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A Generalized Fermat-Torricelli Tree in the Three Dimensional Euclidean Space

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We obtain a generalized Fermat-Torricelli tree for a boundary pentagonal pyramid (closed hexahedron) and a boundary closed enniaedron in the three dimensional Euclidean Space by placing two mobile vertices with positive weights at the interior of their convex hull and by introducing a method of differentiation of the objective function with respect to two variable dihedral angles and four variable length of linear segments which connect each mobile vertex with two boundary vertices. The generalized Fermat-Torricelli tree with degree at most four with respect to a closed hexahedron or a closed enniaedron is an Euclidean minimal tree structure which may be considered as an intermediate class of a tree structure between a weighted Fermat-Torricelli (tree) structure and a weighted Steiner minimal tree structure with respect to a closed hexahedron or closed enniaedron.

Keywords: Generalized Fermat-Torricelli tree, generalized Fermat-Torricelli constant, pyramids, closed polyhedra

1991 Mathematics Subject Classification: 51E10, 52A15, 52B10

1. Introduction

We start by stating the Gauss tree problem in $\mathbb{R}^2$ (see in [2, Chapter II], [9]) which is a generalization of the Fermat problem (1643):

Problem 1.1 (Gauss tree problem). *How to find a railway network of minimal total length which connects the four cities Bremen, Harburg (today part of the city of Hamburg), Hannover, and Braunschweig.*

If we set the non-negative weight that correspond to Braunschweig (for instance the population) equal to zero, then we obtain the following Fermat-Torricelli tree problem which is a particular case of the Gauss tree problem.

Problem 1.2 (Fermat-Torricelli tree problem). *How to find a railway network of minimal total length which connects the three cities Bremen, Harburg (today part of the city of Hamburg) and Hannover.*

Definition 1.3. The common point which meet the linear segments (edges)

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of the Fermat-Torricelli tree is called a Fermat-Torricelli point.

We note that C. F. Gauss posed this geometric problem of practical importance (Gauss tree problem) to the astronomer Schumacher in 1836. The solution to Gauss tree problem is that the cities Bremen (A_1), Harburg (A_2) and Hannover (A_3) are connected with a Fermat-Torricelli point A_0 and the cities Hannover and Braunschweig A_4 are connected with a linear segment (see in [2, Chapter II]). The solution to Gauss tree problem is considered as a particular case of the Jarnik-Kossler tree problem or Courant-Robbins-Steiner tree problem for $n = 4$ in $\mathbb{R}^2$, because the angle $\angle A_{0,123}A_3A_4$ is $120°$ (see in [8], [9], [6, Page 361]).

Problem 1.4 (Jarnik-Kossler-Courant-Robbins-Steiner tree problem). *Given n points $A_1A_2A_3...A_n$ to find a connected system of linear segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system.*

The definitions of a tree topology, a Fermat-Torricelli tree topology, the degree of a boundary vertex in $\mathbb{R}^3$ and the degree of the two weighted Fermat-Torricelli points which are located at the interior of the convex hull of the hexagonal pyramid or a closed enniaedron are necessary, in order to describe the structure of generalized Fermat-Torricelli trees of a boundary hexagonal pyramid or a closed enniaedron in $\mathbb{R}^3$.

Definition 1.5 ([7]). A tree topology is a connection matrix specifying which pairs of points from the list $A_1, A_2, ..., A_m, A_{0,1}, A_{0,2}, ..., A_{0,m-2}$ have a connecting linear segment (edge).

Definition 1.6 ([9], [4]). The degree of a vertex corresponds to the number of connections of the vertex with linear segments.

Definition 1.7 ([7]). A Fermat-Torricelli tree topology of degree at most four is a tree topology with all boundary vertices of an hexagonal pyramid or a closed enniaedron and two mobile vertices having at most degree four.

Definition 1.8. A tree of minimum length with a Fermat-Torricelli tree topology of degree at most four is called a Fermat-Torricelli tree.

Definition 1.9. A weighted Fermat-Torricelli tree of weighted minimum length with a Fermat-Torricelli tree topology of degree at most four is called a generalized Fermat-Torricelli tree.

In 1879, K. Bopp was the first that solved completely the Gauss tree problem for any four given points in $\mathbb{R}^2$ (see in [3], [2], [9]).

In 1968, E. Gilbert and H. Pollak clarify the set of solutions of the Jarnik-Kossler-Steiner tree problem ([7], [2, pages 328–329]) and A. O. Ivanov and A.

