Journal of Convex Analysis
Volume 24 (2017), No. 1, 239–259

On Holditch's Theorem and Holditch Curves

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Received: March 28, 2015
Accepted: June 27, 2015

Holditch's Theorem and results due to A. Broman are revisited and extended under geometric hypotheses. In addition, smoothness and convexity properties of Holditch curves are derived, with emphasis on the necessity of smoothness hypotheses on the parent curve.

Keywords: Holditch's Theorem, smoothness and convexity of Holditch curves, convex analysis, Broman's condition, local obtuseness, Mamikon's theorem

2000 Mathematics Subject Classification: Primary 26A27, 49J52

1. Introduction

In 1858, Reverend Hamnet Holditch, president of Gonville and Caius College, Cambridge, published a very interesting result pertaining to the curve traced out by a point on a chord of fixed length whose endpoints travel around a given closed curve [5].

Theorem 1.1 (Holditch's Theorem (original statement)). *If a chord of a closed curve, of constant length $a + b$, be divided into two parts of lengths a and b respectively, the difference between the areas of the closed curve, and of the locus of the dividing point, will be πab.*

Reverend Holditch's result lacked required hypotheses and a cogent proof. Prior

*A. Stancu's research was supported by the Natural Sciences Engineering Research Council of Canada, Grant 327635-2012.

ISSN 0944-6532 / $ 2.50

to stating a corrected version of the theorem due to Broman [2], let us first specify the type of curve that surely was intended. Throughout this article, $\mathbb{C}$ will denote a convex curve in $\mathbb{R}^2$, meaning that $\mathbb{C}$ is a planar curve enclosing a convex region. A function $C : \mathbb{R} \to \mathbb{R}^2$ provides a parametrization. We assume that

(i) $C(\cdot)$ is continuous.
(ii) $C(t)$ moves counterclockwise as t increases.
(iii) $C(t) = C(t+1)$ for all $t \in \mathbb{R}$. Note that in particular, one complete traverse is made for $0 \leq t < 1$, and $C(0) = C(1)$.
(iv) If t_1 and t_2 are distinct values in $[0, 1)$, then $C(t_1) \neq C(t_2)$.

We will make reference to the following hypothesis involving a positive number L (referred to in what follows as the *chordlength*) and the existence of a certain function $B(\cdot)$. It makes precise Holditch's notion of a chord of length $L > 0$ which travels "properly" around the parent curve $\mathbb{C}$.

Good Travel Hypothesis $\mathbb{GT}(L)$: There exists a function $B : \mathbb{R} \to \mathbb{R}^2$ with the following properties:

(a) $B(\cdot)$ enjoys the same properties (i)–(iv) as $C(\cdot)$ does.
(b) $\|C(t) - B(t)\| = L \ \forall t \in \mathbb{R}$.
(c) The direction angle $\theta(\cdot)$ of the chord $\overline{CB}$ (from $C(\cdot)$ to $B(\cdot)$), measured counterclockwise from the horizontal is monotone nondecreasing.

Remark 1.2. Note that (a) implies that $\theta(\cdot)$ is continuous and $\theta(t+1) = \theta(t) + 2\pi$ for all $t \in \mathbb{R}$.

The following result is a corrected version of Theorem 1.1 due to A. Broman [2]. Notationally, $L = a + b$, and D denotes the dividing point of the traveling chord $\overline{CB}$ so that and $\|\overline{DB}\| = a$ and $\|\overline{CD}\| = b$.

Theorem 1.3 (Broman's formulation of Holditch's Theorem). *Suppose that $\mathbb{GT}(L)$ holds and that the locus of D is a Jordan curve. Then the area of the region between the curve $\mathbb{C}$ and locus of D is πab, where a and b are the division parameters.*

Remark 1.4.

1. We call the locus of any dividing point D of the traveling chord $\overline{CB}$ a *Holditch curve* (for chordlength L and division parameters a and b).
2. Note that it is the Jordan property which permits us to speak of the region *between* a Holditch curve and $\mathbb{C}$.
3. Theorem 1.3 is a special case of a more modern version proven by Broman via line integral techniques, relying upon the fact that the relevant functions have C^1 approximations. (This is a consequence of the rectifiability of $\mathbb{C}$ as well as the monotonicity and boundedness of $\theta(\cdot)$ on $[0, 1)$.)

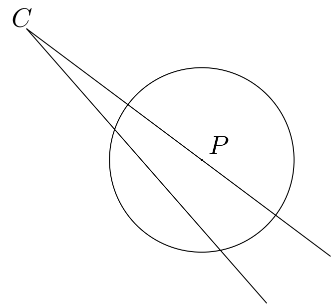

Figure 1.1: Acute polyhedral corner

In the next section, we will investigate interesting consequences of two geometric hypotheses vis-à-vis Theorem 1.3. The first of these was introduced by A. Broman in [3] for a given $L > 0$.:

Hypothesis $\mathbb{P}(L)$: Every circle with radius L and centered on the curve $\mathbb{C}$, intersects $\mathbb{C}$ at precisely two points.

Remark 1.5.

1. It is readily noted that Property $\mathbb{P}(L)$ fails for *every* $L > 0$ if $\mathbb{C}$ possesses an acute (polyhedral) corner C, as in in Figure 1.1. Referring to the picture, suppose that the radius of the circle centered at P is reduced so that there are only two intersection points of the circle with $\mathbb{C}$. Then if one slides this smaller circle along the line containing P towards C, the picture with four intersection points is re-created.
2. One can also directly show that "good chord travel" as in $\mathbb{GT}(L)$ is impossible for every chordlength $L > 0$ if $\mathbb{C}$ possesses an acute corner.

In the next section, we will prove the following:

Theorem 1.6. *Suppose that $\mathbb{P}(L^*)$ holds for some $L^* > 0$. Then $\mathbb{GT}(L^*)$ holds, and if $L > 0$ is sufficiently small, every Holditch curve corresponding to L has the Jordan property.*

The other geometric hypothesis is of a different type. For a given $C \in \mathbb{C}$, the *normal cone* to the compact convex region $\widehat{\mathbb{C}}$ enclosed by $\mathbb{C}$ (the convex hull of $\mathbb{C}$) is given by

$$N_{\widehat{\mathbb{C}}}(C) := \{\zeta \in \mathbb{R}^2 : \langle \zeta, W - C \rangle \leq 0 \;\forall\; W \in \widehat{\mathbb{C}}\} ,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. The normal cone is closed, convex and consists of normals to supporting hyperplanes (simply lines in $\mathbb{R}^2$) to $\widehat{\mathbb{C}}$ at C We go on to say that $N_{\widehat{\mathbb{C}}}(C)$ is *acute* provided that

$$\langle \zeta_1, \zeta_2 \rangle > 0 \;\; \forall\, \zeta_1, \zeta_2 \in N_{\widehat{\mathbb{C}}}(C) \backslash \{0\}. \tag{1}$$

In other words, the maximum angle subtended by the normal cone at C is less than $\frac{\pi}{2}$ radians. This can be viewed as a *local obtuseness* condition on the curve $\mathbb{C}$. We now introduce a geometric hypothesis based on this idea.

