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Convergence of Weighted Sums and Strong Law of Large Numbers for Convex Compact Integrable Random Sets and Fuzzy Random Sets*

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We prove several results in the a.s. convergence of weighted sums and strong law of large numbers for convex compact integrable random sets in separable Banach spaces via the norm compactness of the expectation of convex compact integrable random sets and an embedding method. The techniques are applied to prove strong law of large numbers for fuzzy random variables and other related results. Tightness conditions are discussed in comparison with earlier results in the literature.

Keywords: Expectation, fuzzy convex, strong law of large numbers, tight, Toeplitz sequence, weighted sum

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1. Introduction

Let E be a separable Banach space and let $\{X_k\}$ be a sequence of convex compact integrable random sets in E. Using the norm compactness of the expectation of convex compact integrable random sets and an embedding method, we prove several results in the almost sure convergence with respect to the Hausdorff distance for strong law of large numbers (SLLN)

$$d_H\left(\frac{1}{n}\sum_{k=1}^{n} X_k, \frac{1}{n}\sum_{k=1}^{n} \mathbf{E}[X_k]\right)$$

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and weighted sums of the form

$$S_n = \sum_{k=1}^{n} a_{nk} X_k.$$

We apply our techniques to prove the SLLN for fuzzy random variables and other related results in this framework.

The paper is organized as follows. In Sect. 2 we set our notations and definitions, and summarize needed results. In Sect. 3 we present the norm compactness of the Aumann expectation of convex compact integrable random set attached with an embedding method. Sect. 4 is devoted to the a.s. convergence of weighted sums and the SLLN for convex compact valued integrable random sets. In Sect. 5 the a.s. convergence of weighted sums of level sets associated with random fuzzy variables is presented and various convergence results on the SLLN for fuzzy random sets are provided.

Our results generalize some related results in the literature under conditions weaker or non comparable to tightness. We refer to Klement et al. [17], Puri and Ralescu [21], Taylor and Inoue [24], Inoue [15], Joo et al. [16], Li and Ogura [18], Li et al. [19] for related results.

2. Preliminaries

Some basic definitions and properties will be briefly listed. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(E, \|.\|)$ be a real separable Banach space. We denote by $\mathrm{c}(E)$ the family of all nonempty closed subsets of E, $\mathrm{wk}(E)$ (resp. $\mathrm{k}(E)$) the family of all nonempty weakly compact (resp. compact) subsets of E and $\mathrm{cwk}(E)$ (resp. $\mathrm{ck}(E)$) is the family of all nonempty convex weakly compact (resp. convex compact) subsets of E. We define the Minkowski's addition and scalar multiplication respectively in $\mathrm{k}(E)$ (or $\mathrm{ck}(E)$) by

$$A + B := \{a + b : a \in A, b \in B\},$$
$$\lambda A := \{\lambda a : a \in A\}$$

where $A, B \in \mathrm{k}(E)$ (or $\mathrm{ck}(E)$) and $\lambda \in \mathbf{R}$. The Hausdorff metric on $\mathrm{k}(E)$ is defined by

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$$

for $A, B \in \mathrm{k}(E)$. The metric space $(\mathrm{k}(E), d_H)$ is complete and separable [8]. For every $A \in \mathrm{k}(E)$, denote $|A| = d_H(A, \{0\}) = \sup_{a\in A} \|a\|$. A $\mathrm{c}(E)$-valued mapping $X : \Omega \rightrightarrows E$ is a closed valued random set, if X is measurable, i.e., for every $B \in \mathrm{c}(E)$, the set $X^-(B) = \{\omega \in \Omega : X(\omega) \cap B \neq \emptyset\} \in \mathcal{F}$ [8, 14]. A $\mathrm{cwk}(E)$-valued random variable $X : \Omega \rightrightarrows E$ is integrable if $|X|$ is integrable. According to [8], a $\mathrm{cwk}(E)$-valued mapping $X : \Omega \rightrightarrows E$ is measurable iff for every $x^* \in E^*$, the mapping $\delta^*(x^*, X)$ is measurable, where $\delta^*(x^*, X)$ denotes the support function of X. The Hausdorff distance between $A, B \in \mathrm{cwk}(E)$ is given by the formula $d_H(A,B) = \sup_{x^* \in \overline{B}_{E^*}} |\delta^*(x^*, A) - \delta^*(x^*, B)|$ where $\overline{B}_{E^*}$ denotes the closed unit ball in E^*. For every $\mathrm{cwk}(E)$-valued integrable random set X, the expectation of X, $\mathbf{E}[X]$, is defined by the Aumann integral of X: $\mathbf{E}[X] = \{\int_\Omega f dP : f \in S^1_X\}$ where $\int_\Omega f dP$ is the usual Bochner integral and S^1_X is the set of all integrable selections of X. Let us denote by $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ the space of all convex compact valued integrable random sets.

Let us recall some well known notions. A Toeplitz sequence is a double array of non-negative numbers $\{a_{nk}\}$ satisfying:

(a) For each k, $\lim_{n\to\infty} a_{nk} = 0$,

(b) There exists a positive constant C such that $\sup_n \sum_{k=1}^{\infty} a_{nk} \le C$.

A sequence $\{X_n\}$ in $L^1_E(\Omega, \mathcal{F}, P)$ is tight if for every $\varepsilon > 0$, there exists a compact subset Γ_ε of E such that $P[X_n \notin \Gamma_\varepsilon] < \varepsilon$ for all $n \in \mathbf{N}$. A sequence $\{X_n\}$ in $L^1_E(\Omega, \mathcal{F}, P)$ is compactly uniformly integrable if for every $\varepsilon > 0$, there exists a compact subset K_ε of E such that $\mathbf{E}[\|X_n\| I_{[X_n \notin K_\varepsilon]}] < \varepsilon$ for all $n \in \mathbf{N}$.

A sequence $\{X_n\}$ in $L^1_E(\Omega, \mathcal{F}, P)$ is said to be compactly uniformly integrable in the Cesàro sense [2] if for every $\varepsilon > 0$, there exists a compact subset K_ε of E such that

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \mathbf{E}[\|X_k\| I_{[X_n \notin K_\varepsilon]}] < \varepsilon.$$

We note that the compactly uniform integrability is stronger than the compactly uniform integrability in the Cesàro sense.

3. On the compactness of the Aumann integral $\mathbf{E}[X]$ and the space $L^1_{C(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$

For the convenience of the reader, we recall and summarize some needed results.

Theorem 3.1. *Let E be a separable Banach space and X is a* $\mathrm{cwk}(E)$*-valued integrable* ($\mathbf{E}[|X|] < \infty$) *random set. Then the following properties hold*

(*a*) *The set S^1_X of all integrable selections of X is convex $\sigma(L^1_E, L^\infty_{E^*})$-compact,*

(*b*) $\mathbf{E}[X] := \{\mathbf{E}f : f \in S^1_X\}$ *is convex weakly compact,*

(*c*) $\delta^*(x^*, \mathbf{E}X) = \mathbf{E}[\delta^*(x^*, X)]$, *for all $x^* \in E^*$,*

(*d*) *If X is a* $\mathrm{ck}(E)$*-valued integrable random set, then* $\mathbf{E}[X] := \{\mathbf{E}f : f \in S^1_X\}$ *is convex compact.*[1]

Proof. We only sketch the proof. See [4, 8] for details. As $\mathbf{E}[|X|] < \infty$, the set S^1_X of all selections of X is convex weakly compact in L^1_E, cf. [1, 4, 8, 7]

[1] The compactness of $\mathbf{E}X$ according to Debreu integral is not available here, see also the remarks of Theorem 4.5 in Hiai-Umegaki [14].

so that $\int_\Omega XdP$ is convex weakly compact in E. Making use of the Strassen formula (cf. [8]) we note

$$\delta^*(x^*, \mathbf{E}[X]) = \mathbf{E}[\delta^*(x^*, X)], \quad \forall x^* \in E^*.$$

Let E_c^* denote the vector space E^* endowed with the topology of compact convergence, τ_c. Since X is convex compact, $\delta^*(., X)$ is continuous on E_c^*. We need to show that

$$x^* \mapsto \delta^*\left(x^*, \int_\Omega XdP\right) = \int_\Omega \delta^*(x^*, X)dP$$

is continuous on E_c^*. Next we show that

$$x^* \mapsto \delta^*\left(x^*, \int_\Omega XdP\right) = \int_\Omega \delta^*(x^*, X)dP$$

is continuous on $\overline{B}^c_{E^*}$ Note that $\overline{B}^c_{E^*}$ is compact metrizable. Let x_n^* τ_c converges to $x^* \in \overline{B}^c_{E^*}$. Then we have

$$\lim_{n\to\infty} \delta^*(x_n^*, X) = \delta^*(x^*, X).$$

We have $|\delta^*(x^*, X)| \le |X|$ for all $x^* \in \overline{B}^c_{E^*}$. By Lebesgue dominated convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty} \delta^*\left(x_n^*, \int_\Omega XdP\right) &= \lim_{n\to\infty} \int_\Omega \delta^*(x_n^*, X)dP \\
&= \int_\Omega \lim_{n\to\infty} \delta^*(x_n^*, X)dP = \int_\Omega \delta^*(x^*, X)dP \\
&= \delta^*\left(x^*, \int_\Omega XdP\right).
\end{aligned}$$