A. Tuzhilin study the weighted Jarnik-Kossler-Steiner trees in $\mathbb{R}^n$ (see in [9], [10]).

In 2011, a solution of the generalized Fermat-Torricelli tree (or weighted Steiner tree) in the K-plane (two dimensional sphere, hyperbolic plane and Euclidean plane) is given in [16] by introducing a method of cyclical differentiation of the objective function with respect to four variable angles and recently an explicit solution of the generalized Fermat-Torricelli tree has been derived in $\mathbb{R}^2$ ([15]).

A detailed study for classes of closed polyhedra which are derived from prescribed rays which meet at a reference point (weighted Fermat-Torricelli point) is given in the classical book of A. D. Aleksandrov for convex polyhedra in [1, Chapter 9]. We note that by a closed enniaedron we refer to a closed polyhedron which is derived by gluing two tetrahedra at two non-neighboring sides of a tetragonal pyramid (excluding the base) in $\mathbb{R}^3$.

In this paper, we introduce a method of symmetrical differentiation of the objective function which corresponds to a boundary hexahedron (pentagonal pyramid) and a closed enniaedron, in order to obtain a generalized Fermat-Torricelli tree having two interior vertices (weighted Fermat-Torricelli points) at the interior of their convex hull. By this way, we generalize the weighted Steiner tree structure, which has been derived in the K-plane ([16], [15]), in $\mathbb{R}^3$.

2. A generalized Fermat-Torricelli tree for a boundary pentagonal pyramid and for a closed enniaedron in $\mathbb{R}^3$.

Let $A_1A_2A_3A_4A_5A_6$ be a pentagonal pyramid (closed hexahedron, Figure 2.1) having as a base the convex pentagon $A_1A_2A_3A_5A_6$ and as a vertex the point A_4 in $\mathbb{R}^3$, A_0 and $A_{0'}$ be two mobile vertices that belong to the interior of the convex hull of the pyramid B'_i be a positive number (weight) that correspond to a boundary vertex A_i for $i = 1, 2, 3, 4, 5, 6$ and by B_j a positive number (weight) that correspond to a mobile vertex A_j for $j = 0, 0'$.

We denote by a_i the length of the linear segment A_0A_i, for $i = 1, 2, 4$ by a_j the length of the linear segment $A_{0'}A_j$, for $j = 3, 5, 6$, by a_{ij} the length of the linear segment A_iA_j, by $a_{00'}$ the length of the linear segment $A_0A_{0'}$, by $h_{0,ik}$ the length of the height of $\triangle A_0A_iA_k$ from A_0 with respect to A_iA_j, for $i, k = 1, 2, 4$, by $h_{0,jl}$ the length of the height of $\triangle A_0A_jA_l$ for $j, l = 3, 5, 6$, by $A_{0,ij}$ the intersection of $h_{0,ij}$ with A_iA_j, by $h_{q,ijk}$ the distance of A_q from the plane defined by $\triangle A_iA_jA_k$ for $q = 0, 0'$ by $x_{p,q}$ the length of the linear segment $A_pA_{0,123}$, by $x'_{p,q}$ the length of the linear segment $A_pA_{0,124}$, by $H_{4,12356}$ the distance of A_4 from the plane $A_1A_2A_3A_5A_6$ and by $h_{0,1235}$ the distance of A_0 from the base of $A_1A_2A_3A_4A_5A_6$ for $i, j, k, p = 1, 2, 3, 4, 5, 6$.

We denote by α_{klm} the angle $\angle A_kA_lA_m$, by α, the dihedral angle which is formed between the planes defined by $\triangle A_1A_2A_3$ and $\triangle A_1A_2A_0$, with α_{g_i} the dihedral angle which is formed by the planes defined by $\triangle A_1A_2A_i$ and $\triangle A_1A_2A_0$, with α'_2 the angle $\angle A_1A_2A_{0,123}$, α''_2 the angle $\angle A_1A_2A_{0,124}$, $\alpha_{i,r0s}$ the angle which is formed by a_i and the linear segment which connects A_0 with the trace from the orthogonal projection of a_i to the plane defined by $\triangle A_0A_rA_s$, for $k, l, m, = 0, 0', 1, 2, 3, 4, 5, 6$ and $i, r, s = 0', 1, 2, 3, 4, 5, 6$.

We assume that all boundary vertices of the pentagonal pyramid (closed hexahedron) have degree one and the two mobile vertices that belong to the interior of the convex hull have degree four.

We state the generalized Fermat-Torricelli tree problem for $A_1A_2A_3A_4A_5A_6$ in $\mathbb{R}^3$, considering a two way communication weighted network.