Hypothesis $\mathbb{O}_{\mathbb{C}}$: $N_{\widehat{\mathbb{C}}}(C)$ is acute for all $C \in C$.

In the next section we will also obtain the following result:

Theorem 1.7. *Hypothesis $\mathbb{O}_{\mathbb{C}}$ implies that $\mathbb{P}(L^*)$ holds for all sufficiently small $L^* > 0$.*

Of course, the upshot of Theorems 1.6 and 1.7 is the following corollary to Theorem 1.3:

Corollary 1.8. *Suppose the local obtuseness condition $\mathbb{O}_{\mathbb{C}}$ holds. Then if $L > 0$ is sufficiently small, $\mathbb{GT}(L)$ holds and every Holditch curve corresponding to L has the Jordan property. Furthermore, the area between any Holditch curve and $\mathbb{C}$ is πab, where a and b are the division parameters.*

Remark 1.9. The local obtuseness hypothesis is not quite minimal when it comes to obtaining the desired conclusions of Corollary 1.8, since Theorem 1.3 also applies to sets where the normal cone subtends an angle of $\frac{\pi}{2}$ radians. (The Gauss-Bonnet theorem implies that there cannot be more than four points in $\mathbb{C}$ where this occurs.) See for example Broman's analysis of the case where $\mathbb{C}$ is the unit square ([2]).

In Section 3 we will derive smoothness and convexity properties of Holditch curves when $\mathbb{C}$ is assumed to possess sufficient smoothness and L is sufficiently small; examples are provided which illustrate the need for smoothness hypotheses in such results.

Section 4 is of independent interest. There, we offer a proof of the following version of Holditch's Theorem, where local obtuseness is not assumed.

Theorem 1.10. *Suppose that $\mathbb{GT}(L)$ holds. Then upon reduction of L if required, every Holditch curve corresponding to L has the Jordan property, and the area between any Holditch curve and $\mathbb{C}$ is πab, where a and b are the division parameters.*

The proof is as faithful as possible to Holditch's original technique. His nineteenth century proof depended on the (unverified) smoothness of a certain convex curve which he called the *envelope*, taken to mean the boundary of a set which is the intersection of a family of halfspaces generated by linear extensions of the traveling chord. Our proof deals with possible non-smoothness of the envelope, which is an issue that Broman's proof of a generalized version of Holditch's Theorem was able to circumvent.

2. Geometric considerations

The following result is due to Broman [3].

Proposition 2.1.

(a) *Property $\mathbb{P}(L)$ implies that $\mathbb{GT}(L)$ holds.*
(b) *If Property $\mathbb{P}(L')$ holds, then $\mathbb{P}(L)$ holds for all $L \in (0, L')$.*

Our goal now is to provide a converse to Proposition 2.1. First we require the following lemma, whose proof is straightforward and left to the reader.

Lemma 2.2. *Suppose that Hypothesis $\mathbb{GT}(L)$ holds. Then the following hold:*

1. *If one writes $B(t) = C(\tau(t))$, then the function $\tau : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing.*
2. *Define a mapping $T_L : \mathbb{C} \to \mathbb{C}$ via*

$$T_L(C(t)) = B(t) = C(\tau(t)),$$

as t varies in $[0, 1)$. Then $T_L(\cdot)$ is bijective and continuous (and is in fact a homeomorphism on $\mathbb{C}$).

We are now in position to prove the desired converse result:

Proposition 2.3. *Suppose that Hypothesis $\mathbb{GT}(L^*)$ holds for some $L^* > 0$. Then Property $\mathbb{P}(L)$ holds for all sufficiently small $L > 0$.*

Proof. We will show that $L < \frac{L^*}{\sqrt{2}}$ implies that $\mathbb{P}(L)$ holds. To this end, let $C \in \mathbb{C}$, with $A = T_{L^*}(C)$ and $B = T_{L^*}^{-1}(C)$. Consider the line $\mathbb{L}$ from C into $\widehat{\mathbb{C}}$ which bisects the angle $\angle(ACB)$ (whose value in radians is clearly in the interval $(0, \pi]$), and let $\mathbb{L}$ intersect $\mathbb{C}$ again at D. Since the chord of length L^* must be parallel to $\mathbb{L}$ twice as it traverses $\mathbb{C}$ (or must coincide with segment $\overline{CD}$ at some instant), it follows that $\|\overline{CD}\| \geq L^*$, and therefore the distance d from C to the line segment $\overline{AD}$ satisfies $d \geq \frac{L^*}{\sqrt{2}}$. By symmetry the same is true of the distance from C to the line segment $\overline{BD}$.

Next, consider a circle $\mathbb{W}$ centered at C, with radius strictly less than $\frac{L^*}{\sqrt{2}}$. We claim that $\mathbb{W}$ intersects $\mathbb{C}$ at precisely two points, which will complete the proof.

First, note that the part of $\mathbb{W}$ in the quadrilateral $ADBC$ is in the quadrilateral's interior except at two points on the segments $\overline{CA}$ and $\overline{CB}$, respectively. Also, it is clear that the curve $\mathbb{C}$ has empty intersection with this interior.

Now, upon a shift and rotation of $\mathbb{C}$, without loss of generality we can view the point C as the origin, with the y-axis spanned by $\overline{CD}$. Then arc(ACB) of $\mathbb{C}$ is a portion of the graph of a convex function with chords $\overline{CA}$ and $\overline{CB}$, and this graph has only two points in common with the circle $\mathbb{W}$. Note also that

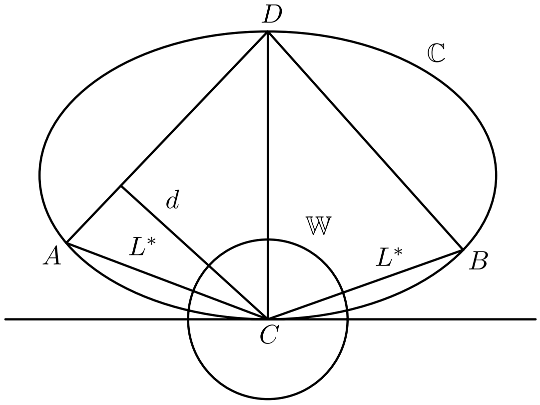

Figure 2.1: Proof of Proposition 2.3

no point of $\mathbb{W}$ can be strictly between arc(ACB) and the quadrilateral $ADBC$ (should such points exist) and have points in common with $\mathbb{C}$. This verifies the aforementioned claim. $\square$

The proof of the preceding proposition is illustrated in Figure 2.1.

The preceding proposition will be applied in proving a result on the Jordan property of Holditch curves. We first require the following lemma:

Lemma 2.4. *Suppose that Property $\mathbb{P}(L')$ holds. Let $P \in \mathbb{C}$ and let P_1 and P_2 be the unique pair of points where the circle of radius L', centered at P, intersects $\mathbb{C}$. Then the set*

$$S := \widehat{\mathbb{C}} \cap \{x \in \mathbb{R}^2 : \|x - P\| \leq L'\}$$

has only the arc $\mathbb{A}$ containing P with endpoints P_1 and P_2 in common with $\mathbb{C}$.

