in \mathbb{N}).$$

Hence

$$-2e \leq k_n z \leq 2e \quad (\forall k \in \mathbb{N}).$$

Since $k_n \uparrow \infty$, then $z \in N_V$.

Now, let $A \subseteq V$ be an arbitrary ideal. Since e is an order unit in V, then $[e]$ is an order unit in V/A. So, $N_{V/A} = I_{V/A}$ as it was shown above.

(d) Denote by $u \vee v = \sup(u, v)$ and by $u \wedge v = \inf(u, v)$ the supremum and infimum of $u, v \in V$ in (V, V_+). Take $u, v \in V$. Since $u \vee v - u, u \vee v - v \in V_+$ and $V_+ \subseteq D_V$, then $u \vee v - u \in D_V$ and $u \vee v - v \in D_V$.

Let $w - u\,, w - v \in D_V$, then

$$0 \leq n(w-u) + \xi_1\,, \quad 0 \leq n(w-v) + \xi_2 \quad (\forall n \in \mathbb{N})$$

for some $\xi_1, \xi_2 \in V_+$. Denoting $\xi = \xi_1 + \xi_2$, we obtain

$$\begin{aligned} 0 &\leq (n(w-u) + \xi) \wedge (n(w-v) + \xi) = n((w-u) \wedge (w-v)) + \xi \\ &= n(w + (-u) \wedge (-v)) + \xi = n(w - u \vee v) + \xi \quad (\forall n \in \mathbb{N})\,, \end{aligned}$$

which means that $w - u \vee v \in D_V$. As $N_V = D_V \cap (-D_V)$, the supremum of $[u], [v]$ exists, and is equal to $[u \vee v]$ in the quotient OVS $(V/N_V, D_V + N_V)$. Since $u, v \in V$ are taken arbitrary, $(V/N_V, D_V + N_V)$ is a vector lattice. □

Proposition 3.2. *Given a POVS (V, V_+). The following assertions hold true.*

(a) *Let $A \subseteq V$ be an order ideal. Then*

$$V/A \textit{ is } \text{almost } \textit{Archimedean} \Rightarrow A \textit{ is uniformly closed}.$$

(b) *V is almost Archimedean $\Leftrightarrow N_V = \{0\} \Leftrightarrow I_V = \{0\}$.*

(c) *If (V, V_+) possesses an order unit then $(V/N_V, D_V + N_V)$ is an Archimedean OVS.*

(d) *If $(V/N_V, D_V + N_V)$ is Archimedean then $(V/N_V, V_+ + N_V)$ is almost Archimedean. In particular, if (V, V_+) possesses an order unit then $(V/N_V, V_+ + N_V)$ is almost Archimedean.*

Proof. (a) Take a sequence $(y_n) \subseteq A$ such that $y_n \stackrel{(u)}{\to} x$ for some $u \in V_+$, and $x \in V$. We may assume that

$$n(y_n - x) \in [-u, u] \quad (\forall n \in \mathbb{N}).$$

Thus

$$n[-x] = n[y_n - x] \in [-[u], [u]] \quad (\forall n \in \mathbb{N}).$$

Since V/A is almost Archimedean, $[-x] = [0]$, and hence $x \in A$.

(b) The implication "V is almost Archimedean $\Rightarrow \{0\}$ is uniformly closed" follows from (a).

The implication "$\{0\}$ is uniformly closed $\Rightarrow I_V = \{0\}$" is true due to the definition of I_V.

The implication "$I_V = \{0\} \Rightarrow N_V = \{0\}$" is true by the part (b) of Proposition 3.1.

The implication "$N_V = \{0\} \Rightarrow V$ almost Archimedean" follows from Definition 2.4.

(c) It has been proved in [4, Thm. 2.35].

(d) It follows from Assertion 2.7, as $V_+ + N_V$ is a subcone of an almost Archimedean cone $D_V + N_V$ in V/N_V. $\square$

The next example shows that, for the cone $D_V + N_V$ in V/N_V to be Archimedean, existence of an order unit in (V, V_+) is not essential.

Example 3.3. Let V be the vector space $\mathcal{P}(\mathbb{R})$ of all real polynomials on $\mathbb{R}$ with the positive cone

$$V_+ := \{p \in V | (\forall t \in \mathbb{R}) p(t) > 0\} \cup \{p \,| p(t) \equiv 0\}.$$

The OVS (V, V_+) has no order unit, $N_V = \{0\}$, and $D_V := \{p \in V | (\forall t \in \mathbb{R}) p(t) \geq 0\}$. Take $x \equiv 1$, $y = -t^2$ in V. Then

$$x - ny \in V_+ \quad (\forall n \in \mathbb{N})$$

but $y \notin -V_+$. Thus (V, V_+) is not Archimedean, however $N_V = \{0\}$. The OVS $(V/N_V, D_V + N_V) = (V, D_V)$ is clearly Archimedean.