Problem 2.1. *Given a weighted pentagonal pyramid $A_1A_2A_3A_4A_5A_6$ in $\mathbb{R}^3$ having two interior weighted mobile vertices A_0, A_0' find a connected weighted system of linear segments of shortest total weighted length such that any two of the given points can be joined by a polygon consisting of linear segments of the system:*

$$\begin{aligned} f(X) &= B_1'a_1 + B_2'a_2 + B_3'a_3 + B_4'a_4 + B_5'a_5 + B_6'a_6 + \frac{B_0 + B_0'}{2}a_{00'} \\ &= minimum. \end{aligned} \tag{1}$$

Definition 2.2. We call the solution of the generalized Fermat-Torricelli tree problem a generalized Fermat-Torricelli tree.

Proposition 2.3. *The generalized Fermat-Torricelli tree with respect to the boundary vertices of $A_1A_2A_3A_4A_5A_6$ exists.*

Proof of Proposition 2.3. Without loss of generality, we set

$$B_i = \frac{B_i'}{\sum_{i=1}^{6} B_i'},$$

for $i = 1, 2, 3, 4, 5, 6$ such that:

$$\sum_{i=1}^{6} B_i = 1.$$

By dividing (1) by $\sum_{i=1}^{6} B_i'$, we get:

$$f(X) = \sum_{i=1}^{6} B_i a_i + \frac{B_0 + B_{0'}}{2} a_{00'}. \tag{2}$$

Taking into consideration that the closed polyhedron $A_1A_2A_3A_4A_5A_6$ is a subset of a compact set D, we obtain that the continuous function $f : D \to \mathbb{R}$ attains a minimum in $\mathbb{R}^3$. $\square$

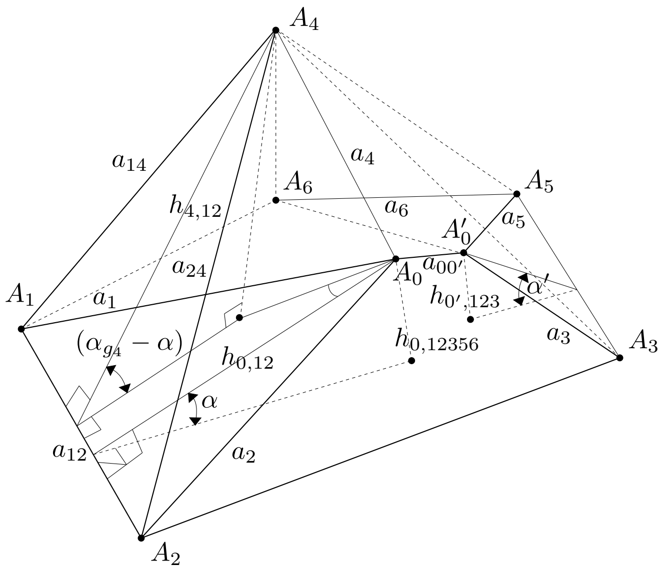

Figure 2.1: A generalized Fermat-Torricelli tree of a pentagonal pyramid

Theorem 2.4. *A generalized Fermat-Torricelli tree of $A_1A_2A_3A_4A_5A_6$ consists of two weighted Fermat-Torricelli points A_0 and A'_0 which are located at the interior convex hull with corresponding weights B_0, $B_{0'}$ and minimizes the objective function:*

$$\sum_{i=1}^{6} B_i a_i + \frac{B_0 + B_{0'}}{2} a_{00'} \tag{3}$$

Proof of Theorem 2.4. We need to prove that the objective function (3) depend on a_1, a_2, α, a_3, a_5 and α', because the location of A_0 depends on a_1, a_2, α, and the location of $A_{0'}$ depends on a_3, a_5, α'.

We apply the same technique that was used in [12] and [13], in order to express the variable lengths a_4, and $a_{00'}$ as functions of a_1, a_2 and α with respect to the pentahedron $A_1A_2A_3A_4A_{0'}$.