Proof. We only need to show that $\mathbb{C} \cap S^0 = \phi$, where S^0 denotes the interior of S. Suppose that there existed $Q \in \mathbb{C} \cap S^0$. Consider the circle of radius $\|P - Q\|$ centered at P. Since $\|P - Q\| < L'$, Proposition 2.1(b) implies that this circle intersects $\mathbb{C}$ at only two points. But it necessarily intersects $\mathbb{C}$ at Q as well as two points in $\mathbb{A}$, which is a contradiction. $\square$

Proposition 2.5. *Again assume that Hypothesis $\mathbb{GT}(L^*)$ holds for some $L^* > 0$. Then if $L > 0$ is sufficiently small, all Holditch curves corresponding to L have the Jordan property.*

Proof. Suppose that there existed a non-Jordan Holditch curve parametrized by $D(\cdot)$. We will show that this is impossible if the associated chordlength L is sufficiently small.

For division parameters a and b, we define $\lambda := \frac{b}{a+b}$. Then $D(t) = \lambda B(t) + (1 - \lambda)C(t)$ for $t \in [0, 1)$. Since the curve is presumed to not possess the Jordan

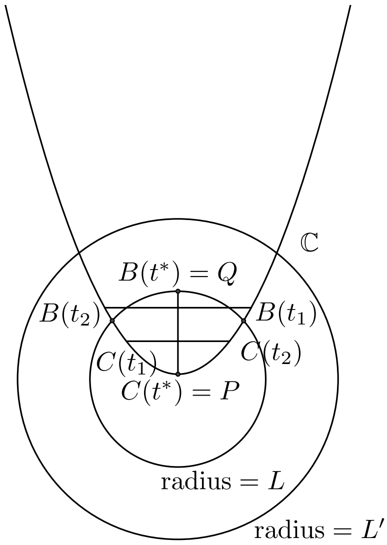

Figure 2.2: Proof of Proposition 2.5. (Impossible picture)

property, it self-intersects. Hence there exist t_1 and t_2 in $[0, 1)$ such that $D(t_1) = D(t_2)$. Consequently

$$\lambda[B(t_1) - B(t_2)] = (1 - \lambda)[C(t_2) - C(t_1)].$$

The four points $C(t_1), C(t_2), B(t_1)$ and $B(t_2)$ constitute the vertices of a quadrilateral Q (in fact, an "isosceles trapezoid") whose diagonals have length L, and each of these four vertices lies on the curve $\mathbb{C}$.

According to Proposition 2.3, Property $\mathbb{P}(L')$ holds if $L' > 0$ is sufficiently small. Let $0 < L < L'$.

Consider the motion of the chord $\overline{C(t)B(t)}$. It is clear that for some $t^* \in (t_1, t_2)$, the chord is perpendicular to the two parallel sides of the trapezoid. Denote $P = C(t^*)$, and note that $Q := B(t^*)$ is in S^0, in the notation of Lemma 2.4. Since $Q \in \mathbb{C}$, the lemma implies this is not possible, completing the proof. □

The idea of the preceeding proof is illustrated in Figure 2.2.

Next we will prove that local obtuseness suffices for Property $\mathbb{P}(L)$ to hold for sufficiently small chordlengths. Note that this jibes with Remark 1.5.

We denote $\widehat{N}_{\widehat{\mathbb{C}}}(C) := N_{\widehat{\mathbb{C}}}(C) \cap S$, where $S := \{\zeta \in \mathbb{R}^2 : \|\zeta\| = 1\}$ is the unit circle in the plane (where $\|\cdot\|$ denotes the euclidean norm).

Theorem 2.6. *Suppose that $\mathbb{O}_C$ holds. Then Property $\mathbb{P}(L)$ holds for all sufficiently small $L > 0$.*

Proof. It is not difficult to see that for every $C \in \mathbb{C}$ and every $\zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(C)$, the acuteness hypothesis on the normal cone implies that the open half-line

$$L_{C,\zeta} := \{C - \alpha\zeta : \alpha > 0\}$$

intersects $\mathbb{C}$ at precisely one point $P(C,\zeta) = C - \alpha(C,\zeta)\zeta$, and the line segment $\overline{C, P(C,\zeta)}$, which is of length $\alpha(C,\zeta) > 0$, is contained in the interior of $\widehat{\mathbb{C}}$. Also, as is readily noted, the function $p(C,\cdot) : \widehat{N}_{\widehat{\mathbb{C}}}(C) \to \mathbb{C}$ is continuous, and therefore the function $\alpha(C,\cdot) : \widehat{N}_{\widehat{\mathbb{C}}}(C) \to \mathbb{R}$ is continuous as well. Hence for any $C \in \mathbb{C}$ we can define

$$\begin{aligned} m(C) &:= \min\{\alpha(C,\zeta) : \zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(C)\} \\ &= \min\{\|p(C,\zeta) - C\| : \zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(C) \cap S\}. \end{aligned}$$

(The minimum is attained due to the continuity of $\alpha(C,\cdot)$ and $p(C,\cdot)$ along with the compactness of $\widehat{N}_{\widehat{\mathbb{C}}}(C)$.) Also, it is clear that $m(C) > 0$ for all $C \in \mathbb{C}$.

We now claim that the function $m : \mathbb{C} \to \mathbb{R}$ is lower semicontinuous. To this end, let C_i be any sequence in $\mathbb{C}$ converging to C and assume that $m(C_i)$ is convergent. We are to show that

$$\lim_{i\to\infty} m(C_i) \geq m(C). \tag{2}$$

For each i, choose $\zeta_i \in \widehat{N}_{\widehat{\mathbb{C}}}(C_i)$ such that $m(C_i) = \alpha(C_i,\zeta_i)$. We may without loss of generality assume that ζ_i converges (due to the compactness of $\overline{B}$). We have

$$m(C_i)\zeta_i = C_i - p(C_i,\zeta_i) \to [\lim_{i\to\infty} m(C_i)]\zeta = C - P \tag{3}$$

for some $P \in \mathbb{C}$.

Recall now that the the multifunction $\widehat{N}_{\widehat{\mathbb{C}}}(\cdot) : \mathbb{C} \to \overline{B}$ possesses the closed graph property. (See e.g. Clarke, Ledyaev, Stern and Wolenski [4].) Hence $\zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(C)$, and (3) implies that $\lim_{i\to\infty} m(C_i) = \alpha(C,P)$. But $\alpha(C,\zeta) \geq m(C)$. Hence (2) holds and $\alpha(\cdot)$ is lower semicontinuous, as claimed. It follows that the minimum of $m(\cdot)$ on $\mathbb{C}$ is attained, and is positive. We denote it by Λ.

Figure 2.3 illustrates the rest of the proof of the theorem.