Notice that if we take $V = \mathcal{P}[0, 1]$ with the same ordering, then V becomes a non-Archimedean but almost Archimedean OVS with an order unit and uniformly closed order ideal $N_V = \{0\}$. Moreover, it is well known that this OVS satisfies the *Riesz decomposition property*, however is not a vector lattice.

In connection with Example 3.3, a question arises, whether or not any almost Archimedean OVS (V, V_+) satisfies the property that (V, D_V) is Archimedean. Fortunately, it has a positive answer as the following proposition shows.

Proposition 3.4. *Let (V, V_+) be an almost Archimedean OVS. Then the OVS (V, D_V) is Archimedean.*

Proof. Assume in contrary that the cone D_V is not Archimedean. Then for some $x, y \in V$, $x \in D_V$,

$$(\forall r > 0) \;\; rx + y \in D_V \;\; \text{but } y \notin D_V\,.$$

Since $x \in D_V$, then for some $v \in V_+$

$$(\forall \alpha > 0)\ \ x + \alpha v \in V_+ \,.$$

Take

$$K = \mathrm{span}(x, v, y) \cap V_+ \,.$$

Denote by $\mathrm{cl}(C)$ the closure in Euclidean topology on $\mathrm{span}(x, v, y)$ of $C \subseteq \mathrm{span}(x, v, y)$. Then $x \in \mathrm{cl}(K)$ and $rx + y \in \mathrm{cl}(K)$ for all $r > 0$ and hence $y \in \mathrm{cl}(K) \subseteq D_V$ which is impossible. Therefore, D_V is Archimedean. □

Corollary 3.5. *Any almost Archimedean cone can be embedded into an Archimedean cone. Moreover, given a POVS* (V, V_+)*, then the OVS* $(V/N_V, D_V{+}N_V)$ *is Archimedean if and only if the OVS* $(V/N_V, V_+{+}N_V)$ *is almost Archimedean.*

Proof. The first part is immediate by Proposition 3.4. For the second part, denote $N = N_V$ and $D = D_V$.

The necessity follows from Assertion 2.7, since $V_+ + N$ is a subcone of the Archimedean cone $D + N$ in V/N.

For the sufficiency, assume that $(V/N, V_+ + N)$ is almost Archimedean. Then the cone $D_{V/N}$ is Archimedean in V/N, by Proposition 3.4. Since

$$\begin{aligned} D + N = [D]_N &= \big[\{u \in V | (\exists\ \xi \in V_+)(\forall n \in \mathbb{N})\ nu + \xi \in V_+\}\big]_N \\ &\subseteq \{[u] \in V/N | (\exists\ [\xi] \in V_+ + N)(\forall n \in \mathbb{N})[nu + \xi] \in V_+ + N\} \\ &= D_{V/N}, \end{aligned}$$

the cone $D + N$ is almost Archimedean in $V/N_V = V/N$. □

The following example which is due to T. Nakayama (see, for instance, [3, p. 436]) shows that the cone $D_V + N_V$ in V/N_V is not even almost Archimedean in general.

Example 3.6. Consider

$$V = \{a = (a_k^1, a_k^2)_k |\ (a_k^1, a_k^2) \in (\mathbb{R}^2, \leq_{lex})\,, a_k^1 \neq 0 \text{ for finitely many } k\}\,.$$

Then V is a vector lattice with respect to the pointwise ordering and operations. Denote

$$\begin{gathered} A = \{a \in V |\ a_k^1 = 0 \text{ for all } k\,, a_k^2 \neq 0 \text{ for finitely many } k\}\,, \\ \text{and}\ \ B := \{a \in V |\ a_k^1 = 0 \text{ for all } k\}\,. \end{gathered}$$