We express a_4 as a function of a_1, a_2 and α by using the following equations:

$$h_{0,124} = h_{0,12}\sin(\alpha_{g_4} - \alpha), \tag{4}$$

$$x_{20}'^2 = a_2^2 - h_{0,124}^2, \tag{5}$$

$$\sin(\angle A_1A_2A_{0,124}) = \frac{h_{0,12}\cos(\alpha_{g_4} - \alpha)}{x'_{20}}, \tag{6}$$

$$x_{40}'^2 = x_{20}'^2 + a_{24}^2 - 2x'_{20}a_{24}\cos(\alpha_{124} - \angle A_1A_2A_{0,124}), \tag{7}$$

$$a_4^2 = x_{40}'^2 + h_{0,124}^2, \tag{8}$$

or

$$a_4^2 = a_2^2 + a_{24}^2 - 2a_{24}[\sqrt{a_2^2 - h_{0,12}^2}\cos(\alpha_{124}) + h_{0,12}\sin(\alpha_{124})\cos(\alpha_{g_4} - \alpha)]. \tag{9}$$

From the cosine law in $\triangle A_1A_0A_2$ we get:

$$\cos(\alpha_{102}) = \frac{a_1^2 + a_2^2 - a_{12}^2}{2a_1a_2}, \tag{10}$$

From the sine law in $\triangle A_0A_{0,12}A_{0,123}$ we get:

$$h_{0,12} = \frac{a_1a_2\sin(\alpha_{102})}{a_{12}}, \tag{11}$$

Therefore, by replacing (11) in (10), we derive that:

$$h_{0,12} = h_{0,12}(a_1, a_2) = \sqrt{\frac{4a_1^2a_2^2 - (a_1^2 + a_2^2 - a_{12}^2)^2}{4a_{12}^2}} \tag{12}$$

Following the same process for the calculation of a_4, we express $a_{00'}$ a function with respect to a_1, a_2, α, $a_{20'}$, $\alpha_{120'}$ and $\alpha_{g_{0'}}$.

$$a_{00'}^2 = a_2^2+a_{20'}^2-2a_{20'}[\sqrt{a_2^2 - h_{0,12}^2}\cos(\alpha_{120'})+h_{0,12}\sin(\alpha_{120'})\cos(\alpha_{g_{0'}}-\alpha)]. \tag{13}$$

Similarly, we express the variable length a_6, as a function of a_3, a_5 and α' with respect to the pentahedron $A_2A_3A_5A_6A_0$ by using the following equations:

$$h_{0',2356} = h_{0',35}\sin(\alpha'), \tag{14}$$

$$x_{60'}^2 = a_6^2 - h_{0',2356}^2, \tag{15}$$

$$\sin(\angle A_3A_5A_{0',2356}) = \frac{h_{0',35}\cos(\alpha')}{x_{60'}}, \tag{16}$$

$$x_{60'}^2 = x_{50'}^2 + a_{56}^2 - 2x_{50'}a_{56}\cos(\alpha_{356} - \angle A_3A_5A_{0',2356}) \tag{17}$$

$$a_6^2 = x_{60'}^2 + h_{0',2356}^2, \tag{18}$$

or

$$a_6^2 = a_5^2 + a_{56}^2 - 2a_{56}[\sqrt{a_5^2 - h_{0',35}^2}\cos(\alpha_{356}) + h_{0',35}\sin(\alpha_{356})\cos(\alpha')]. \tag{19}$$

where

$$h_{0',35} = h_{0',35}(a_3, a_5) = \sqrt{\frac{4a_3^2a_5^2 - (a_3^2 + a_5^2 - a_{35}^2)^2}{4a_{35}^2}}. \tag{20}$$

We express $a_{00'}$ as a function of a_1, a_2, α, a_3, a_5 and α'. We shall prove that (13) takes the following form:

$$\begin{aligned} a_{00'}^2 = {} & a_2^2 + (a_{20'}(a_3, a_5, \alpha'))^2 \\ & - 2a_{20'}((a_3, a_5, \alpha'))\Big[\sqrt{a_2^2 - h_{0,12}^2}\cos(\alpha_{120'}(a_3, a_5, \alpha')) \\ & + h_{0,12}\sin(\alpha_{120'}(a_3, a_5, \alpha'))\cos(\alpha_{g_{0'}}(a_3, a_5, \alpha') - \alpha)\Big]. \end{aligned} \tag{21}$$

By following the same process with the calculation of a_4 and by changing the indices $0 \to 0'$ $1 \to 3$, $2 \to 5$ we express $a_{20'}$ and $a_{10'}$ as a function of a_3, a_5, and α', respectively:

$$a_{20'}^2 = a_5^2 + a_{52}^2 - 2a_{52}[\sqrt{a_5^2 - h_{0',35}^2}\cos(\alpha_{352}) + h_{0',35}\sin(\alpha_{352})\cos(\alpha')] \tag{22}$$

and

$$a_{10'}^2 = a_5^2 + a_{51}^2 - 2a_{51}[\sqrt{a_5^2 - h_{0',35}^2}\cos(\alpha_{351}) + h_{0',35}\sin(\alpha_{351})\cos(\alpha')]. \tag{23}$$