Let $0 < r < \Lambda$. For a given $C \in C$, consider a circle of radius r centered at C, and let Q and W be points on $\mathbb{C}$ such that the arc (QCW) contains no other intersection points of the circle with $\mathbb{C}$. Choose any $\zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(C)$, and consider the sector $PQCWP$, where $P = P(C,\zeta)$. This sector is a convex subset of $\widehat{\mathbb{C}}$, whose boundary has the arc (QCW) in common with $\mathbb{C}$, along with possibly other points. Now consider any point $U \in \mathbb{C}$ strictly between Q and W on the arc (QCW), with U not on either of the lines spanned by the line segments $\overline{QP}$ or $\overline{WP}$. (In other words, we choose $U \in \mathbb{C}$ strictly between Q and W such that the line segment $\overline{UP}$ is, in a sense, strictly between $\overline{QP}$ and $\overline{WP}$.) One such point U is shown in the simple scenario depicted in Figure 2.3, and C itself is another.) Now note that the relative interior of the line segment $\overline{UP}$ is contained in the interior of $\widehat{\mathbb{C}}$. (If not, the line spanned by U and P would

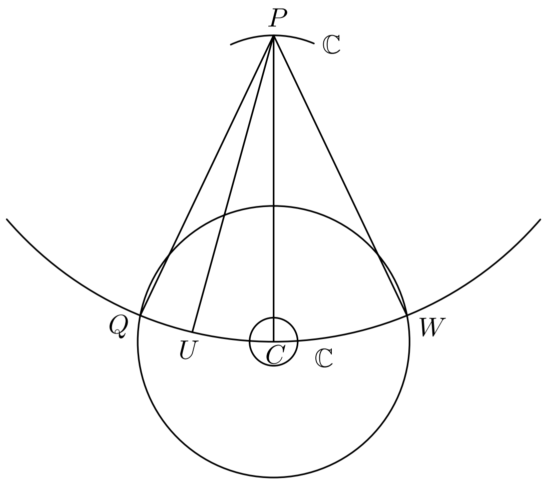

Figure 2.3: Part of proof of Theorem 2.6

support $\widehat{\mathbb{C}}$ with Q and W on opposite sides of the line, which is impossible.) It follows that the relative interior of $\overline{UP}$ contains no points of $\mathbb{C}$. Hence the interior of the sector $PQCWP$ contains no points of $\mathbb{C}$.

Now note that the distance from C to the line segment $\overline{QP}$ is bounded below by the distance from C to the line spanned by that segment. An elementary trigonometric argument shows that this distance is in turn bounded below by $\frac{r}{\sqrt{2}}$. An analogous bound holds for the segment $\overline{WP}$. It follows that if $0 < L < \frac{r}{\sqrt{2}}$, then for any $C \in \mathbb{C}$, the circle of radius L intersects $\mathbb{C}$ at precisely two points. $\square$

Remark 2.7. Actually, the proof shows that any circle with radius $L < \frac{\Lambda}{\sqrt{2}}$ and centered on $\mathbb{C}$ intersects $\mathbb{C}$ at precisely two points.

It is now clear that Theorems 1.6 and 1.7 as well as Corollary 1.8 in the introduction all follow immediately from the results derived thus far.

3. Smoothness and convexity of Holditch curves

Terminology:

- We say that a planar curve $\mathbb{G}$ is *smooth* on an open interval $\mathbb{J}$, if it possesses a C^1-parametrization $G : \mathbb{J} \to \mathbb{R}^2$ with non-vanishing derivative. In particular, smoothness precludes the existence of corners.
- Note also that when $\mathbb{C}$ is smooth, then for every $C \in \mathbb{C}$, $N_{\widehat{\mathbb{C}}}(C)$ is a half-line and is therefore acute. (Hence local obtuseness obtains when $\mathbb{C}$ is smooth.)
- If the parametrization of $\mathbb{G}$ is k times continuously differentiable with non-vanishing derivative, then we say that $\mathbb{G}$ is C^k-*smooth*.

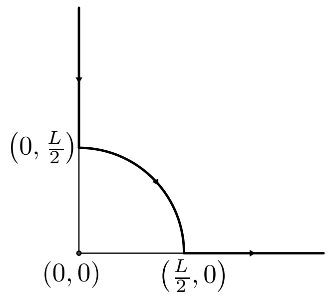

Figure 3.1: Square corner: Locus of midpoint

The following two examples show that in the absence of smoothness assumptions on $\mathbb{C}$, Holditch curves might be neither smooth nor convex for any (arbitrarily small) chordlength $L > 0$, even when, as is easily verified in these examples, the hypotheses of Theorem 1.3 hold.

Example 3.1. Suppose that the curve $\mathbb{C}$ is the boundary of a a square, positioned so that near the origin, $\mathbb{C}$ is given by the square corner

$$\{(x,0) : x \geq 0\} \cup \{(0,y) : y \geq 0\}.$$

For sufficiently small chordlengths $L > 0$, a straightforward computation shows that the Holditch curve corresponding to the midpoint of the chord is given by

$$\left\{(0,y) : y \geq \frac{L}{2}\right\} \cup \left\{(x,0) : x \geq \frac{L}{2}\right\} \\ \cup \left\{(x,y) : x^2 + y^2 = \frac{L^2}{4}\,,\, x \geq 0\,,\, y \geq 0\right\}.$$

Note that the third term in the union describes the part of a circle with radius $\frac{L}{2}$, centered at the origin, which is in the first quadrant and bridges the horizontal and vertical parts of the locus of the midpoint of the chord. From this the nonsmoothness and nonconvexity of the locus are evident; see Figure 3.1.

An example exhibiting a similar pathology to the previous one is the following. (Note that in this example, we have local obtuseseness, which was not the case in Example 3.1):

Example 3.2. Suppose now that the curve $\mathbb{C}$ is the boundary of a regular octagon, positioned so that near the origin, $\mathbb{C}$ is given by the graph of the function

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 0 & \text{if } x > 0. \end{cases}$$

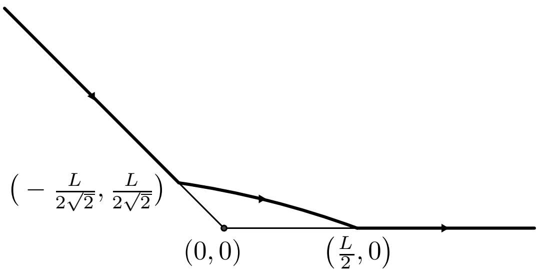

Figure 3.2: Obtuse corner: Locus of midpoint

Here, for small $L > 0$, the locus of the midpoint of the chord is given by

$$\left\{(x,-x) : x \leq -\frac{L}{2\sqrt{2}}\right\} \cup \left\{(x,0) : x \geq \frac{L}{2}\right\}$$
$$\cup \left\{(x,y) : x^2 + 5y^2 + 4xy = \frac{L^2}{4}\, , -\frac{L}{2\sqrt{2}} \leq x \leq \frac{L}{2}\, , y \geq 0\right\}.$$

The third term in the union gives a portion of an ellipse centered at the origin, bridging the horizontal and vertical parts of the locus; see Figure 3.2.