Thus $\delta^*(., \int_\Omega XdP)$ is continuous on $\overline{B}^c_{E^*}$. By virtue of the Banach-Dieudonné theorem we conclude that $\delta^*(., \int_\Omega XdP)$ is continuous on E_c^*. Thus $\mathbf{E}[X]$ is convex and compact. □

Now we present the relationships between the space $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ and the space $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$. Given $X \in \mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$, the function $\delta^*(., X)$ belongs to $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$ since the function $\delta^*(x^*, X(\omega))$ defined on $\Omega \times \overline{B}^c_{E^*}$ is a Caratheodory integrable mapping, that is, $\delta^*(., X(\omega))$ is continuous on $\overline{B}^c_{E^*}$ for every $\omega \in \Omega$, $\delta^*(x^*, X(.))$ is measurable on Ω and $|\delta^*(x^*, X(\omega))| \le |X(\omega)|$ for all $(\omega, x^*) \in \Omega \times \overline{B}^c_{E^*}$ with $|X| \in L^1(\Omega, \mathcal{F}, P)$, shortly $\delta^*(., X) \in L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$. More generally, the space $\mathcal{G}$ of all Caratheodory integrable integrands on $\Omega \times S$ where S is a compact metric plays a central role in the convergence of Young measures, see e.g. [7, 26] this space can be identified with the space $L^1_{\mathcal{C}(S)}(\Omega, \mathcal{F}, P)$ where $\mathcal{C}(S)$ is the space of all continuous functions on S endowed with the norm of uniform convergence. Similarly, in the present situation, it turns out that the space $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$ associated with space $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ plays also a useful role in the a.s. convergence problem in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ with respect to the Hausdorff distance. This fact is illustrated by establishing the relationships between these spaces that are summarized in the following.

Theorem 3.2. *Let $X \in \mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ be given. Then the following hold*

(1) $\delta^*(.,X) \in L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$,

(2) $\mathbf{E}[\delta^*(.,X)] \in \mathcal{C}(\overline{B}^c_{E^*})$ *and the value of the function* $\mathbf{E}[\delta^*(.,X)]$ *at the point* $x^* \in \overline{B}^c_{E^*}$ *is given by*

$$\begin{aligned} \mathbf{E}[\delta^*(.,X)](x^*) &= \langle \mathbf{E}[\delta^*(.,X)], \varepsilon_{x^*}\rangle \\ &= \int_\Omega \delta^*(x^*, X(\omega))dP = \delta^*(x^*, \mathbf{E}[X]) \end{aligned} \tag{1}$$

where ε_{x^*} is the Dirac mass at the point $x^* \in \overline{B}^c_{E^*}$.

(3) *Let $X, Y \in \mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ be given. Then*

$$\begin{aligned} & \|\delta^*(.,X) - \mathbf{E}[\delta^*(.,Y)]\| \\ = & \sup_{x^* \in \overline{B}^c_{E^*}} |\delta^*(x^*, X) - \mathbf{E}[\delta^*(.,Y)](x^*)| \\ = & \sup_{x^* \in \overline{B}^c_{E^*}} |\delta^*(x^*, X) - \delta^*(x^*, \mathbf{E}[Y])| = d_H(X, \mathbf{E}[Y]). \end{aligned} \tag{2}$$

Proof. *(1)* It has already seen that $\delta^*(.,X) \in L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$.

(2) Since $\overline{B}^c_{E^*}$ is compact (metrizable) and $\mathbf{E}[\delta^*(.,X(.))] \in \mathcal{C}(\overline{B}^c_{E^*})$, we have

$$\mathbf{E}[\delta^*(.,X)](x^*) = \langle \mathbf{E}[\delta^*(.,X)], \varepsilon_{x^*}\rangle = \int_\Omega \delta^*(x^*, X(\omega))dP$$

where ε_{x^*} is the Dirac mass at the point $x^* \in \overline{B}^c_{E^*}$. By Theorem 3.1($c$) $\delta^*(x^*, \mathbf{E}X) = \mathbf{E}[\delta^*(x^*, X)]$, for all $x^* \in E^*$.

(3) follows from *(2)* and the Hormander formula ([8], Theorem II-14). □

Embedding. It is easy to see that ck(E) is embedded naturally as a convex cone of the separable Banach space $(\mathcal{C}(\overline{B}^c_{E^*}), \|.\|)$ via the mapping

$$X \mapsto \delta^*(.,X)$$

by observing that $\delta^*(.,X)$ is continuous on E_c^* where E_c^* is the vector space E^* endowed with the topology of compact convergence[2]. This embedding is isometric, the addition is preserved and the multiplication by nonnegative real numbers is preserved.

Now let $X \in \mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$. By Theorem 3.2(*2*) we have $\mathbf{E}[\delta^*(.,X)] \in \mathcal{C}(\overline{B}^c_{E^*})$ and the value of the function $\mathbf{E}[\delta^*(.,X)]$ at the point $x^* \in \overline{B}^c_{E^*}$ is given by

$$\mathbf{E}[\delta^*(.,X)](x^*) = \langle \mathbf{E}[\delta^*(.,X)], \varepsilon_{x^*}\rangle = \int_\Omega \delta^*(x^*, X(\omega))dP = \delta^*(x^*, \mathbf{E}[X])$$

where ε_{x^*} is the Dirac mass at the point $x^* \in \overline{B}^c_{E^*}$. Thus by the above considerations and Theorems 3.1–3.2, we note that $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ is embedded as a convex cone of the Banach space $L^1_{C(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$ via the mapping $j : X \mapsto \delta^*(.,X)$ so that

$$\mathbf{E}[j(X)] = j(\mathbf{E}[X]), \quad \forall X \in \mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P) \tag{3}$$

$$\|j(X) - \mathbf{E}[j(Y)]\| = d_H(X, \mathbf{E}[Y]), \quad \forall X, Y \in \mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P). \tag{4}$$

Remark. Actually (3) and (4) are the main properties of this embedding we use in further applications. That is why we need to provide a concise verification of these properties.

Let $\{X_n\}$ be a sequence of random sets in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ and let $\{a_{nk}\}$ be a Toeplitz sequence. Then $\{\delta^*(.,X_n)\}$ is a sequence of random elements in $L^1_{C(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$. By Theorem 3.2 we have

$$\begin{aligned}
&\left\| \sum_{k=1}^n a_{nk}\delta^*(.,X_k) - \sum_{k=1}^n a_{nk}\mathbf{E}[\delta^*(.,X_k)] \right\| &(5)\\
&= \sup_{x^* \in \overline{B}^c_{E^*}} \left| \sum_{k=1}^n a_{nk}\delta^*(x^*,X_k) - \sum_{k=1}^n a_{nk}\mathbf{E}[\delta^*(.,X_k)](x^*) \right| \\
&= \sup_{x^* \in \overline{B}^c_{E^*}} \left| \delta^*\left(x^*, \sum_{k=1}^n a_{nk}X_k\right) - \delta^*\left(x^*, \sum_{k=1}^n a_{nk}\mathbf{E}[X_k]\right) \right| \\
&= d_H\left(\sum_{k=1}^n a_{nk}X_k, \sum_{k=1}^n a_{nk}\mathbf{E}[X_k]\right).
\end{aligned}$$

When $a_{nk} = \frac{1}{n}$ if $k = 1, ..., n$ and 0 otherwise we have

$$d_H\left(\frac{1}{n}\sum_{k=1}^n X_k, \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k]\right) = \left\| \frac{1}{n}\sum_{k=1}^n \delta^*(.,X_k) - \frac{1}{n}\sum_{k=1}^n \mathbf{E}[\delta^*(.,X_k)] \right\|. \tag{6}$$

[2] Note that the topology of compact convergence and the weak star topology coincide on $\overline{B}_{E^*}$.

4. Convergence of weighted sums of convex compact valued random sets

In this section we develop some convergence results of weighted sums of convex compact valued integrable random sets with respect to the Hausdorff distance d_H. For the convenience of the reader we summarize a convergence result of weighted sums of E-valued integrable random variables.

Theorem 4.1. *Let $\{X_n\}$ be an independent, compactly uniformly integrable sequence in $L^1_E(\Omega, \mathcal{F}, P)$. Let X be a nonnegative random variable such that $P(|X_n| \geq x) \leq P(X \geq x)$ for all n and all $x > 0$, and $\mathbf{E}X^{1+\frac{1}{\gamma}} = \Gamma < \infty$ for some $\gamma > 0$. If $\{a_{nk}\}$ is a Toeplitz sequence and $\max_{1\leq k\leq n} a_{nk} = O(n^{-\gamma})$, then*

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} a_{nk}(X_k - \mathbf{E}[X_k]) \right\| = 0 \quad a.s.$$

Proof. See Cuesta and Matran [11], Taylor and Inoue [24], Wang and Rao [27]. □

Now we proceed to the a.s. convergence for the weighted sums and the SLLN for convex compact valued integrable random sets.