By replacing (22) and (23) in the cosine law of $\triangle A_1A_2A_{0'}$ we get:

$$\cos\alpha_{120'} = \frac{a_{12}^2 + (a_{20'}(a_3, a_5, \alpha'))^2 (a_{10'}(a_3, a_5, \alpha'))^2}{2a_{12}a_{20'}(a_3, a_5, \alpha')} \tag{24}$$

or

$$\alpha_{120'} = \alpha_{120'}(a_3, a_5, \alpha'). \tag{25}$$

From the sine law in $\triangle A_{0',12356}A_{0',35}A_{0'}$ and $\triangle A_{0',12356}A_{0',12}A_{0'}$ we obtain the following two relations:

$$h_{0',12356} = h_{0',35}\sin\alpha' \tag{26}$$

and

$$h_{0',12356} = h_{0',12}\sin\alpha_{g_{0'}}. \tag{27}$$

where

$$h_{0',12} = h_{0',12}(a_{10'}, a_{20'}) = \sqrt{\frac{4a_{10'}^2 a_{20'}^2 - (a_{10'}^2 + a_{20'}^2 - a_{12}^2)^2}{4a_{12}^2}} \tag{28}$$

From (26) and (27) we get:

$$\alpha_{g_{0'}} = \alpha_{g_{0'}}(a_3, a_5, \alpha'). \tag{29}$$

Taking into account (22), (25), and (29) we obtain (21).

From (9), (21) and (19) we have:

$$a_4 = a_4(a_1, a_2, \alpha), \quad a_{00'} = a_{00'}(a_1, a_2, \alpha, a_3, a_5, \alpha'), \quad a_6 = a_6(a_3, a_5, \alpha'). \tag{30}$$

From (3) and (30) the following equation is obtained:

$$\begin{aligned} &B_1a_1 + B_2a_2 + B_3a_3 + B_5a_5 + B_4a_4(a_1, a_2, \alpha) \\ &\quad + B_6a_6(a_3, a_5, \alpha') + \frac{B_0 + B_{0'}}{2}a_{00'}(a_1, a_2, \alpha, a_3, a_5, \alpha') = \text{minimum}. \end{aligned} \tag{31}$$

By differentiating (31) with respect to the variables a_1, a_2 and α, respectively, we get:

$$B_1 + B_4\frac{\partial a_4}{\partial a_1} + \frac{B_0 + B_{0'}}{2}\frac{\partial a_{00'}}{\partial a_1} = 0, \tag{32}$$

$$B_2 + B_4 \frac{\partial a_4}{\partial a_2} + \frac{B_0 + B_{0'}}{2} \frac{\partial a_{00'}}{\partial a_2} = 0, \tag{33}$$

and

$$B_4 \frac{\partial a_4}{\partial \alpha} + \frac{B_0 + B_{0'}}{2} \frac{\partial a_{00'}}{\partial \alpha} = 0. \tag{34}$$

We proceed by extending the parametrization technique given in [14] and [5] in $\mathbb{R}^3$ and by applying the first variation formula we shall calculate (32) and (33) ([11, Problem 1.5.3, pp. 16–17]).

Let $\gamma_1(s) = \gamma_{12}(s)$ be a linear segment which passes from A_1 and it degenerates to the fixed point A_2, $\gamma_2(s) = \gamma_{20}(s)$ be a linear segment which passes from A_2 and A_0 and $\gamma(s,t) = \gamma(A_2, \gamma_2(s), t)$.

The derivative of the length of the linear segment $a_2(s) = a_2(\gamma(s,t))$ with respect to s is given by ([11, Problem 1.5.3, pp. 16–17]):

$$\frac{\partial a_2}{\partial s} = \cos \alpha_{102} \tag{35}$$

Similarly, we get:

$$\frac{\partial a_4}{\partial s} = \cos \alpha_{104} \tag{36}$$

and

$$\frac{\partial a_{00'}}{\partial s} = \cos \alpha_{100'}. \tag{37}$$

Taking into account (35), (36) and (37) and by choosing the parametrization $s = a_1$, we get:

$$\frac{\partial a_2}{\partial a_1} = \cos \alpha_{102}, \tag{38}$$

$$\frac{\partial a_4}{\partial a_1} = \cos \alpha_{104}, \tag{39}$$

and

$$\frac{\partial a_{00'}}{\partial a_1} = \cos \alpha_{100'}. \tag{40}$$

By replacing (38), (39) and (40) in (32), we derive that:

$$\sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{10i}) = 0, \tag{41}$$

Similarly, from (33) by choosing the parametrization $s' = a_2$, we deduce that:

$$\sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{20i}) = 0. \tag{42}$$

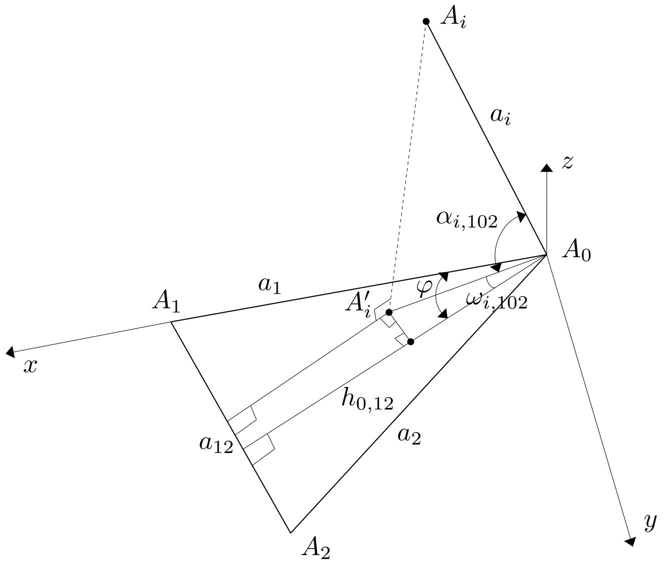

Figure 2.2:

Without loss of generality, we replace $B_0 = B_{0'} = \frac{B_0+B_{0'}}{2}$, in order to be consistent with (32) and (33).

We use the following parametrization with respect to the unit vectors u_i (see Fig. 2 with the orthogonal coordinate system Oxyz, O in A_0, x-axis on A_0A_1), for $i = 1, 2, 4, 0'$:

$$\vec{u}_1 = (1, 0, 0), \tag{43}$$

$$\vec{u}_2 = (\cos(\alpha_{102}), \sin(\alpha_{102}), 0), \tag{44}$$

$$\vec{u}_i = (\cos(\alpha_{i,102})\cos(\varphi - \omega_{i,102}), \cos(\alpha_{i,102})\sin(\varphi - \omega_{i,102}), \sin(\alpha_{i,102})), \tag{45}$$

From (43), (44), (45), we get:

$$\cos(\alpha_{10i}) = \cos(\alpha_{i,102})\cos(\varphi - \omega_{i,102}) \tag{46}$$

and

$$\cos(\alpha_{20i}) = \cos(\alpha_{i,102})\cos((\alpha_{102} - \varphi) + \omega_{i,102}). \tag{47}$$

By multiplying (46) by B_i and taking the sum for $i = 1, 2, 4, 0'$, we get:

$$\sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{10i}) = \cos(\varphi) \sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102})\cos(\omega_{i,102}) + \sin(\varphi) \sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102})\sin(\omega_{i,102})$$

or

$$\cos(\varphi) \sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102})\cos(\omega_{i,102}) + \sin(\varphi) \sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102})\sin(\omega_{i,102}) = 0. \tag{48}$$

Similarly, by multiplying (47) by B_i and taking the sum for $i = 1, 2, 4, 0'$, we get:

$$\sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{20i}) = \cos(\alpha_{102} - \varphi) \sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102}) \cos(\omega_{i,102}) - \sin(\alpha_{102} - \varphi) \sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102}) \sin(\omega_{i,102}),$$

or

$$\cos(\alpha_{102} - \varphi) \sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102}) \cos(\omega_{i,102}) - - \sin(\alpha_{102} - \varphi) \sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102}) \sin(\omega_{i,102}) = 0. \tag{49}$$

From (48) and (49), we derive that:

$$\sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102}) \cos(\omega_{i,102}) = 0 \tag{50}$$

and

$$\sum_{i=\{1,2,4,0'\}} B_i \cos(\alpha_{i,102}) \sin(\omega_{i,102}) = 0. \tag{51}$$

We need to apply a method of differentiation with respect to dihedral angle which has been developed in [17], in order to deduce the third equation.