Clearly $A \subseteq B$, $A \neq B$, and $A \subseteq N_V$. Furthermore, $B = I_V$ and $A = N_V$. Indeed, take an $a \in B$, then

$$-u \leq n(a - a_n) \leq u \ \ (\forall n \in \mathbb{N})$$

for

$$a_n := \{(0, a_1^2), (0, a_2^2), \ldots, (0, a_1^n), (0, 0,), (0, 0,), \ldots\} \in A$$

and $u := (0, k|a_k^2|)_{k \in \mathbb{N}} \in V$. So, $a_n \stackrel{(u)}{\to} a$ and $a \in I_V$, since $a_n \in N_V \subset I_V$. Therefore, $B \subseteq I_V$. Denote $M_k = \{a \in V|\ a_k^1 = 0\}$ for $k \in \mathbb{N}$. Clearly, any M_k is a uniformly closed ideal in V, and hence

$$I_V \subseteq \bigcap_{k \in \mathbb{N}} M_k = B\,.$$

So, $B = I_V$. In particular, $N_V \subseteq B$. But if $a \in N_V$ then

$$-b \leq na \leq b \in V \quad (\forall n \in \mathbb{N})\,,$$

and $b = (b_k^1, b_k^2)_{k \in \mathbb{N}}$ with only finitely many $b_k^1 \neq 0$. We may assume that if $b_k^1 = 0$ then also $b_k^2 = 0$. Hence $a = (a_k^1, a_k^2)_{k \in \mathbb{N}}$ has all $a_k^1 = 0$ and only finitely many $a_k^2 \neq 0$ Therefore $a \in A$ and, finally, $A = N_V$. Notice that

$$N_V = \{a \in V|\ (\forall k \in \mathbb{N})\ a_k^1 = 0 \text{ and } a_k^2 \neq 0 \text{ for finitely many } k\}.$$
$$D_V = \{a \in V|\ (\forall k \in \mathbb{N})\ a_k^1 \geq 0 \text{ and } a_k^2 < 0 \text{ for finitely many } k\}.$$

The vector lattice $(V/N_V, D_V + N_V)$ is not almost Archimedean. To see this, consider $c_n \in V$ with

$$[c_n]_k = \begin{cases} (0, -\frac{1}{2^k}) & 1 \leq k \leq n \\ (0, 0) & k > n. \end{cases}$$

Then $c_n \in N_V$, and c_n converges uniformly to $d \in V$, where

$$d = \left(\left(0, -\frac{1}{2^k}\right)\right)_{k=1}^{\infty} \notin N_V.$$

Thus, N_V is not uniformly closed in (V, V_+). Moreover, c_n converges uniformly to d in (V, D_V), however $d \notin D_V$. In view of part (b) of Proposition 3.1, $(V/N_V, D_V + N_V)$ is a vector lattice. So, by Veksler's result [5], the vector lattice $(V/N_V, D_V + N_V)$ is not Archimedean and therefore not almost Archimedean.

Although, the first part of Corollary 3.5 gives already an Archimedeanization of any almost Archimedean space (V, V_+) (one only needs to replace V_+ by the intersection of all Archimedean cones containing V_+), the main result in our paper, Theorem 1.1 does not require any restrictions on the cone V_+ and it covers also such OVS as V in Example 3.6 where V/N still has nonzero infinitely small elements.

Proof of Theorem 1.1. Let $V = (V, V_+)$ be an OVS. Denote $N_0 = \{0\}$,

$$N_1 = N_V = \{x \in V | [x]_{N_0} \text{ is infinitely small in } V/N_0 = V\},$$
$$N_{n+1} = \{x \in V | [x]_{N_n} \text{ is infinitely small in } V/N_n\},$$

and more generally

$$N_\alpha = N_\alpha(V) = \{x \in V | [x]_{\cup_{\beta<\alpha} N_\beta} \text{ is infinitely small in } V/_{\cup_{\beta<\alpha} N_\beta}\}$$

for an arbitrary ordinal $\alpha > 0$. It follows directly from the definition of N_α that

$$N_{\alpha_1} \subseteq N_{\alpha_2} \quad (\forall \alpha_1 \le \alpha_2).$$

Take the first ordinal, say λ_V, such that $N_{\lambda_V+1} = N_{\lambda_V}$. The OVS $(V/N_{\lambda_V}, [V_+]_{N_{\lambda_V}})$ has no nonzero infinitely small elements and hence is almost Archimedean. By Corollary 3.5, the OVS

$$(V/N_{\lambda_V}, [D_{V/N_{\lambda_V}}]_{N_{\lambda_V}}) = (V/N_{\lambda_V}, D_{V/N_{\lambda_V}})$$