Theorem 4.2. *Let $\{X_n\}$ be a sequence in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$. Assume that the sequence $\{\delta^*(., X_n)\}$ is independent, compactly uniformly integrable in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$. Let X be a nonnegative random variable such that $P(|X_n| \geq x) \leq P(X \geq x)$ for all n and all $x > 0$, and $\mathbf{E}[X^{1+\frac{1}{\gamma}}] = \Gamma < \infty$ for some $\gamma > 0$. If $\{a_{nk}\}$ be a Toeplitz sequence and $\max_{1\leq k\leq n} a_{nk} = O(n^{-\gamma})$ then*

$$\lim_{n\to\infty} d_H \left(\sum_{k=1}^{n} a_{nk} X_k, \sum_{k=1}^{n} a_{nk}\mathbf{E}[X_k] \right) = 0 \quad a.s.$$

Proof. By our hypotheses, the sequence $\{\delta^*(., X_n)\}$ satisfies the conditions of Theorem 4.1, here the separable Banach space is $\mathcal{C}(\overline{B}^c_{E^*})$. By (5) we have

$$\begin{aligned} &\lim_{n\to\infty} d_H \left(\sum_{k=1}^{n} a_{nk} X_k, \sum_{k=1}^{n} a_{nk}\mathbf{E}[X_k] \right) \\ &= \lim_{n\to\infty} \left\| \sum_{k=1}^{n} a_{nk}\delta^*(., X_k) - \sum_{k=1}^{n} a_{nk}\mathbf{E}[\delta^*(., X_k)] \right\| = 0 \quad a.s. \end{aligned}$$ □

We provide some new results on the SLLN of pairwise independent sequence of ck(E)-valued integrable random sets illustrating the embedding method we presented here.

Theorem 4.3. *Let $\{X_n\}$ be a sequence in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$. Assume that*

(a) $\sum_{n=1}^{\infty} \frac{\mathbf{E}[|X_n|^p]}{n^p} < \infty$, *for some* $1 \leq p \leq 2$,

(b) *the sequence* $\{\delta^*(., X_n)\}$ *is pairwise independent and compactly uniformly integrable in the Cesàro sense in* $\mathcal{C}(\overline{B}^c_{E^*})$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq m\leq n} d_H\left(\frac{1}{n}\sum_{k=1}^{m} X_k, \frac{1}{n}\sum_{k=1}^{m}\mathbf{E}[X_k]\right) > \varepsilon\right) < \infty.$$

In particular, $\{X_n\}$ *verifies the SLLN with respect to the Hausdorff distance*

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[X_i]\right) = 0 \quad a.s.$$

Proof. By our assumption (a)–(b), the sequence $\{\delta^*(., X_n)\}$ in $L^1_{C(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$ satisfies the conditions of Theorem 3.1 in Bai et al. [2], therefore we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq m\leq n}\left\|\frac{1}{n}\sum_{k=1}^{m}\delta^*(., X_k) - \frac{1}{n}\sum_{k=1}^{m}\mathbf{E}[\delta^*(., X_k)]\right\| > \varepsilon\right) < \infty \tag{7}$$

for all $\varepsilon > 0$ and

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n}\delta^*(., X_i) - \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[\delta^*(., X_i)]\right\| = 0 \quad a.s. \tag{8}$$

that is $\{\delta^*(., X_n)\}$ satisfies the SLLN in $(\mathcal{C}(\overline{B}^c_{E^*}), \|.\|)$. We have

$$\begin{aligned}
& d_H\left(\frac{1}{n}\sum_{i=1}^{m} X_i, \frac{1}{n}\sum_{i=1}^{m}\mathbf{E}[X_i]\right) \\
= & \sup_{x^*\in\overline{B}^c_{E^*}}\left|\delta^*\left(x^*, \frac{1}{n}\sum_{i=1}^{m} X_i\right) - \delta^*\left(x^*, \frac{1}{n}\sum_{i=1}^{m}\mathbf{E}[X_i]\right)\right| \\
= & \sup_{x^*\in\overline{B}^c_{E^*}}\left|\frac{1}{n}\sum_{i=1}^{m}\delta^*(x^*, X_i) - \frac{1}{n}\sum_{i=1}^{m}\mathbf{E}[\delta^*(x^*, X_i)]\right| \\
= & \sup_{x^*\in\overline{B}^c_{E^*}}\left|\frac{1}{n}\sum_{i=1}^{m}\delta^*(x^*, X_i) - \frac{1}{n}\sum_{i=1}^{m}\mathbf{E}[\delta^*(., X_i)](x^*)\right| \\
= & \left\|\frac{1}{n}\sum_{i=1}^{m}\delta^*(., X_i) - \frac{1}{n}\sum_{i=1}^{m}\mathbf{E}[\delta^*(., X_i)]\right\|
\end{aligned}$$

and by (6) we have

$$d_H\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[X_i]\right) = \left\|\frac{1}{n}\sum_{i=1}^{n}\delta^*(., X_i) - \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[\delta^*(., X_i)]\right\|.$$

Then from (7) for all $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P \left(\max_{1 \le m \le n} d_H \left(\frac{1}{n} \sum_{k=1}^{m} X_k, \frac{1}{n} \sum_{k=1}^{m} \mathbf{E}[X_k] \right) > \varepsilon \right) < \infty$$

and from (8) $\{X_n\}$ verifies the SLLN with respect to the Hausdorff distance

$$\lim_{n\to\infty} d_H \left(\frac{1}{n} \sum_{i=1}^{n} X_i, \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[X_i] \right) = 0 \quad a.s. \qquad \square$$

Since the compactly uniform integrability is stronger than the compactly uniform integrability in the Cesàro sense, the following corollary is obtained.

Corollary 4.4. *Let $\{X_n\}$ be a sequence in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$. Assume that*

(a) $\sum_{n=1}^{\infty} \frac{\mathbf{E}[|X_n|^p]}{n^p} < \infty$, *for some $1 \le p \le 2$,*
(b) *the sequence $\{\delta^*(., X_n)\}$ is pairwise independent and compactly uniformly integrable in $L^1_{C(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$.*

Then $\{X_n\}$ verifies the SLLN with respect to the Hausdorff distance

$$\lim_{n\to\infty} d_H \left(\frac{1}{n} \sum_{i=1}^{n} X_i, \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[X_i] \right) = 0 \quad a.s.$$

For the sake of completeness we produce an important result that is borrowed from Castaing and Raynaud de Fitte ([9], Theorem 4.8).

Theorem 4.5. *Let E be a separable Banach space. Let $\{X_n\}$ be a pairwise independent identically distributed sequence of integrably bounded* $\mathrm{ck}(E)$*-valued, such that $g := \sup_{n\in\mathbf{N}} |X_n| \le \alpha$ is integrable, then*

$$\mathbf{E}[X_n] = \mathbf{E}[X_1] \in \mathrm{ck}(E), \quad \forall n \in \mathbf{N}$$

and

$$\lim_{n\to\infty} d_H \left(\frac{1}{n} \sum_{i=1}^{n} X_i, \mathbf{E}[X_1] \right) = 0 \quad a.s.$$

Remarks. It is important to have the norm compactness of $\mathbf{E}[X_n] = \mathbf{E}[X_1]$ because $\mathbf{E}[X_1]$ is the norm compactness limit in our SLLN. It is also worth to note that if $\{X_n\}$ is a sequence of convex compact valued integrably bounded i.i.d. random sets, the random variable $g = \sup_n |X_n|$ is necessarily constant (with finite value). See [9] for details.

It is worth to mention that we can recover Theorem 4.5 by applying the embedding method described in Theorem 3.2 and Etemadi theorem [13].

Theorem 4.6. *Let $\{X_n\}$ be a sequence of pairwise independent and identically distributed in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ with $\mathbf{E}[X_n] = \mathbf{E}[X_1]$ for all $n \in \mathbf{N}$. Then $\{X_n\}$ satisfies the SLLN w.r.t. the Hausdorff distance*

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \mathbf{E}[X_1]\right) = 0 \quad a.s.$$

Proof. By our assumption $\{\delta^*(., X_n)\}$ is a sequence of pairwise independent and identically distributed random elements in $L^1_{C(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$, then by Theorem 3.1, we have $\mathbf{E}[X_n] = \mathbf{E}[X_1] \in \mathrm{ck}(E)$ for all $n \in \mathbf{N}$. By (3) we have

$$\mathbf{E}[\delta^*(., X_n)] = \delta^*(., \mathbf{E}[X_n]) = \delta^*(., \mathbf{E}[X_1]) = \mathbf{E}[\delta^*(., X_1)]$$

for all $n \in \mathbf{N}$. Applying Etemadi's Theorem we have

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{i=1}^{n} \delta^*(., X_i) - \mathbf{E}[\delta^*(., X_1)] \right\| = 0 \quad a.s.$$

By (3)–(4) we have

$$\begin{aligned} d_H\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \mathbf{E}[X_1]\right) &= \sup_{x^*\in\overline{B}^c_{E^*}} \left| \delta^*\left(x^*, \frac{1}{n}\sum_{i=1}^{n} X_i\right) - \delta^*(x^*, \mathbf{E}[X_1]) \right| \\ &= \sup_{x^*\in\overline{B}^c_{E^*}} \left| \frac{1}{n}\sum_{i=1}^{n} \delta^*(x^*, X_i) - \mathbf{E}[\delta^*(x^*, X_1)] \right| \\ &= \sup_{x^*\in\overline{B}^c_{E^*}} \left| \frac{1}{n}\sum_{i=1}^{n} \delta^*(x^*, X_i) - \mathbf{E}[\delta^*(., X_1)](x^*) \right| \\ &= \left\| \frac{1}{n}\sum_{i=1}^{n} \delta^*(., X_i) - \mathbf{E}[\delta^*(., X_1)] \right\| \end{aligned}$$

so that $\{X_n\}$ satisfies the SLLN w.r.t. the Hausdorff distance. □

Comments and Remarks. Some comments are in order on the above results. Theorem 4.1 is the vector-valued version of Theorem 3.3 in Inoue and Taylor [24] and Theorem 4.2 in Wang and Rao [27]. Using Theorem 4.1 and the embedding method described in Theorems 3.1 and 3.2, we get Theorem 4.2 that is a convex compact valued version of Theorem 3.3 in Inoue and Taylor [24]. However in Theorem 3.3 of Inoue and Taylor, the tightness condition is imposed on the metric space $(\mathrm{ck}(E), d_H)$ while in Theorem 4.2, the tightness condition is imposed on the space $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$.