By differentiating (9) and (13) with respect to α we get:

$$\frac{\partial a_i}{\partial \alpha} = -\frac{a_{2i}}{a_i} h_{0,12} \sin(\alpha_{12i}) \sin(\alpha_{g_i} - \alpha), \tag{52}$$

for $i = 1, 2, 4, 0'$

By replacing (52) in (34), we get:

$$\sum_{i=\{1,2,4,0'\}}^{n} B_i \frac{a_{2i}}{a_i} \sin(\alpha_{12i}) \sin(\alpha_{g_i} - \alpha) = 0. \tag{53}$$

By multiplying both members of (53) with $\frac{1}{6}a_{12}$, we get:

$$\sum_{i=\{1,2,4,0'\}} B_i \frac{1}{6} a_{12} \frac{a_{2i}}{a_i} h_{0,12} \sin(\alpha_{12i}) \sin(\alpha_{g_i} - \alpha) = 0,$$

or

$$\sum_{i=\{1,2,4,0'\}} B_i \operatorname{sgn}(i) \frac{\mathrm{VOL}(A_{012i})}{a_i} = 0,$$

or

$$\sum_{i=\{1,2,4,0'\}} B_i \operatorname{sgn}(i) S(A_1A_2A_0)\frac{h_{i,120}}{a_i} = 0,$$

or

$$\sum_{i=\{1,2,4,0'\}} B_i \operatorname{sgn}(i) \sin(\alpha_{i,102}) = 0, \tag{54}$$

where $\mathrm{VOL}(A_{012i})$ is the volume of the tetrahedron $A_0A_1A_2A_i$, $S(A_1A_2A_0)$ is the area of the triangle $A_1A_2A_0$, and $\operatorname{sgn}(i)$ is given by the function: $\operatorname{sgn}(i) =$

$$\begin{cases} +1, & \text{if } A_i \text{ is upper from the plane } A_1A_0A_2 \ , \\ 0, & \text{if } A_i \text{ belongs to the plane } A_1A_0A_2, \\ -1, & \text{if } A_i \text{ is lower from the plane } A_1A_0A_2 \end{cases}$$

We note that $B_{0'}$ corresponds to the weight $\frac{B_0+B_{0'}}{2}$ for the consistency of (50), (51) and (54).

Thus, (50), (51) and (54) inherit a weighted Fermat-Torricelli tree structure with respect to the objective function

$$B_1a_1 + B_2a_2 + B_4a_4 + \frac{B_0+B_{0'}}{2}a_{00'} = \text{minimum}$$

of the boundary tetrahedron $A_1A_2A_{0'}A_4$, such that A_0 is the corresponding weighted Fermat-Torricelli point and the following weighted floating equilibrium condition holds:

$$B_1\vec{u}_1 + B_2\vec{u}_2 + B_4\vec{u}_4 + \frac{B_0+B_{0'}}{2}\vec{u}_{0'} = \vec{0}. \tag{55}$$

By differentiating (31) with respect to the variables a_3, a_5 and α' we get:

$$B_3 + B_6\frac{\partial a_6}{\partial a_3} + \frac{B_0+B_{0'}}{2}\frac{\partial a_{00'}}{\partial a_3} = 0, \tag{56}$$

$$B_5 + B_6\frac{\partial a_6}{\partial a_5} + \frac{B_0+B_{0'}}{2}\frac{\partial a_{00'}}{\partial a_5} = 0, \tag{57}$$

$$B_6\frac{\partial a_6}{\partial \alpha'} + \frac{B_0+B_{0'}}{2}\frac{\partial a_{00'}}{\partial \alpha'} = 0. \tag{58}$$

By working similarly, (56), (57) and (58) yield

$$\sum_{i=\{3,5,6,0\}} B_i \cos(\alpha_{i,30'5}) \cos(\omega_{i,30'5}) = 0, \tag{59}$$

$$\sum_{i=\{3,5,6,0\}} B_i \cos(\alpha_{i,30'5}) \sin(\omega_{i,30'5}) = 0, \tag{60}$$

and

$$\sum_{i=\{3,5,6,0\}} \operatorname{sgn}(i) B_i \sin(\alpha_{i,30'5}) = 0. \tag{61}$$

Thus, (59), (60) and (61) inherit a weighted Fermat-Torricelli tree structure with respect to the objective function

$$B_3 a_3 + B_5 a_5 + B_6 a_6 + \frac{B_0 + B_{0'}}{2} a_{00'} = \text{minimum},$$

of the boundary tetrahedron $A_3A_5A_0A_6$ and $A_{0'}$ is the corresponding weighted Fermat-Torricelli point, because we express $a_{00'}$ in the following form:

$$\begin{aligned} a_{00'}^2 &= a_5^2 + (a_{50}(a_1, a_2, \alpha))^2 \\ &\quad - 2a_{50}(a_1, a_2, \alpha)\Big[\sqrt{a_5^2 - h_{0',35}^2}\cos(\alpha_{350}(a_1, a_2, \alpha)) \\ &\quad + h_{0',35}\sin(\alpha_{350}(a_1, a_2, \alpha))\cos(\alpha_{g_0}(a_1, a_2, \alpha) - \alpha')\Big]. \end{aligned} \tag{62}$$