is Archimedean. Denote by p_V the quotient map $V \to V/N_{\lambda_V}$, then p_V is positive and linear. For any other pair $\langle U, \phi\rangle$, where $U = (U, U_+)$ is an Archimedean OVS and $\phi : V \to U$ is a positive linear map, we have that $\phi(x) \in N_U$ for every $x \in N_\alpha$, where α is an arbitrary ordinal. Since every Archimedean OVS is almost Archimedean, $N_U = \{0\}$ and therefore $N_{\lambda_V} \subseteq \ker(\phi)$. So, the map $\tilde{\phi} : V/N_{\lambda_V} \to U$ is well defined by $\tilde{\phi}([x]_{N_{\lambda_V}}) = \phi(x)$ and satisfies $\tilde{\phi} \circ p_V = \phi$. Moreover, if $[x]_{N_{\lambda_V}} \in D_{V/N_{\lambda_V}}$, then

$$n[x]_{N_{\lambda_V}} + [\xi]_{N_{\lambda_V}} \in [V_+]_{N_{\lambda_V}} \quad (\forall n \in \mathbb{N})$$

for some $\xi \in V_+ + N_{\lambda_V}$. Applying ϕ gives

$$n\phi(x) + \phi(\xi) \in U_+ \quad (\forall n \in \mathbb{N}),$$

and, since (U, U_+) is Archimedean, it follows that $\phi(x) \in U_+$. Thus

$$\tilde{\phi}([x]_{N_{\lambda_V}}) = \phi(x) \in U_+ ,$$

and $\tilde{\phi}$ is a positive linear map. To show that $\tilde{\phi}$ is unique, take any $\psi : V/N_{\lambda_V} \to U$, that satisfies $\psi \circ p_V = \phi$. Then

$$\psi([y]_{N_{\lambda_V}}) = \psi(p_V(y)) = \phi(y) = \tilde{\phi}(p_V(y)) = \tilde{\phi}([y]_{N_{\lambda_V}}) \quad (\forall y \in V),$$

and hence $\psi = \tilde{\phi}$. Thus, $\big\langle (V/N_{\lambda_V}, D_{V/N_{\lambda_V}}), p_V \big\rangle$ is an initial object of the category $\mathcal{C}_{Arch}(V)$. Hence, the OVS $(V/N_{\lambda_V}, D_{V/N_{\lambda_V}})$ is an *Archimedization* of the OVS (V, V_+). $\square$

In connection with the proof of Theorem 1.1, the following question arises naturally. Whether or not for any ordinal α there is an OVS (V, V_+) for which α is the first ordinal such that $N_{\alpha+1}(V) = N_\alpha(V)$?

4. Proofs of Theorem 1.2 and Theorem 1.3

Now we complete the paper by proving Theorems 1.2 and 1.3.

4.1. Proof of Theorem 1.2

(i) $\Rightarrow$ *(ii)*: Let V_+ be Archimedean. Take a decreasing net $\{x_\tau\}_\tau$ in V which is bounded below by some $d \in V$. Assume $z \in V$ satisfies $z \le x_\tau - y$ for all indexes τ and for all $y \in L$. Since $0 \le x_\tau - y$ for all τ, in order to prove the implication, we have to show that $z \le 0$.

As $y + z \le x_\tau$ for all τ and all $y \in L$, we obtain that $y + z \in L$ for every $y \in L$. It follows by the induction,

$$y + nz \in L \quad (\forall y \in L)(\forall n \in \mathbb{N}) .$$

In particular, $d + nz \le x_{\tau_0}$ (and hence, $nz \le x_{\tau_0} - d$) for some $\tau_0 \in \{\tau\}$ and all $n \in \mathbb{N}$. By the condition

$$nz \le x_{\tau_0} - d \ \ (\forall n \in \mathbb{N}) ,$$

the Archimedean property of V_+ implies $z \le 0$, what is required.

(ii) $\Rightarrow$ *(i)*: Let $x \in V_+$, $ny \le x$ for all $n \in \mathbb{N}$. We have to show that $y \le 0$. Denote

$$L = \left\{ w \in V : (\forall n \ge 1) \ w \le \frac{1}{n} x \right\} .$$

Clearly, $y \in L$. Given $u \in L$, then $0 \le \frac{1}{n}x - u$ for all $n \ge 1$. By the hypothesis applied to the decreasing sequence $\frac{1}{n}x \downarrow\ge y$, the following infimum exists, and

$$\inf_{n\ge1, u\in L} \left(\frac{2}{n}x - u\right) = \inf_{n\ge1, u\in L} \left(\frac{1}{n}x - u\right) = 0 .$$