3.1. Smoothness of Holditch curves

The following results deal with certain geometric possibilities that can occur when a chord of constant length travels along a smooth convex curve, resulting in non-differentiability in the motion of one of its endpoints.

For convenience, we introduce the following hypothesis:

Hypothesis $\mathbb{S}(L)$: The convex curve $\mathbb{C}$ is smooth and $L > 0$ is such that condition $\mathbb{P}(L)$ holds.

Let $\tau(\cdot)$ be as in Lemma 2.2.

Proposition 3.3. *Suppose that Hypothesis $\mathbb{S}(L)$ holds, and let $\hat{t}$ be such that the chord $\overline{CB}$ is perpendicular to the curve $\mathbb{C}$ at $B(\hat{t}) = C(\tau(\hat{t}))$, but not at $C(\hat{t})$. Then $B(\cdot)$ is not differentiable at $\hat{t}$ and neither is $D(\cdot)$ for any dividing point of $\overline{CB}$. (That is, no Holditch curve is differentiable at $\hat{t}$.)*

Proof. The geometric hypotheses assert that

$$\langle B(\hat{t}) - C(\hat{t}), \dot{C}(\tau(\hat{t}))\rangle = 0. \tag{4}$$

and

$$\langle B(\hat{t}) - C(\hat{t}), \dot{C}(\hat{t})\rangle \neq 0. \tag{5}$$

By way of contradiction, suppose that $\tau(\cdot)$ is differentiable at $\hat{t}$. After differentiating the equation

$$\|C(t) - C(\tau(t))\|^2 = L^2$$

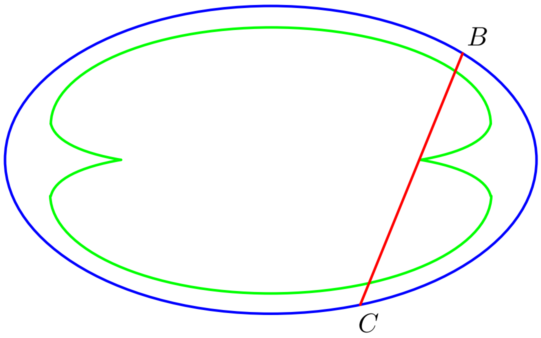

Figure 3.3: Locus of midpoint with corners

and gathering terms, one obtains

$$\dot{\tau}(t)\left[\langle B(t)-C(t),\dot{C}(\tau(t))\rangle\right]=\langle B(t)-C(t),\dot{C}(t)\rangle.$$

Upon taking $t=\hat{t}$ and applying (4)–(5), the left side is 0 and the right side is not. This contradiction shows that $\tau(\cdot)$ cannot be differentiable at $\hat{t}$. Now, since $B(\cdot)=C(\tau(\cdot))$ and $C(\cdot)$ is smooth, the non-differentiability of $B(\cdot)$ at $\hat{t}$ follows.

The non-differentiability of the locus of any dividing point is now immediate, since $C(\cdot)$ is differentiable at $\hat{t}$ and $B(\cdot)$ is not, which implies that we also have non-differentiability at $\hat{t}$ of any convex combination of $C(\hat{t})$ and $B(\hat{t})$. $\square$

Remark 3.4. Let us consider some possibilities for Holditch curves when $\mathbb{C}$ is an ellipse.

- The development of corners in the loci of dividing points of the traveling chord is illustrated on the Wolfram Demonstrations Project website

  http://demonstrations.wolfram.com/HolditchCurvesInsideAnEllipse/.

  In particular, Figure 3.3 shows an ellipse with eccentricity .894, where $L>0$ has been chosen so that $\mathbb{P}(L)$ holds and two corners in the locus of the midpoint occur as the chord travels. (The situation for only the right corner is shown.) Similarly, Figure 3.4 illustrates what transpires for the same ellipse and same L, but when the dividing point is such that $\frac{a}{a+b}=.7$. In both pictures, the position of the chord when corners occur illustrate Proposition 3.3; that is, a corner develops at a time $\hat{t}$ at which the chord is perpendicular to $\mathbb{C}$ at $B(\hat{t})$ but not at $C(\hat{t})$.
- If one has perpendicularity of the chord to Γ at *both* of its endpoints, then no *a priori* conclusion can be drawn. Figure 3.5 from the Wolfram Demonstratons Project website shows an ellipse with eccentricity .77, where we have perpendicularity at both endpoints, and the locus of the dividing point with $\frac{a}{a+b}=.7$ is nonsmooth. In this example, L is the

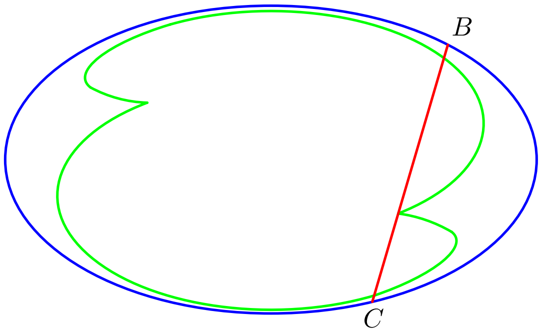

Figure 3.4: Another locus with corners

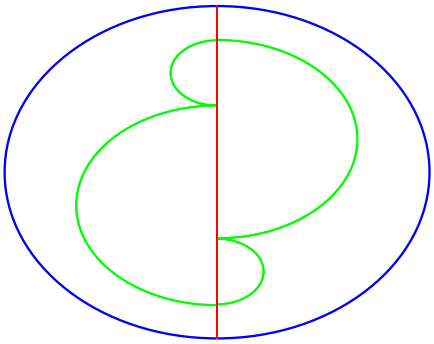

Figure 3.5: Indeterminate geometry leading to corners

length of the minor axis. One can show that $\mathbb{P}(L)$ fails to hold, but there does exist a function $B(\cdot)$ such that (a)–(c) of Theorem 1.3 still hold.

- Now consider the case of a circle. It is easy to see that if L is the diameter, $\mathbb{P}(L)$ fails to hold, but there exists $B(\cdot)$) (simply the antipode of $C(\cdot)$) such that (a)–(c) of Theorem 1.3 still hold. Here, all Holditch curves other than the locus of the midpoint, which gives only the center of the circle, are circles themselves and therefore smooth. Clearly we again have perpendicularity at both endpoints. Hence in general, the "double perpendicular" property is *indeterminate* as far as the differentiability of Holditch curves is concerned.
- Again consider Figure 3.5 but now take the dividing point to be the midpoint. Then the Holditch curve is not only nonsmooth, but also not Jordan. This is illustrated in Figure 3.6. (The case of a circle also exhibits this behaviour, but in a degenerate fashion.) Note that together, Figures 3.5 and 3.6 show that whether or not the Jordan curve property holds can depend on the choice of the dividing point of the chord (if L is not small enough).