The reader is refereed to the study of tightness for measurable multifunctions (alias random sets) in Castaing et al. [6] where several concepts of tightness are examined for sequences of multifunctions with values in a separable Banach space E. Let $\mathcal{C} := \mathrm{ck}(E)$ for notational convenience. These notions are denoted by $\mathcal{I}(\mathcal{C})_\varepsilon$, $\mathcal{S}(\mathcal{C})_\varepsilon$, $\mathcal{D}(\mathcal{C})_\varepsilon$ and $\mathcal{D}'(\mathcal{C})_\varepsilon$, $\mathcal{I}_+(\mathcal{C})_\varepsilon$, $\mathcal{S}_+,(\mathcal{C})_\varepsilon$.

$\mathcal{I}(\mathcal{C})_\varepsilon$: for every $\varepsilon > 0$, there exists a $\mathcal{C}$-valued measurable multifunction Γ_ε such that if we set $A_{n\varepsilon} = \{X_n \bigcap \Gamma_\varepsilon \neq \emptyset\}$, we have

$$P(\limsup_{n\to+\infty} A_{n\varepsilon}) \geq 1 - \varepsilon.$$

$\mathcal{S}(\mathcal{C})_\varepsilon$: for every $\varepsilon > 0$, there exists a $\mathcal{C}$-valued measurable multifunction Γ_ε such that if we set $A_{n\varepsilon} = \{X_n \subseteq \Gamma_\varepsilon\}$, we have

$$P(\limsup_{n\to+\infty} A_{n\varepsilon}) \geq 1 - \varepsilon.$$

$\mathcal{D}(\mathcal{C})_\varepsilon$: for every $\varepsilon > 0$, there exists a $\mathcal{C}$-valued measurable multifunction Γ_ε such that

$$\Omega_\varepsilon = \{\omega \in \Omega : \liminf_{n\to+\infty} d(0, X_n(\omega) \bigcap \Gamma_\varepsilon(\omega)) < +\infty\}$$

satisfies $P(\Omega_\varepsilon) \geq 1 - \varepsilon$.

$\mathcal{D}'(\mathcal{C})_\varepsilon$: for every $\varepsilon > 0$, there exists a $\mathcal{C}$-valued measurable multifunction Γ_ε such that

$$\Omega_\varepsilon = \{\omega \in \Omega : \limsup_{n\to+\infty} d(0, X_n(\omega) \bigcap \Gamma_\varepsilon(\omega)) < +\infty\}$$

satisfies $P(\Omega_\varepsilon) \geq 1 - \varepsilon$.

$\mathcal{I}_+(\mathcal{C})_\varepsilon$: for every $\varepsilon > 0$, there is a $\mathcal{C}$-valued measurable multifunction Γ_ε such that if we set $A_{n\varepsilon} = \{X_n \cap \Gamma_\varepsilon \neq \emptyset\}$, we have

$$\inf_{n\geq 1} \mu(A_{n\varepsilon}) \geq 1 - \varepsilon.$$

$\mathcal{S}_+(\mathcal{C})_\varepsilon$: for every $\varepsilon > 0$, there is a $\mathcal{C}$-valued measurable multifunction Γ_ε such that if we set $A_{n\varepsilon} = \{X_n \subseteq \Gamma_\varepsilon\}$, we have

$$\inf_{n\geq 1} P(A_{n\varepsilon}) \geq 1 - \varepsilon.$$

We note that when the X_n are single valued, $\mathcal{I}(\mathcal{C})_\varepsilon$ and $\mathcal{S}(\mathcal{C})_\varepsilon$ are equivalent and $\mathcal{I}_+(\mathcal{C})_\varepsilon$ and $\mathcal{S}_+(\mathcal{C})_\varepsilon$ are equivalent. In the present context, condition $\mathcal{S}_+(\mathcal{C})_\varepsilon$ seems suitable.

Taking into the above considerations, we present the relationships between the tightness condition and CUI condition in the space $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ and the space $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$.

We will provide at first results needed on tightness and CUI conditions.

Lemma 4.7. *If $\{X_n\}$ is a sequence of convex compact random sets satisfying the condition: for every $\varepsilon > 0$ there is an equilibrated convex compact set K in E with $0 \in K$ such that $P[X_n \subseteq K] \geq 1 - \varepsilon$ $\forall n \geq 1$, then $\{\delta^*(., X_n)\}$ is tight in $\mathcal{C}(\overline{B}^c_{E^*})$.*

Proof. Let $\varepsilon > 0$ and let K be given in this $\mathcal{S}_+(\mathcal{C})_\varepsilon$ condition. We note that the inclusion $X_n \subseteq K$ implies $\delta^*(., X_n) \in \mathcal{K}$ where $\mathcal{K}$ is a compact subset in $\mathcal{C}(\overline{B}^c_{E^*})$ given by

$$\mathcal{K} := \big\{\varphi \in \mathcal{C}(\overline{B}^c_{E^*}) : \varphi(0) = 0,$$

$$|\varphi(x^*) - \varphi(y^*)| \leq \max\{\delta^*(x^* - y^*, K), \delta^*(y^* - x^*, K)\},\ \forall x^*, y^* \in \overline{B}^c_{E^*}\big\}$$

so that $[X_n \subseteq K] \subset [\delta^*(., X_n) \in \mathcal{K}]$. Thus we have

$$P[\delta^*(., X_n) \in \mathcal{K}] \geq P[X_n \subseteq K] \geq 1 - \varepsilon \quad \forall n \geq 1.$$

□

Lemma 4.8. *If $\{X_n\}$ is a sequence of convex compact integrable random sets in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ satisfying the CUI condition: for every $\varepsilon > 0$ there is an equilibrated convex compact set with $0 \in K$ such that*

$$\mathbf{E}[|X_n| 1_{[X_n \subseteq K]^c}] \leq \varepsilon \quad \forall n \geq 1$$

where $[X_n \subseteq K]^c$ is the complementary of $[X_n \subseteq K]$, then, $\{\delta^*(., X_n)\}$ is compactly uniformly integrable in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$.

Proof. Let $\varepsilon > 0$ and let K be given in the CUI condition for $\{X_n\}$. Applying the argument given in the proof of Lemma 4.7, provides a compact subset $\mathcal{K}$ in $\mathcal{C}(\overline{B}^c_{E^*})$ such that $[X_n \subseteq K] \subset [\delta^*(., X_n) \in \mathcal{K}]$. Thus we get

$$\mathbf{E}[|X_n| 1_{[\delta^*(., X_n) \notin \mathcal{K}]}] \leq \mathbf{E}[|X_n| 1_{[X_n \subset K]^c}] \leq \varepsilon \quad \forall n \geq 1$$

showing that $\{\delta^*(., X_n)\}$ is compactly uniformly integrable in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$.

□

A sequence $\{X_n\}$ in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ is Cesàro CUI if for every $\varepsilon > 0$, there exists an equilibrated convex compact set K in E with $0 \in K$ such that

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \mathbf{E}[|X_k| I_{[X_n \subseteq K]^c}] < \varepsilon.$$

Then similarly we have

Lemma 4.9. *Assume that $\{X_n\}$ is a Cesàro CUI sequence in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$, then $\{\delta^*(., X_n)\}$ is compactly uniformly integrable in the Cesàro sense in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P))$.*

There are some applications illustrating the above facts.