□

Corollary 2.5. *An unweighted (or equally weighted) generalized Fermat-Torricelli tree of $A_1A_2A_3A_4A_5A_6$ consists of two equally weighted Fermat-Torricelli points A_0 and $A_{0'}$ which are located at the interior convex hull with corresponding weights $B_0 = B_{0'} = \frac{1}{6}$ and minimizes the objective function:*

$$\frac{1}{6}\sum_{i=1}^{6} a_i + \frac{1}{6} a_{00'}. \tag{63}$$

Proof of Corollary 2.5: The result follows directly from Theorem 2.4 by replacing in (3) $B_0 = B_i = \frac{1}{6}$, for $i = 1, 2, 3, 4, 5, 6$. □

Definition 2.6. We call the variable weight B_0 which corresponds to the weighted Fermat-Torricelli points A_0 and A_0' the generalized Fermat-Torricelli variable of degree four in $\mathbb{R}^3$.

Definition 2.7. We call the constant weight $c_G = \frac{1}{6}$ the generalized Fermat-Torricelli constant of degree four in $\mathbb{R}^3$.

Remark 2.8. The generalized Fermat-Torricelli constant of degree three in $\mathbb{R}^3$ for a tetrahedron is $c_G = 0.25$ which is greater than the generalized Fermat-Torricelli constant of degree four in $\mathbb{R}^3$.

By setting $B_0 = B_{0'} = 0$ in (3) and by assuming that the weighted floating case inequalities are satisfied such that A_0 is an interior point of the convex hull of $A_1A_2A_3A_4A_5A_6$, (see for the weighted floating case in [2, Chapter II]) we derive directly the following three results for degenerate generalized Fermat-Torricelli trees:

Proposition 2.9. *A degenerate generalized Fermat-Torricelli tree of $A_1A_2A_3A_4A_5A_6$ consists of one weighted Fermat-Torricelli point A_0 which is located at the interior of the convex hull and minimizes the objective function:*

$$\sum_{i=1}^{6} B_i a_i. \tag{64}$$

Corollary 2.10 ([13, pentahedra]). *For $B_6 = 0$, a degenerate generalized Fermat-Torricelli tree of $A_1A_2A_3A_4A_5A_6$ consists of one weighted Fermat-Torricelli point A_0 which is located at the interior of the convex hull and coincides with a degenerate generalized Fermat-Torricelli tree of the pentahedron $A_1A_2A_3A_4A_5$.*

Corollary 2.11 ([12, tetrahedra]). *For $B_5 = B_6 = 0$, a degenerate generalized Fermat-Torricelli tree of $A_1A_2A_3A_4A_5A_6$ consists of one weighted Fermat-Torricelli point A_0 which is located at the interior of the convex hull and coincides with a degenerate generalized Fermat-Torricelli tree of the tetrahedron $A_1A_2A_3A_4$.*

Remark 2.12. We mention that a generalized Fermat-Torricelli tree for a pentahedron is a weighted Fermat-Torricelli tree with respect to a boundary pentahedron and has been studied in [13]. A generalized Fermat-Torricelli tree for a tetrahedron is a weighted Fermat-Torricelli tree with respect to a boundary tetrahedron and has been studied in [12].

Proposition 2.13. *A generalized Fermat-Torricelli tree of the enniahedron $A_1A_2A_3A_4A_5A_6$ consists of two weighted Fermat-Torricelli points A_0 and A_0' which are located at the interior convex hull with corresponding weights B_0 and $B_{0'}$ and minimizes the objective function:*

$$\sum_{i=1}^{6} B_i a_i + \frac{B_0 + B_{0'}}{2} a_{00'} \tag{65}$$

Proof of Proposition 2.13. It is a direct consequence of Theorem 2.4 taking into account in the calculation of a_6 that A_6 does not belong to the plane defined by the base $A_1A_2A_3A_5$ and it is upper from the plane defined by $A_{0'}A_3A_5$:

$$a_6^2 = a_5^2 + a_{56}^2 - 2a_{56}[\sqrt{a_5^2 - h_{0',35}^2}\cos(\alpha_{356}) + h_{0',35}\sin(\alpha_{356})\cos(\alpha_{g_6} - \alpha')]. \tag{66}$$

where α_{g_6} is the given dihedral angle between the planes $A_1A_2A_3A_5$ and $A_3A_5A_6$. $\square$

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