In the next result, we consider another possibility for the traveling chord's geometric relation to the curve $\mathbb{C}$; specifically, when we have non-perpendicularity

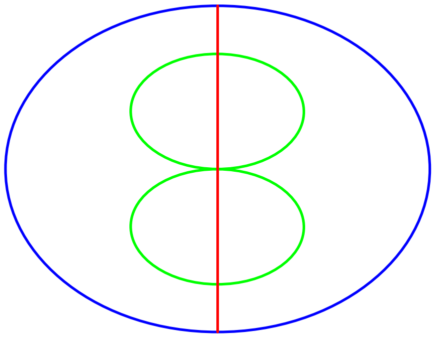

Figure 3.6: Non-Jordan Holditch curve

of the chord at both endpoints.

Proposition 3.5. *Suppose that Hypothesis $\mathbb{S}(L)$ holds, and assume that $\hat{t}$ is such that the chord $\overline{CB}$ is not perpendicular to the curve $\mathbb{C}$ at both $B(\hat{t}) = C(\tau(\hat{t}))$ and $C(\hat{t})$. Then the following hold:*

1. *There exists an open interval containing $\hat{t}$ upon which $B(\cdot)$ is continuously differentiable with non-vanishing derivative; that is, $B(\cdot)$ is smooth on that interval.*
2. *If the present assumptions hold, but now with $\mathbb{C}$ taken to be C^k-smooth, then there exists an open interval containing $\hat{t}$ upon which $B(\cdot)$ is C^k-smooth.*

Proof. Consider the continuously differentiable function of (t, τ) given by

$$g(t, \tau) := \|C(t) - C(\tau)\|^2 - L^2. \tag{6}$$

The non-perpendicularity condition at $B(\hat{t})$ is

$$\langle \dot{C}(\tau(\hat{t})), \left[C(\hat{t}) - C(\tau(\hat{t}))\rangle\right] \neq 0,$$

which is equivalent to

$$\frac{\partial g}{\partial \tau}(\hat{t}, \tau(\hat{t})) \neq 0.$$

Since $g(\hat{t}, \tau(\hat{t}) = 0$, the Implicit Function Theorem asserts that on a neighborhood of $\hat{t}$ there is defined a unique continuously differentiable function $h(\cdot)$, such that $g(t, h(t)) = 0$. But since $g(t, \tau(t)) = 0$ for all t, we have that $h(\cdot) = \tau(\cdot)$ near $\hat{t}$. Hence $B(\cdot) = C(\tau(\cdot))$ is continuously differentiable on that neighborhood.

The Implicit Function Theorem also asserts that for t near $\hat{t}$ one has

$$\dot{\tau}(t) = -\frac{\frac{\partial g}{\partial t}(t, \tau(t))}{\frac{\partial g}{\partial \tau}(t, \tau(t))}.$$

The non-perpendicularity condition at $C(\hat{t})$ is

$$\langle \dot{C}(\hat{t}), C(\hat{t}) - C(\tau(\hat{t}))\rangle \neq 0,$$

which is equivalent to the numerator in the above expression being nonzero. Hence $\dot{\tau}(\cdot)$ is nonzero near $\hat{t}$. (In fact, since $\tau(\cdot)$ is increasing, we have positivity of this derivative near $\hat{t}$.) We therefore have continuous differentiability of $B(\cdot)$ near $\hat{t}$, and also

$$\dot{B}(t) = \dot{C}(\tau(t))\dot{\tau}(t) \neq 0,$$

since the derivative of $C(\cdot)$ is non-vanishing. This proves (1).

In order to prove (2), note that (6) is now a k times continuously differentiable function of (t, τ). Then (2) follows from the proof of (1), upon noting that the Implicit Function Theorem provides for $\tau(\cdot)$ to be k times continuously differentiable. □

Remark 3.6. If $\mathbb{S}(L)$ holds, then there cannot exist $\hat{t}$ for which one has perpendicularity of the chord at $C(\hat{t})$ but not at $B(\hat{t}) = C(\tau(\hat{t}))$. By way of contradiction, suppose such a $\hat{t}$ existed. Then by the first part of Proposition 3.5, we would have differentiability of $B(\cdot)$ at $\hat{t}$. Upon reversing the roles of the endpoints of the chord in the proof of Proposition 3.3, we would then have non-differentiability of $C(\cdot)$ at $\hat{t}$, violating the smoothness assumption for $\mathbb{C}$.

We shall require the following geometric lemma.

Lemma 3.7. *For a given smooth convex planar curve $\mathbb{C}$, there exists $\Lambda > 0$ such that any chord of $\mathbb{C}$ with length less than Λ is such that the following hold:*

1. *The chord is non-perpendicular to $\mathbb{C}$ at both its endpoints.*
2. *The tangent directions to $\mathbb{C}$ at the chord's endpoints are not parallel.*

Proof. Let $x \in \mathbb{C}$, and denote by $n(x)$ the (unique) outer unit normal to the convex region $\widehat{\mathbb{C}}$ enclosed by $\mathbb{C}$. Consider the half-line

$$L_x := \{x - \alpha n(x) : \alpha > 0\}.$$

The hypotheses on $\mathbb{C}$ imply that L_x intersects $\mathbb{C}$ only at one point, which we label $y(x)$. Since the segment $(x, y(x))$ is in the interior of $\widehat{\mathbb{C}}$, we have

$$m(x) := \|y(x) - x\| > 0 \;\; \forall\, x \in C.$$

We claim that $m(\cdot)$ is continuous on $\mathbb{C}$. To this end, we need to show that $y(\cdot)$ is continuous. By way of contradiction, suppose that $y(\cdot)$ failed to be continuous at $\tilde{x} \in C$. Then there would necessarily exist a sequence x_i in $\mathbb{C}$ with $x_i \to \tilde{x}$ such that $y(x_i) \to \tilde{y} \in C$ and $\tilde{y} \neq y(\tilde{x})$. Now, the continuity of $n(\cdot)$ implies that $\tilde{y} - \tilde{x}$ is colinear with $n(\tilde{x})$. But then $\tilde{y} \in C$ implies $\tilde{y} = y(\tilde{x})$, a contradiction.

The continuity of the positive valued function $m(\cdot)$ implies that on the compact set $\mathbb{C}$ it attains its minimum value, $\Lambda > 0$. The first part of the lemma's

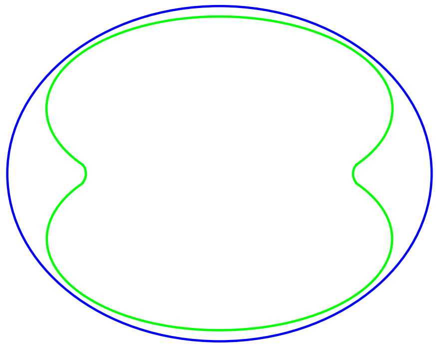

Figure 3.7: Smooth locus of midpoint

statement is now immediate. As for the second part, note that for a given $x \in C$, there are precisely two support lines to $\widehat{\mathbb{C}}$ which are perpendicular to $n(x)$. As x varies in $\mathbb{C}$, the infimal distance Γ between these lines is clearly positive, since $\widehat{\mathbb{C}}$ has nonempty interior. Upon noting that $\Lambda \leq \Gamma$, the proof is complete. □

We can now prove a general result on the smoothness of Holditch curves.