Corollary 4.10. *Let $\{X_n\}$ be a pairwise independent sequence in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ satisfying:*

(i) $\sum_{n=1}^{\infty} \frac{\mathbf{E}[|X_n|^p]}{n^p} < \infty$, *for some* $1 \le p \le 2$,

(ii) $\{X_n\}$ *is Cesàro CUI.*

Then $\{X_n\}$ verifies the SLLN with respect to the Hausdorff distance

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[X_i]\right) = 0 \quad a.s.$$

Proof. According to Lemma 4.9, (ii) implies that $\{\delta^*(., X_n)\}$ is compactly uniformly integrable in the Cesàro sense in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$. Thus Corollary 4.10 is direct consequence of Theorem 4.3. $\square$

Corollary 4.11. *Let $\{X_n\}$ be an independent $\mathcal{S}_+(\mathcal{C})_\varepsilon$-tight sequence in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$such that $\mathbf{E}[|X_n|^p] \le M < \infty$ for all n where $p > 1$. Then, the SLLN holds for $\{X_n\}$*

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[X_i]\right) = 0 \quad a.s.$$

Proof. Since $\{X_n\}$ is independent and $\mathcal{S}_+(\mathcal{C})_\varepsilon$-tight, the sequence $\{\delta^*(., X_n)\}$ is independent and tight in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$ in view of Lemma 4.7 with

$$\mathbf{E}[\|\delta^*(., X_n)\|^p] = \mathbf{E}[\sup_{x^* \in \overline{B}^c_{E^*}} |\delta^*(x^*, X_n)|^p] = \mathbf{E}[|X_n|^p] \le M < \infty$$

for all n when $p > 1$. Applying theorem of Taylor and Wei ([25], Theorem 2) or Corollary 4.5 of Wang and Rao [27] to the independent tight sequence $\{\delta^*(., X_n)\}$ in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$ yields

$$\lim_{n\to\infty} \left\|\frac{1}{n}\sum_{i=1}^{n} \delta^*(., X_i) - \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[\delta^*(., X_i)]\right\| = 0 \quad a.s.$$

By (6) we have

$$d_H\left(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n \mathbf{E}[X_i]\right) = \left\|\frac{1}{n}\sum_{i=1}^n \delta^*(.,X_i) - \frac{1}{n}\sum_{i=1}^n \mathbf{E}[\delta^*(.,X_i)]\right\|$$

so that $\{X_n\}$ satisfies the SLLN w.r.t. the Hausdorff distance. □

We finish this section with

Corollary 4.12. *Let $\{X_n\}$ be an independent sequence in $\mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ such that*

(i) *For every $\varepsilon > 0$, there is a compact set $\mathcal{K} \in (\mathrm{ck}(E), d_H)$ such that*

$$\frac{1}{n}\sum_{k=1}^n \mathbf{E}[|X_k| 1_{[X_k \notin \mathcal{K}]}] < \varepsilon \quad \forall n \in \mathbf{N},$$

(ii) $\{|X_n|\}$ *satisfies to the SLLN*

$$\lim_n \frac{1}{n}\sum_{k=1}^n (|X_k| - \mathbf{E}[|X_k|]) = 0 \quad a.s.$$

Then, the SLLN holds for $\{X_n\}$

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n \mathbf{E}[X_i]\right) = 0 \quad a.s.$$

Proof. We will use some arguments developed in Taylor and Inoue ([24], Theorem 2.1) with appropriate modifications. Let $\varepsilon > 0$ and let $\mathcal{K}$ be given according to the condition (i). Since $\mathcal{K}$ is compact in $(\mathrm{ck}(E), d_H)$ we have

$$\mathcal{K} \subset \cup_{i=1}^m N(K_i, \varepsilon)$$

when

$$N(K_i, \varepsilon) = \{Y \in \mathrm{ck}(E) : d_H(K_i, Y) < \varepsilon\}$$

denotes the open ball in $(\mathrm{ck}(E), d_H)$ of center $K_i \in \mathrm{ck}(E)$ and radius $\varepsilon > 0$, $i = 1, 2, ..., m$. Define the convex compact integrable random sets which take the values in $\{K_1, ..., K_m\}$ in $\mathrm{ck}(E)$ by

$$Y_n = K_1 1_{[X_n \in N(K_1,\varepsilon)]} + \sum_{i=2}^m K_i 1_{[X_n \in N(K_i,\varepsilon)] \cap [\cup_{j=1}^{i-1}[X_n \in N(K_j,\varepsilon)]]^c}$$

and $Z_n = Y_n 1_{[X_n \in \mathcal{K}]}$. Then for each n

$$\begin{aligned}
& d_H\left(\frac{1}{n}\sum_{k=1}^n X_k, \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k]\right) \\
\leq\ & d_H\left(\frac{1}{n}\sum_{k=1}^n X_k, \frac{1}{n}\sum_{k=1}^n X_k 1_{[X_k\in\mathcal{K}]}\right) && (I) \\
& \quad + d_H\left(\frac{1}{n}\sum_{k=1}^n X_k 1_{[X_k\in\mathcal{K}]}, \frac{1}{n}\sum_{k=1}^n Z_k\right) && (II) \\
& \quad + d_H\left(\frac{1}{n}\sum_{k=1}^n Z_k, \frac{1}{n}\sum_{k=1}^n \mathbf{E}[Z_k]\right) && (III) \\
& \quad + d_H\left(\frac{1}{n}\sum_{k=1}^n \mathbf{E}[Z_k], \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k 1_{[X_k\in\mathcal{K}]}]\right) && (IV) \\
& \quad + d_H\left(\frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k 1_{[X_k\in\mathcal{K}]}], \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k]\right) && (V).
\end{aligned}$$

For (I), we have

$$\begin{aligned}
& \limsup_n d_H\left(\frac{1}{n}\sum_{k=1}^n X_k, \frac{1}{n}\sum_{k=1}^n X_k 1_{[X_k\in\mathcal{K}]}\right) \\
\leq\ & \limsup_n \frac{1}{n}\sum_{k=1}^n |X_k| 1_{[X_k\notin\mathcal{K}]} \\
=\ & \limsup_n \frac{1}{n}\sum_{k=1}^n (|X_k| - \mathbf{E}[|X_k|]) 1_{[X_k\notin\mathcal{K}]} + \limsup_n \frac{1}{n}\sum_{k=1}^n \mathbf{E}[|X_k| 1_{[X_k\notin\mathcal{K}]}] \\
\leq\ & \limsup_n \frac{1}{n}\sum_{k=1}^n \mathbf{E}[|X_k| 1_{[X_k\notin\mathcal{K}]}] < \varepsilon \quad a.s.
\end{aligned}$$

using (i) and (ii). For (II), by the construction of Z_n we have

$$d_H\left(\frac{1}{n}\sum_{k=1}^n X_k 1_{[X_k\in\mathcal{K}]}, \frac{1}{n}\sum_{k=1}^n Z_k\right) < \varepsilon.$$

For (III) apply the SLLN in Corollary 4.11 to Z_n by noting that Z_n takes finitely many values in $\{\{0\}, K_1, .., K_m\}$ in $(\mathrm{ck}(E), d_H)$ gives

$$\lim_n d_H\left(\frac{1}{n}\sum_{k=1}^n Z_k, \frac{1}{n}\sum_{k=1}^n \mathbf{E}[Z_k]\right) = 0 \quad a.s.$$

For (IV) as $d_H(\mathbf{E}X, \mathbf{E}Y) \le \mathbf{E}[d_H(X,Y)]$ for $X, Y \in \mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$ so that from (II)

$$\begin{aligned} & d_H\left(\frac{1}{n}\sum_{k=1}^n \mathbf{E}[Z_k], \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k 1_{[X_k\in\mathcal{K}]}]\right) \\ \le\ & \mathbf{E}\left[d_H\left(\frac{1}{n}\sum_{k=1}^n Z_k, \frac{1}{n}\sum_{k=1}^n X_k 1_{[X_k\in\mathcal{K}]}\right)\right] < \varepsilon. \end{aligned}$$

For (V) we note that for any $X \in \mathcal{L}^1_{\mathrm{ck}(E)}(\Omega, \mathcal{F}, P)$, $\mathbf{E}[X]$ is convex compact by Theorem 3.1 and

$$\delta^*(x^*, \mathbf{E}[X]) = \mathbf{E}[\delta^*(x^*, X)]$$

so that by Hormander formula we have

$$\begin{aligned} & d_H\left(\frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k 1_{[X_k\in\mathcal{K}]}], \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k]\right) \\ =\ & \sup_{x^*\in\overline{B}^c_{E^*}} \left|\delta^*\left(x^*, \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k 1_{[X_k\in\mathcal{K}]}]\right) - \delta^*\left(x^*, \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k]\right)\right| \\ =\ & \sup_{x^*\in\overline{B}^c_{E^*}} \left|\mathbf{E}\left[\delta^*\left(x^*, \frac{1}{n}\sum_{k=1}^n X_k 1_{[X_k\in\mathcal{K}]}\right)\right] - \mathbf{E}\left[\delta^*\left(x^*, \frac{1}{n}\sum_{k=1}^n X_k\right)\right]\right| \\ =\ & \sup_{x^*\in\overline{B}^c_{E^*}} \left|\mathbf{E}\left[\delta^*\left(x^*, \frac{1}{n}\sum_{k=1}^n X_k 1_{[X_k\in\mathcal{K}]}\right) - \delta^*\left(x^*, \frac{1}{n}\sum_{k=1}^n X_k\right)\right]\right| \\ \le\ & \sup_{x^*\in\overline{B}^c_{E^*}} \left|\mathbf{E}\left[\left|\delta^*\left(x^*, \frac{1}{n}\sum_{k=1}^n X_k 1_{[X_k\notin\mathcal{K}]}\right)\right|\right]\right| \le \frac{1}{n}\sum_{k=1}^n \mathbf{E}[|X_k| 1_{[X_k\notin\mathcal{K}]}] < \varepsilon \end{aligned}$$

using (i). Thus

$$\limsup_n d_H\left(\frac{1}{n}\sum_{k=1}^n X_k, \frac{1}{n}\sum_{k=1}^n \mathbf{E}[X_k]\right) < 4\varepsilon \quad a.s.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n \mathbf{E}[X_i]\right) = 0 \quad a.s. \qquad \square$$

5. Convergence of weighted sums to fuzzy random sets

Puri and Ralescu [20] defined fuzzy random variables for $\mathbf{R}^p$. Motivated by the definition and the application, we define a fuzzy random variable in separable Banach space by attaching the measurability and compactness to the level sets. According to [28] a *fuzzy convex upper semicontinuous variable* is a mapping $X : E \to [0, 1]$ such that

(i) X is upper semicontinuous,
(ii) $\{x \in E : X(x) = 1\} \neq \emptyset$,
(iii) X is fuzzy convex, that is $X(\lambda x + (1 - \lambda)y) \geq \min(X(x), X(y))$, for all $\lambda \in [0, 1]$ and for all $x, y \in E$.