Hence,

$$\inf_{n\ge1, u\in L} \left[\left(\frac{2}{n}x - u\right) - y\right] = -y + \inf_{n\ge1, u\in L} \left(\frac{2}{n}x - u\right) = -y .$$

Since

$$0 \le \left(\frac{1}{n}x - u\right) + \left(\frac{1}{n}x - y\right) = \left(\frac{2}{n}x - u\right) - y$$

for all $n \ge 1$, then

$$0 \le \inf_{n\ge1, u\in L} \left[\left(\frac{2}{n}x - u\right) - y\right] = -y .$$

Then, $y \le 0$ and hence V_+ is Archimedean. $\square$

4.2. Proof of Theorem 1.3

$(i) \Rightarrow (ii)$: Although this implication follows immediately from well known extension result in the case if the OVS (V, V_+) is Archimedean (see for example, [1, Lem. 1.26]) and from Corollary 3.5, we prefer to give a direct proof of this implication.

Since an extension of T on an algebraic complement U_0 of $U_+ - U_+$ in U can be chosen as an arbitrary linear operator, we only need to obtain an extension of an additive mapping $U_+ \xrightarrow{T} V_+$ to a linear operator $(U_+ - U_+) \xrightarrow{\bar{T}} V$.

For each $y \in (U_+ - U_+)$, pick $y_1, y_2 \in U_+$ with $y = y_1 - y_2$ and put

$$\bar{T}y := T(y_1) - T(y_2).$$

It is routine to see that $\bar{T}$ is well defined and additive. The additivity of $(U_+ - U_+) \xrightarrow{\bar{T}} V$ implies that $\bar{T}$ is $\mathbb{Q}$-homogeneous. To complete the proof, it is enough to show that $\bar{T}$ is $\mathbb{R}_+$-homogeneous on U_+, where it coincides with T. So we need to show only that $T : U_+ \to V_+$ is $\mathbb{R}_+$-homogeneous. We shall use the following elementary remark:

$$\begin{aligned} &\big[x, y \in [q, p]\big] \Rightarrow \big[x - q, y - q \in [0, p - q]\big] \\ &\Rightarrow \big[x - y = (x - q) - (y - q) \in [-(p - q), p - q]\big]. \end{aligned} \tag{8}$$

Let $\mathbb{Q}_+ \ni r_n \uparrow a \in \mathbb{R}$, $\mathbb{Q}_+ \ni r'_n \downarrow a$, $u \in U_+$. Then

$$r_n T(u) = T(r_n u) \le T(au) \le T(r'_n u) = r'_n T(u) .$$

But also

$$r_n T(u) \le aT(u) \le r'_n T(u) .$$

That is

$$T(au), aT(u) \in [r_n T(u), r'_n T(u)] . \tag{9}$$

Applying (8) in (9), we get

$$T(au) - aT(u) \in [-(r'_n - r_n)Tu, (r'_n - r_n)Tu] \quad (\forall n \in \mathbb{N}) .$$

Since V_+ is almost Archimedean and $r'_n - r_n \downarrow 0$, we obtain $T(au) - aT(u) = 0$, what is required.

$(ii) \Rightarrow (iii)$: It is trivial.

$(iii) \Rightarrow (i)$: Let $x \in V_+$, $y \in V$ be such that

$$-\frac{1}{n}x \le y \le \frac{1}{n}x \quad (\forall n \ge 1). \tag{10}$$

Take a function $f : \mathbb{R} \to \mathbb{R}$ which is $\mathbb{Q}$-linear but not $\mathbb{R}$-linear and define an additive mapping $T : \mathbb{R}_+ \to V$ by

$$T(a) = ax + f(a)y \quad (a \in \mathbb{R}_+) .$$

Then T maps $\mathbb{R}_+$ into V_+. Indeed, $T(0) = 0$ and if $0 < a$ then $0 \leq a - \frac{|f(a)|}{n_0}$ for a large enough n_0. It follows from (10) that $-\frac{|f(a)|}{n}x \leq \pm f(a)y \leq \frac{|f(a)|}{n}x$ for all $n \geq 1$. In particular, $-\frac{|f(a)|}{n_0}x \leq f(a)y$, and therefore

$$0 \leq \left(a - \frac{|f(a)|}{n_0}\right) x \leq ax + f(a)y = T(a)$$

Take a linear extension $\bar{T}$ of T to all of $\mathbb{R}$. Then $\bar{T}$ (and hence T) must be $\mathbb{R}_+$-homogeneous on $\mathbb{R}_+$ which is only possible if $y = 0$. $\square$

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