Theorem 3.8. *Suppose that $\tilde{L} > 0$ is such that Hypothesis $\mathbb{S}(\tilde{L})$ holds. Then if*

$$0 < L < \min\{\tilde{L}, \Lambda\}, \tag{7}$$

$\mathbb{S}(L)$ holds, and all Holditch curves for chordlength L are smooth. Furthermore, if $\mathbb{C}$ is taken to be C^k-smooth, then all Holditch curves for chordlength L are C^k-smooth.

Proof. The fact that $\mathbb{S}(L)$ holds follows from Theorem 2.6. Consider the motion of the chord $\overline{CB}$ as the smooth convex curve $\mathbb{C}$ is traversed. In view of part 1 of the preceding lemma, the non-perpendicularity hypotheses of Proposition 3.5 are satisfied. Hence $B(\cdot)$ is locally smooth, and therefore smooth. Hence any convex combination $D(\cdot)$ of $B(\cdot)$ and $C(\cdot)$ is differentiable. Furthermore, part 2 of the lemma implies that $\dot{B}(\cdot)$ and $\dot{C}(\cdot)$ are not colinear, and therefore $\dot{D}(\cdot)$ is non-vanishing. It follows that all Holditch curves for chordlength L are smooth. The "furthermore" assertion now follows immediately from part (2) of Proposition 3.5. □

Figure 3.7 illustrates the theorem. The ellipse should be compared to the one in Figure 3.3. Again, the dividing point is the midpoint of the traveling chord, but now the chordlength has been reduced. This results in a smoothing of the corners of the Holditch curve, but does not yield convexity. Similar behaviours occur in Figures 3.4, 3.5 and 3.6 upon reducing the chordlength. As will be seen in the next section, further reduction is required in order to obtain convexity of Holditch curves.

3.2. Convexity of Holditch curves

A smooth convex curve Γ is said to be *strictly convex* if at each point $\gamma \in \Gamma$, the tangent line through γ meets Γ only at γ. In case Γ is C^2-smooth, nonzero curvature at each point of Γ is sufficient for strict convexity. (That the reverse implication is false can be seen by considering the curve (t, t^4) at the origin.) Since curvature varies continuously as Γ is traversed, nonzero curvature is equivalent to the curvature being bounded away from zero and never changing sign.

The next result requires a possible further reduction in chordlength, beyond the bound (7), in order to obtain strict convexity of Holditch curves.

Theorem 3.9. *Suppose that the convex planar curve $\mathbb{C}$ is C^2-smooth with non-vanishing curvature. Then for all sufficiently small chordlengths $L > 0$, each Holditch curve is C^2-smooth with non-vanishing curvature, and is therefore strictly convex.*

Proof. Let a dividing point of the chord of length L be denoted

$$D(t) = C(t) + \alpha L s(t),$$

where $0 < \alpha < 1$ and $s(t)$ is the unit vector in the direction $\overline{CB}$; that is, $s(t) = \frac{1}{L}(B(t) - C(t))$. Note that in view of Proposition 3.5, $s(\cdot)$ is C^2-smooth. Also, Theorem 3.8 implies that $D(\cdot)$ is C^2-smooth.

The sign of the curvature of $D(\cdot)$ is determined by the quantity $\det(D', D'')$, where now differentiation is indicated with primes. We calculate

$$\det(D', D'') = \det(C', C'') + \alpha^2 L^2 \det(s', s'') + \alpha L[\det(C', s'') - \det(C'', s')].$$

Since the determinant $\det(C', C'')$ is bounded away from zero throughout the motion of the chord, and the other determinants on the right are bounded, it follows that $\det(D', D'')$ is nonzero for sufficiently small $L > 0$, which completes the proof. □

Remark 3.10. Figure 3.8 (adapted from [2]) summarizes what has been gleaned regarding properties of Holditch curves. The ellipse $\frac{x^2}{2} + y^2 = 1$ is depicted, along with the loci of the midpoint of traveling chords of different lengths. For length 2.2, the locus is nonsmooth and non-Jordan. For length 1.9, the transition to smoothness has begun, and for length 1.4, one has both smoothness and convexity.

4. Proof of Theorem 1.10

We assume that that the good travel hypothesis $\mathbb{GT}(L)$ holds, and that the chordlength L has been reduced if necessary to ensure the Jordan property of all Holditch curves, as in Proposition 2.5 (and note that by the results of Section 2 that property $\mathbb{GT}(L)$ persists under reduced chordlength).

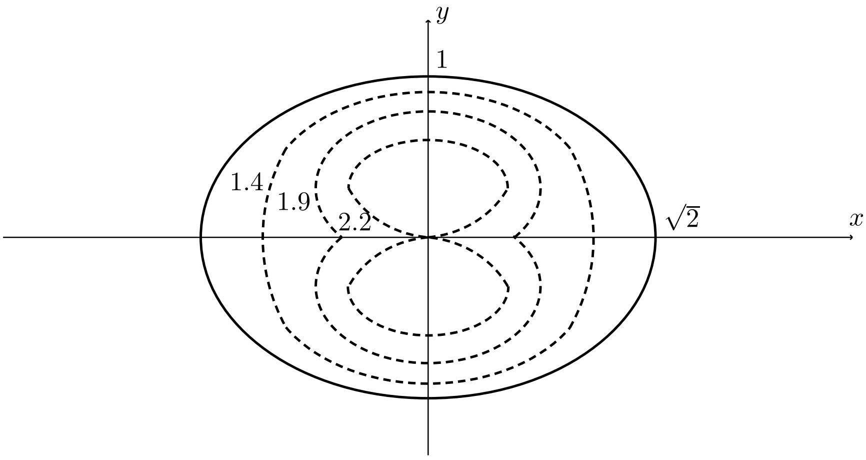

Figure 3.8: Summary

4.1. The envelope

We denote the open unit ball in $\mathbb{R}^2$ centered at the origin by B, and its closure by $\overline{B}$.

For each $t \in [0,1)$, consider the line $\mathbb{L}_t$ through the chord $\overline{C(t)B(t)}$ as it traverses the curve $\mathbb{C}$. Let p be a point in the interior of $\widehat{\mathbb{C}}$. Given (a sufficiently small) $\delta > 0$, it is clear that we can reduce the chordlength L if necessary so that

$$\{p + \delta\overline{B}\} \cap \mathbb{L}_t = \phi \;\; \forall\, t \in [0,1). \tag{8}$$

Let $\mathbb{H}_t$ denote the closed halfspace bounded by $\mathbb{L}_t$ which contains p. We go on to define Holditch's *envelope* as

$$\mathbb{E} := \bigcap_{t\in[0,1)} \mathbb{H}_t. \tag{9}$$

The following provides basic facts about the envelope.