A random fuzzy convex upper semicontinuous variable is an $\mathcal{F} \otimes \mathcal{B}(E)$-measurable mapping $X : \Omega \times E \to [0, 1]$ such that for each $\omega \in \Omega$, the mapping $X_\omega : E \to [0, 1]$ is a fuzzy convex upper semicontinuous variables. By upper semicontinuity and fuzzy convexity, for each $\omega \in \Omega$ and for each $\alpha \in]0, 1]$, the level set

$$X_\alpha(\omega) := L_\alpha(X)(\omega) := \{x \in E : X_\omega(x) \geq \alpha\}$$

is closed convex. It is clear that the graph of this multifunction belongs to $\mathcal{F} \otimes \mathcal{B}(E)$, hence it is measurable by the projection theorem ([8], Theorem III. 23). Similarly the graph of the multifunction

$$\omega \mapsto \{x \in E : X(\omega, x) > \alpha\}$$

belongs to $\mathcal{F} \otimes \mathcal{B}(E)$, hence it is measurable, and so is the mapping

$$X_{\alpha+}(\omega) := L_{\alpha+}(X)(\omega) := \overline{\{x \in E : X(\omega, x) > \alpha\}}$$

for each $\alpha \in [0, 1[$. Further, thank to Lemma III.39 in [8], for any $\mathcal{F} \otimes \mathcal{B}(E)$-measurable mapping $\varphi : \Omega \times E \to [0, +\infty]$, the function $m(\omega) := \sup\{\varphi(\omega, x) : x \in L_\alpha(X)(\omega)\}$ is measurable and so is the function $m_+(\omega) := \sup\{\varphi(\omega, x) : x \in L_{\alpha+}(X)(\omega)\}$. In particular, $L_{0+}(X)$ is measurable where $L_{0+}(X)(\omega) = \overline{\{x \in E : X(\omega, x) > 0\}}$, and $|L_{0+}(X)|(\omega) := \sup\{\|x\| : x \in L_{0+}(X)(\omega)\}$ is measurable, further assume that $L_{0+}(X)(\omega)$ is compact and $|L_{0+}(X)| \in L^1$, then $L_\alpha(X)$ is convex compact valued and integrably bounded: $|L_\alpha(X)| \leq |L_{0+}(X)|$ for all $\alpha \in]0, 1]$, here the inclusion $|L_{0+}(X)| \in L^1$ makes sense because of the above measurable properties. Given a random fuzzy convex upper semicontinuous variable X, the *fuzzy expectation* of X is a fuzzy convex upper semicontinuous variable $\tilde{\mathbf{E}}X$ such that

$$L_\alpha(\tilde{\mathbf{E}}X) = \mathbf{E}[L_\alpha(X)], \quad \forall \alpha \in]0, 1].$$

Now we present some SLLN results for the weighted sums of level sets associated with random fuzzy convex upper semicontinuous variables $\{X^n\}$. We refer to Klement et al. [17], Inoue [15], Joo et al. [16], Li and Ogura [18] for related results.

Theorem 5.1. *Let $\{X^n\}$ be a sequence of random fuzzy convex upper semi-continuous variables defined on $\Omega \times E$ satisfying*

(i) $X^n_{0+}(\omega)$ *is compact, for every* $\omega \in \Omega$ *and for every* $n \in \mathbf{N}$,
(ii) $g := \sup_n |X^n_{0+}|$ *is integrable.*

For each $\alpha \in]0,1]$, *we assume*

(iii) $\{\delta^*(.,X^n_\alpha)\}$ *is independent and compactly uniformly integrable in* $L^1_{C(\overline{B}^c_{E^*})}(\Omega,\mathcal{F},P)$,
(iv) *There exists a nonnegative random variable* X_α *such that* $P(|X^n_\alpha| \geq x) \leq P(X_\alpha \geq x)$ *for all* $n \in \mathbf{N}$ *and all* $x > 0$ *and* $\mathbf{E}[X_\alpha^{1+\frac{1}{\gamma}}] < \infty$ *for some* $\gamma > 0$.

If $\{a_{nk}\}$ *is a Toeplitz sequence and* $\max_{1\leq k\leq n} a_{nk} = O(n^{-\gamma})$, *then*

$$\int_0^1 d_H\left(\sum_{k=1}^n a_{nk}X^k_\alpha, \sum_{k=1}^n a_{nk}\mathbf{E}[X^k_\alpha]\right) d\alpha \to 0 \quad a.s.$$

Proof. Since $\{a_{nk}\}$ is a Toeplitz sequence, there exists a positive number C such that $\sup_n \sum_{k=1}^\infty a_{nk} \leq C$. Let $\alpha \in]0,1]$ be given. As the sequence $\{X^n_\alpha\}$ satisfies the conditions of Theorem 4.2, we have

$$d_H\left(\sum_{k=1}^n a_{nk}X^k_\alpha, \sum_{k=1}^n a_{nk}\mathbf{E}[X^k_\alpha]\right) \to 0 \quad a.s.$$

Also

$$\begin{aligned} d_H\left(\sum_{k=1}^n a_{nk}X^k_\alpha, \sum_{k=1}^n a_{nk}\mathbf{E}[X^k_\alpha]\right) &\leq \left|\sum_{k=1}^n a_{nk}X^k_\alpha\right| + \left|\sum_{k=1}^n a_{nk}\mathbf{E}[X^k_\alpha]\right| \\ &\leq \sum_{k=1}^n a_{nk}|X^k_\alpha| + \sum_{k=1}^n a_{nk}|\mathbf{E}[X^k_\alpha]| \\ &\leq \sum_{k=1}^n a_{nk}\sup_{k\in\mathbf{N}}|X^k_{0+}| + \sum_{k=1}^n a_{nk}\sup_{k\in\mathbf{N}}\mathbf{E}|X^k_{0+}| \\ &\leq C\sup_{k\in\mathbf{N}}|X^k_{0+}| + C\sup_{k\in\mathbf{N}}\mathbf{E}|X^k_{0+}| \end{aligned}$$

and $C\sup_{k\in\mathbf{N}}|X^k_{0+}| + C\sup_{k\in\mathbf{N}}\mathbf{E}|X^k_{0+}|$ is integrable. Thus, by applying Lebesgue Dominated Convergence Theorem we obtain

$$\int_0^1 d_H\left(\sum_{k=1}^n a_{nk}X_\alpha^k, \sum_{k=1}^n a_{nk}\mathbf{E}[X_\alpha^k]\right) d\alpha \to 0 \quad a.s. \qquad \square$$

Next, we prove some convergence results for fuzzy random sets with respect to a generalized Hausdorff distance.

Theorem 5.2. *Let $\{X^n\}$ be a sequence of random fuzzy convex upper semi-continuous variables defined on $\Omega \times E$ satisfying*

(i) *$X_{0+}^n(\omega)$ is compact, for every $\omega \in \Omega$ and for every $n \in \mathbf{N}$,*

(ii) *$g := \sup_n |X_{0+}^n|$ is integrable.*

For each $\alpha \in]0,1]$, we assume

(iii) *$\{X_\alpha^n\}$ and $\{X_{\alpha+}^n\}$ are independent,*

(iv) *$\{\delta^*(., X_\alpha^n)\}$ is compactly uniformly integrable in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$,*

(v) *There exists a nonnegative random variable X_α such that $P(|X_\alpha^n| \geq x) \leq P(X_\alpha \geq x)$ for all $n \in \mathbf{N}$ and all $x > 0$ and $\mathbf{E}[X_\alpha^{1+\frac{1}{\gamma}}] < \infty$ for some $\gamma > 0$.*

(vi) *For every $\varepsilon > 0$ there exists a partition $0 = \alpha_0 < \alpha_1 < ... < \alpha_r = 1$ such that for all $n \geq 1$*

$$\max_{1\leq k\leq r} \mathbf{E}[d_H(X_{\alpha_k}^n, X_{\alpha_{k-1}^+}^n)] < \varepsilon.$$

If $\{a_{nk}\}$ is a Toeplitz sequence and $\max_{1\leq k\leq n} a_{nk} = O(n^{-\gamma})$, then

$$\lim_{n\to\infty} \sup_{\alpha\in]0,1]} d_H\left(\sum_{k=1}^n a_{nk}X_\alpha^k, \sum_{k=1}^n a_{nk}\mathbf{E}[X_\alpha^k]\right) = 0 \quad a.s.$$

Proof. By our assumption, there exists a positive number C such that

$$\sup_n \sum_{i=1}^\infty a_{ni} \leq C.$$

Let $\alpha \in]0,1]$ be given. Then there exists k (depending on α) such that

$$\alpha_{k-1} < \alpha \leq \alpha_k.$$

We have the estimation

$$
\begin{aligned}
& d_H\left(\sum_{i=1}^n a_{ni}X^i_\alpha, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_\alpha]\right) \\
\leq\ & d_H\left(\sum_{i=1}^n a_{ni}X^i_\alpha, \sum_{i=1}^n a_{ni}X^i_{\alpha_k}\right) + d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha_k}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha_k}]\right) \\
& \quad + d_H\left(\sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha_k}], \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_\alpha]\right) \\
\leq\ & \sum_{i=1}^n a_{ni} d_H(X^i_\alpha, X^i_{\alpha_k}) + d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha_k}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha_k}]\right) \\
& \quad + \sum_{i=1}^n a_{ni}\mathbf{E}[d_H(X^i_{\alpha_k}, X^i_\alpha)] \\
\leq\ & \sum_{i=1}^n a_{ni} d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k}) + d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha_k}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha_k}]\right) \\
& \quad + \sum_{i=1}^n a_{ni}\mathbf{E}[d_H(X^i_{\alpha_k}, X^i_{\alpha^+_{k-1}})] \\
\leq\ & \max_{1\leq k\leq r}\sum_{i=1}^n a_{ni}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k}) - \mathbf{E}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k})]] \\
& \quad + \max_{1\leq k\leq r} d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha_k}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha_k}]\right) \\
& \quad + 2\sum_{i=1}^n a_{ni}\max_{1\leq k\leq r}\mathbf{E}[d_H(X^i_{\alpha_k}, X^i_{\alpha^+_{k-1}})] := I_1 + I_2 + I_3.
\end{aligned}
$$

For I_1, for every $k = 1, ..., r$, we note that for all i

$$P(d_H(X^i_{\alpha_k}, X^i_{\alpha^+_{k-1}}) \geq x) \leq P(2X_\alpha \geq x)$$

so that by Rohatgi theorem ([22], Theorem 2) we deduce that

$$\lim_{n\to\infty}\sum_{i=1}^n a_{ni}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k}) - \mathbf{E}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k})]] = 0 \quad a.s.$$

Thus

$$\lim_{n\to\infty} I_1 = 0 \quad a.s.$$

For I_2, for every k, the sequence $\{X^i_{\alpha_k} : i \in \mathbf{N}\}$ satisfies the conditions of Theorem 4.2 because $\{\delta^*(., X^i_{\alpha_k})\}$ is compactly uniformly integrable in

$L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$. Then we have

$$\lim_{n\to\infty} d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha_k}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha_k}]\right) = 0 \quad a.s.$$

Thus

$$\lim_{n\to\infty} I_2 = 0 \quad a.s.$$

We also have $I_3 < 2C\varepsilon$ for n large. Now by combining the arguments above

$$d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha}]\right) < I_1 + I_2 + 2C\varepsilon, \quad \forall\alpha \in]0,1].$$

Since I_1 , I_2 and $C\varepsilon > 0$ do not depend on α then from the above inequality it follows that

$$\sup_{\alpha\in]0,1]} d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha}]\right) \leq I_1 + I_2 + 2C\varepsilon.$$

Passing to the limit when n goes to ∞ in the preceding inequality yields

$$\lim_{n\to\infty} \sup_{\alpha\in]0,1]} d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha}]\right) \leq 2C\varepsilon \quad a.s.$$

Whence

$$\lim_{n\to\infty} \sup_{\alpha\in]0,1]} d_H\left(\sum_{i=1}^n a_{ni}X^i_{\alpha}, \sum_{i=1}^n a_{ni}\mathbf{E}[X^i_{\alpha}]\right) = 0 \quad a.s.$$

since $\varepsilon > 0$ is arbitrary. □

Theorem 5.3. *Let $\{X^n\}$ be a sequence of random fuzzy convex upper semi-continuous variables defined on $\Omega \times E$ satisfying*

(*i*) *$X^n_{0+}(\omega)$ is compact, for every $\omega \in \Omega$ and for every $n \in \mathbf{N}$,*
(*ii*) *$g := \sup_n |X^n_{0+}| \in L^2(\Omega, \mathcal{F}, P)$.*

For each $\alpha \in]0,1]$, we assume

(*iii*) *$\{X^n_\alpha\}$ and $\{X^n_{\alpha+}\}$ are independent,*
(*iv*) *$\{\delta^*(., X^n_\alpha)\}$ is compactly uniformly integrable in the Cesàro sense in $L^1_{\mathcal{C}(\overline{B}^c_{E^*})}(\Omega, \mathcal{F}, P)$,*
(*v*) *For every $\varepsilon > 0$ there exists a partition $0 = \alpha_0 < \alpha_1 < ... < \alpha_r = 1$ such that for all $n \geq 1$,*

$$\max_{1\leq k\leq r} \mathbf{E}[d_H(X^n_{\alpha^+_{k-1}}, X^n_{\alpha_k})] < \varepsilon$$

then

$$\lim_{n\to\infty}\sup_{\alpha\in]0,1]} d_H\left(\frac{1}{n}\sum_{k=1}^{n}X_\alpha^k, \frac{1}{n}\sum_{k=1}^{n}\mathbf{E}[X_\alpha^k]\right)=0 \quad a.s.$$

Proof. Let $\alpha\in]0,1]$ be given. Then there exists k (depending on α) such that $\alpha_{k-1}<\alpha\le\alpha_k$. As in the proof of Theorem 5.2, we have the estimation

$$\begin{aligned}
&\frac{1}{n}d_H\left(\sum_{i=1}^{n}X_\alpha^i, \sum_{i=1}^{n}\mathbf{E}[X_\alpha^i]\right)\\
&\le \frac{1}{n}\max_{1\le k\le r}\sum_{i=1}^{n}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k})-\mathbf{E}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k})]]\\
&\qquad+\frac{1}{n}\max_{1\le k\le r} d_H\left(\sum_{i=1}^{n}X^i_{\alpha_k}, \sum_{i=1}^{n}\mathbf{E}[X^i_{\alpha_k}]\right)\\
&\qquad+\frac{1}{n}2\sum_{i=1}^{n}\max_{1\le k\le r}\mathbf{E}[d_H(X^i_{\alpha_k}, X^i_{\alpha^+_{k-1}})].
\end{aligned}$$

For

$$I_1:=\frac{1}{n}\max_{1\le k\le r}\sum_{i=1}^{n}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k})-\mathbf{E}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k})]]$$

we note that for every $k=1,...,r$ and for every n

$$d_H(X^n_{\alpha^+_{k-1}}, X^n_{\alpha_k})\le 2|X^n_{0+}|$$

so that

$$\mathbf{E}[d_H(X^n_{\alpha^+_{k-1}}, X^n_{\alpha_k})^2]\le 2^2\mathbf{E}[|X^n_{0+}|^2]\le 2^2\mathbf{E}[g^2].$$

Whence by our assumption

$$\sum_{n=1}^{\infty}\frac{\mathbf{E}[d_H(X^n_{\alpha^+_{k-1}}, X^n_{\alpha_k})^2]}{n^2}\le 2^2\sum_{n=1}^{\infty}\frac{\mathbf{E}[|X^n_{0+}|^2]}{n^2}\le 2^2\sum_{n=1}^{\infty}\frac{\mathbf{E}[g^2]}{n^2}<\infty.$$

Applying the Kolmogorov's law of large numbers to the sequence $\{d_H(X^n_{\alpha^+_{k-1}}, X^n_{\alpha_k})\}_{n\ge1}$ yields

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k})-\mathbf{E}[d_H(X^i_{\alpha^+_{k-1}}, X^i_{\alpha_k})]]=0 \quad a.s.$$

so that $I_1\to0$ a.s. For

$$I_2:=\frac{1}{n}\max_{1\le k\le r} d_H\left(\sum_{i=1}^{n}X^i_{\alpha_k}, \sum_{i=1}^{n}\mathbf{E}[X^i_{\alpha_k}]\right)$$

we note that $\{\delta^*(., X^n_\alpha)\}$ is compactly uniformly integrable in the Cesàro sense in $\mathcal{C}(\overline{B}^c_{E^*})$ by (ii) and (iv) and verifies

$$\sum_{n=1}^{\infty} \frac{\mathbf{E}[|X^n_\alpha|^2]}{n^2} \leq \sum_{n=1}^{\infty} \frac{\mathbf{E}[|X^n_{0+}|^2]}{n^2} \leq \sum_{n=1}^{\infty} \frac{\mathbf{E}[g^2]}{n^2} < \infty$$

using (ii). It follows from Theorem 4.3 that

$$\lim_{n\to\infty} \frac{1}{n} d_H \left(\sum_{i=1}^{n} X^i_{\alpha_k}, \sum_{i=1}^{n} \mathbf{E}[X^i_{\alpha_k}] \right) = 0 \quad a.s.$$

so that $I_2 \to 0$ a.s.

$$I_3 := \frac{1}{n} 2 \sum_{i=1}^{n} \max_{1\leq k\leq r} \mathbf{E}[d_H(X^i_{\alpha_k}, X^i_{\alpha^+_{k-1}})].$$

We also have $I_3 < 2\varepsilon$ for n large. Now by combining the arguments above

$$\frac{1}{n} d_H \left(\sum_{i=1}^{n} X^i_\alpha, \sum_{i=1}^{n} \mathbf{E}[X^i_\alpha] \right) < I_1 + I_2 + 2\varepsilon, \quad \forall \alpha \in]0,1].$$

Since I_1 , I_2 and $\varepsilon > 0$ do not depend on α, then from the above inequality it follows that

$$\sup_{\alpha\in]0,1]} \frac{1}{n} d_H \left(\sum_{i=1}^{n} X^i_\alpha, \sum_{i=1}^{n} \mathbf{E}[X^i_\alpha] \right) \leq I_1 + I_2 + 2\varepsilon \quad a.s.$$

Passing to the limit when n goes to ∞ in the preceding inequality yields

$$\lim_{n\to\infty} \sup_{\alpha\in]0,1]} \frac{1}{n} d_H \left(\sum_{i=1}^{n} X^i_\alpha, \sum_{i=1}^{n} \mathbf{E}[X^i_\alpha] \right) \leq 2\varepsilon \quad a.s.$$

Whence

$$\lim_{n\to\infty} \sup_{\alpha\in]0,1]} d_H \left(\frac{1}{n} \sum_{i=1}^{n} X^i_\alpha, \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[X^i_\alpha] \right) = 0 \quad a.s.$$

since $\varepsilon > 0$ is arbitrary. □

We finish this section with the following result that is also new.