Proposition 4.1. *If $L > 0$ is sufficiently small, then $\mathbb{E}$ is a compact convex subset of $\widehat{\mathbb{C}}$ with nonempty interior.*

Proof. That $\mathbb{E}$ is closed and convex is immediate, and that $\mathbb{E}$ has nonempty interior for small enough chordlength follows from (8). We now claim that $\mathbb{E} \subset \widehat{\mathbb{C}}$ (from which it follows that $\mathbb{E}$ is bounded and therefore compact). To this end, suppose that there existed $e \in \mathbb{E}$ such that $e \notin \widehat{\mathbb{C}}$, and let $\mathbb{L}$ be a line strictly separating $\widehat{\mathbb{C}}$ from e. But then, upon "sliding" $\mathbb{L}$ in a normal direction and recalling properties of the chord direction function $\theta(\cdot)$, we see that there exists $t \in [0,1)$ such that the line $\mathbb{L}_t$ is parallel to $\mathbb{L}$, with e and p (as in (8)) in opposite open halfspaces associated with $\mathbb{L}_t$. This contradicts the definition of the envelope in (9). $\square$

If there exists a support line $\mathbb{L}_t$ to $\mathbb{E}$ such that $F_t := \mathbb{E} \cap \mathbb{L}_t$ is not a singleton, then we call F_t a *flat* in the boundary of the envelope. It is readily seen that $\overline{C(t)B(t)} \subset F_t$, and that F_t does not vary for t in a closed interval J, possibly a singleton. (The singleton case corresponds to an instant where a flat coincides with a single chord.) Since the boundary of $\mathbb{E}$ is convex curve (implying it is rectifiable), it has finite length. Therefore $L > 0$ implies that the number of flats is finite. Note also that a flat F_t is characterized by constancy of $\theta(\cdot)$ on J. Observe as well that a flat in the boundary of the envelope is always a proper subset of a flat portion of $\mathbb{C}$.

Let $\mathbb{F}$ denote the union of all the flats in bdry$(\mathbb{E})$ (which is clearly closed), and let

$$\mathbb{Z} := \{p \in \text{bdry}(\mathbb{E}) \,:\, p \notin \mathbb{F}\}\,.$$

Let us first consider the special case where $\mathbb{Z} = \text{bdry}(\mathbb{E})$, meaning there are no flats in the boundary of $\mathbb{E}$. Then $\theta(\cdot)$ is strictly increasing on $[0, 1)$. The general case will be dealt with quite easily after this.

Let $p \in \text{bdry}(\mathbb{E})$, and let

$$T(p) := \{t \in [0, 1) \,:\, p \in \mathbb{L}_t\}\,.$$

If $t \in T(p)$, then $p \in \overline{C(t)B(t)}$, for if not, bdry$(\mathbb{E})$ would contain a flat. Note also that there is a unique point $Q(t)$ in the chord $\overline{C(t)B(t)}$ making contact with the envelope (and so $Q(t) = p$); again, because the envelope contains no flats in the case under consideration.

Suppose $T(p)$ is a singleton $\{t\}$, meaning that the normal cone to $\mathbb{E}$ at p is a single ray. In this case, $\overline{C(t)B(t)}$ makes contact with E at p, and it is the unique such chord (since more than one chord would imply that the normal cone to $\mathbb{E}$ at p has interior).

Now consider the situation where $T(p)$ is not a singleton, which means that the normal cone to the envelope at p has interior; that is, p is a "corner" of the envelope.

Remark 4.2. Typically $T(p) = \{t_1\} \cup \{t_2\}$ where $0 \le t_1 < t_2 < 1$, when p is a corner. This occurs in the example where $\mathbb{C}$ is the unit square with $L = 1$, as presented in [2]. There the envelope has no flats, possesses four non-polyhedral corners, and at each corner p, $T(p)$ consists only of two extremal values. (This is not due to nonsmoothness; the same nonsmooth geometry of the envelope persists in a smooth strictly convex approximation of the square with L slightly reduced.) The reader can verify that when $L \in (\frac{1}{2}, 1)$, the envelope again possesses four non-polyhedral corners and no flats, and that if $L \in (0, \frac{1}{2})$, the envelope is an inner rounding of the square, and flats are present. If $L = \frac{1}{2}$, the envelope is the unit disk.

When $T(p)$ is not a singleton, $Q(t_1)$ and $Q(t_2)$ are defined as above, where t_1 and

t_2 are the extremal values in $T(p)$, and for $t \in (t_1, t_2)$, we define $Q(t) \in \overline{C(t)B(t)}$ as follows:

We have

$$Q(t_i) = \lambda_i C(t_i) + (1 - \lambda_i)B(t_i), \quad i = 1, 2,$$

and for $t \in [t_1, t_2]$ we define $Q(t)$ via the interpolation formula

$$Q(t) = \frac{1}{t_2 - t_1}\Big\{[(t_2-t)\lambda_1+(t-t_1)\lambda_2]C(t)+[(t_2-t)(1-\lambda_1)+(t-t_1)(1-\lambda_2)]B(t)\Big\}.$$

The resulting function $Q(\cdot)$ is continuous on $[0, 1)$.

Following Broman's exposition of Holditch's method, we let $\|\overline{BQ}\| = r$, $\|\overline{BD}\| = a$, and $\|\overline{DC}\| = b$. Since $\theta(\cdot)$ is strictly increasing on $[0, 1)$, we can express the areas swept out by $\overline{BQ}$, $\overline{QC}$ and $\overline{QD}$, respectively as sweep integrals with respect to θ:

$$I_1 = \frac{1}{2}\int_0^{2\pi} r(\theta)^2 d\theta$$

$$I_2 = \frac{1}{2}\int_0^{2\pi} (a + b - r(\theta))^2 d\theta$$

$$I_3 = \frac{1}{2}\int_0^{2\pi} (a - r(\theta))^2 d\theta$$

We have $I_1 = I_2$ since $\overline{BQ}$ and $\overline{CQ}$ sweep out the same region. Then

$$\int_0^{2\pi} r(\theta) d\theta = \pi(a + b),$$

whence $I_2 - I_3 = \pi ab$, as required.

Remark 4.3. Evidently Holditch knew how to compute sweeps as above. Formally, the statements about sweeps are a consequence of Mamikon's Theorem (see e.g. [1]), which asserts that "area of tangent sweep equals area of tangent cluster", in particular for a continuous function valued as the multiple of the tangent to a smooth curve. To see that this viewpoint is adoptable here, note that for $t \in [0, 1)$ we can attach every chord $\overline{C(t)B(t)}$ tangentially to a unit circle with $Q(t) = (\sin\theta(t), -\cos\theta(t))$.

4.2. Completing the proof

Now assume that flats are present in the envelope. Then $\mathbb{Z}$ consists of a finite number of (relatively) open sub-arcs of the boundary of the envelope. Each of these sub-arcs is associated with an open t-interval upon which $\theta(\cdot)$ is strictly monotone increasing. Then we can calculate $I_2 - I_3$ as above on each sub-arc (with appropriate limits on θ in the integrations). Now note that the flats do not contribute to the sweep integrals, because while the traveling chord is confined to a flat, $\theta(\cdot)$ is constant. Therefore the desired result is obtained by adding the results on the sub-arcs. □

Acknowledgements. The authors are grateful to Professor Richard L. Hall for introducing them to Holditch's Theorem.

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