Theorem 5.4. *Assume that E is a separable Banach space. Let $\{X^n\}$ be a sequence of random fuzzy convex upper semicontinuous variables defined on $\Omega \times E$ satisfying*

(i) $\quad X^n_{0+}(\omega)$ *is compact, for every $\omega \in \Omega$ and for every $n \in \mathbf{N}$,* *(ii)* $g := \sup_n |X^n_{0+}| \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$.

Assume that $\{X^n_\alpha\}$ is pairwise independent and identically distributed for every $\alpha \in]0,1]$, then we have

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{k=1}^n X^k_\alpha, \mathbf{E}[X^1_\alpha]\right) = 0 \quad a.s.$$

for every $\alpha \in]0,1]$. In particular, if E is reflexive and $\alpha \mapsto X^1_\alpha$ is scalarly left continuous, then we have

$$\lim_{n\to\infty} \int_0^1 d_H\left(\left[\frac{1}{n}\sum_{k=1}^n X^k\right]_\alpha, [\tilde{\mathbf{E}}X^1]_\alpha\right) d\alpha = 0 \quad a.s.$$

where $\tilde{\mathbf{E}}X^1$ denotes the fuzzy expectation of X^1.

Proof. Since our hypothesis $\{X^n_\alpha\}$ is a pairwise independent identically distributed convex compact valued random sequence for every $\alpha \in]0,1]$, applying Theorem 4.5 gives

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{k=1}^n X^k_\alpha, \mathbf{E}[X^1_\alpha]\right) = 0 \quad a.s. \tag{9}$$

We note that

$$d_H\left(\frac{1}{n}\sum_{k=1}^n X^k_\alpha, \mathbf{E}[X^1_\alpha]\right) \le g + \mathbf{E}[g]$$

for all $n \in \mathbf{N}$ and for all $\alpha \in]0,1]$. By the Lebesgue dominated convergence theorem, it is immediate that

$$\lim_{n\to\infty} \int_0^1 d_H\left(\frac{1}{n}\sum_{k=1}^n X^k_\alpha, \mathbf{E}[X^1_\alpha]\right) d\alpha = 0 \tag{10}$$

Now if E is reflexive (or more generally is a dual of a WCG Banach space) and $\alpha \mapsto X^1_\alpha$ is scalarly left continuous, then by virtue of a recent result of Castaing et al. ([3], Theorem 4.7) on the integration of fuzzy level sets, we may consider the fuzzy expectation $\tilde{\mathbf{E}}X^1$ of X^1, that is a fuzzy convex upper semicontinuous variable defined on E so that $[\tilde{\mathbf{E}}X^1]_\alpha = \mathbf{E}[X^1_\alpha]$ for every $\alpha \in]0,1]$ [3]. It follows from (10) that

$$\lim_{n\to\infty} \int_0^1 d_H\left(\left[\frac{1}{n}\sum_{k=1}^n X^k\right]_\alpha, [\tilde{\mathbf{E}}X^1]_\alpha\right) d\alpha = 0. \qquad \square$$

[3] In this context such a fuzzy expectation theorem is not available in the literature.

A variant of the preceding result is provided by Inoue ([15], Theorem 3.2) in which a tightness condition is imposed.

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References

[1] A. Amrani, C. Castaing, M. Valadier: Méthodes de troncature appliquées à des problèmes de convergence faible ou forte dans L^1, Arch. Ration. Mech. Anal. 117 (1992) 167–191.

[2] P. Bai, P. Y. Chen, S. H. Sung: On complete convergence and the strong law of large numbers for pairwise independent random variables, Acta Math. Hung. 142(2) (2014) 502–518.

[3] C. Castaing, C. Godet-Thobie, T. D. Hoang, P. Raynaud de Fitte: On the integration of fuzzy level sets, in: Advances in Mathematical Economics 19, S. Kusuoka, T. Maruyama (eds.), Springer, Tokyo (2015) 1–32.

[4] C. Castaing: Quelques résultats de compacité liés a l'intégration, C. R. Acad. Sci., Paris, Sér. A 270 (1970) 1732–1735; Bull. Soc. Math. Fr., Suppl., Mém. 31–32 (1972) 73–81.

[5] C. Castaing: Weak compactness and convergences in Bochner and Pettis integration, Vietnam J. Math. 24(3) (1996) 241–286.

[6] C. Castaing, Ch. Hess, M. Saadoune: Tightness conditions and integrability of the sequential weak upper limit of a sequence of multifunctions, in: Advances in Mathematical Economics 11, S. Kusuoka, A. Yamazaki (eds.), Springer, Tokyo (2008) 11–44.

[7] C. Castaing, P. Raynaud de Fitte, M. Valadier: Young Measures on Topological Spaces. With Applications in Control Theory and Probability Theory, Kluwer, Dordrecht (2004).

[8] C. Castaing, M. Valadier: Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics 580, Springer, Berlin (1977).

[9] C. Castaing, P. Raynaud de Fitte: Law of large numbers and Ergodic theorem for convex weak star compact valued Gelfand-integrable mappings, in: Advances in Mathematical Economics 17, S. Kusuoka, T. Maruyama (eds.), Springer, Tokyo (2013) 1–37.

[10] C. Castaing, P. Raynaud de Fitte: $\mathcal{U}$-Uniform scalar integrability and strong laws of large numbers for Pettis-integrable functions with values in a separable locally convex space, J. Theor. Probab. 13(1) (2000) 93–134.

[11] J. A. Cuesta, C. Matran: Strong convergence of weighted sums of random elements through the equivalence of sequences of distributions, J. Multivariate Anal. 25 (1988) 311–322.

[12] G. Debreu: Integration of correspondences, in: Proc. 5th Berkeley Symp. Math. Stat. Probab. (Berkeley, 1965–1966), Vol. II, Part. 1, L. M. Le Cam, J. Neyman (eds.), University of California, Berkeley (1967) 351–372.

[13] N. Etemadi: An elementary proof of the strong law of large numbers, Z. Wahrscheinlichkeitstheor. Verw. Geb. 55 (1981) 119–122.

[14] F. Hiai, H. Umegaki: Integrals, conditional expectations, and martingales of multivalued functions, J. Multivariate Anal. 7 (1977) 149–182.

[15] H. Inoue: A strong law of large numbers for fuzzy random sets, Fuzzy Sets Syst. 41 (1991) 285–291.

[16] S. Y. Joo, Y. K. Kim, J. S. Kwon: Strong convergence for weighted sums of fuzzy random sets, Inf. Sci. 176 (2006) 1086–1099.

[17] E. P. Klement, M. L. Puri, D. A. Ralescu: Limit theorems for fuzzy random variables, Proc. R. Soc. Lond., Ser. A 407 (1986) 171–182.

[18] S. Li, Y. Ogura: Strong laws of large numbers for independent fuzzy set-valued random variables, Fuzzy Sets Syst. 157 (2006) 2569–2578.

[19] S. Li, Y. Ogura, V. Kreinovich: Limit Theorems and Applications of Set-Valued and Fuzzy Set-Valued Random Variables, Kluwer, Dordrecht (2002).

[20] M. L. Puri, D. A. Ralescu: Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409–422.

[21] M. L. Puri, D. A. Ralescu: Strong law of large numbers for Banach space valued random sets, Ann. Probab. 11 (1983) 222–224.

[22] V. K. Rohatgi: Convergence of weighted sums of independent random variables, Proc. Camb. Philos. Soc. 69 (1971) 305–307.

[23] R. L. Taylor, H. Inoue: A strong law of large numbers for random sets in Banach spaces, Bull. Inst. Math., Acad. Sin. 13 (1985) 403–409.

[24] R. L. Taylor, H. Inoue: Convergence of weighted sums of random sets, Stochastic Anal. Appl. 3 (1985) 379–396.

[25] R. L. Taylor, D. Wei: Laws of large numbers for tight random elements in normed linear spaces, Ann. Probab. 7 (1979) 150–155.

[26] M. Valadier: Young measures, in: Methods of Nonconvex Analysis (Varenna, 1989), A. Cellina (ed.), Lecture Notes in Mathematics 1446, Springer, Berlin (1990) 152–188.

[27] X. C. Wang, M. B. Rao: Some results on the convergence of weighted sums of random elements in separable Banach spaces, Stud. Math. 86 (1987) 131–153.

[28] L. H. Zadeh: Fuzzy sets, Inform. Control 8 (1966) 338–353.