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Positive, Extremal and Nodal Solutions for Nonlinear Parametric Problems

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We consider a nonlinear parametric problem driven by the p-Laplace differential operator. For all large enough values of the parameter λ, we show that the problem has a smallest positive solution $u_\lambda^* \in C_0^1(\overline{\Omega})$. We examine the monotonicity and continuity properties of the map $\lambda \longmapsto u_\lambda^*$. Finally we establish the existence of nodal (sign changing) solutions.

Keywords: Extremal positive solutions, nonlinear regularity, nonlinear maximum principle, nodal solutions, p-logistic equation

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Here Δ_p $(1 < p < +\infty)$ denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(\|\nabla u\|^{p-2}\nabla u) \quad \forall u \in W_0^{1,p}(\Omega).$$

For problem (P_λ), first we look for positive solutions. We pay special attention to the existence of a smallest positive solution u_λ^* and we investigate the

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monotonicity and continuity properties of the map $\lambda \longmapsto u_\lambda^*$. Here and in what follows the monotonicity properties are understood with respect to the order structure of the order Banach space $C_0^1(\overline{\Omega})$. Subsequently, we look for nodal (sign changing) solutions. Our hypotheses on the reaction $f(z,\zeta)$ are rather mild and some of our results apply to the p-logistic equation (subdiffusive, equidiffusive or superdiffusive). We refer to the results of Section 2, Proposition 4.2 and Section 5. In Section 3 we investigate the monotonicity properties of the map $\lambda \longmapsto u_\lambda^*$ (u_λ^* being the extremal (smallest) positive solution of problem (P_λ)). The presence of the p-Laplacian (a nonlinear differential operator), implies that we do not have suitable comparison results in our disposal. For this reason we impose a sign condition on $f(z,\cdot)$ (see hypotheses H_2), which excludes from our consideration the p-logistic equation. Similarly, in the second half of Section 4 where we study the continuity of the map $\lambda \longmapsto u_\lambda^*$ in the absence of uniqueness, in order to employ some comparison techniques inspired by the work of Cazenave-Escobedo-Pozio [2], we need to restrict ourself to the semilinear problem (i.e., $p=2$) and assume that $f(z,\cdot)$ is concave. The latter condition excludes from consideration the model logistic equation, in particular the superdiffusive one, which is characterized by lack of uniqueness. We should mention that p-logistic equations were studied by Gasiński-Papageorgiou [12] (subdiffusive and equidiffusive type) and by Dong [5] and Gasiński-Papageorgiou [9] (superdiffusive type).

Our approach is variational and uses also truncation and comparison techniques.

2. Positive and extremal solutions

Our hypotheses on the reaction $f(z,\zeta)$ are the following (recall that $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, if for all $\zeta\in\mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is measurable and for almost all $z\in\Omega$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous).

$\underline{H_1}$ $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for almost all $z\in\Omega$ and

(i) $|f(z,\zeta)| \leqslant a(z)(1+\zeta^{r-1})$ for almost all $z\in\Omega$, all $\zeta\geqslant 0$ with $a\in L^\infty(\Omega)_+$, with $p\leqslant r<p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p<N, \\ +\infty & \text{if } p\geqslant N; \end{cases}$

(ii) $\limsup_{\zeta\to+\infty} \frac{f(z,\zeta)}{\zeta^{p-1}} \leqslant 0$ uniformly for almost all $z\in\Omega$;

(iii) $\liminf_{\zeta\to 0^+} \frac{f(z,\zeta)}{\zeta^{p-1}} \geqslant \eta_0 > 0$ uniformly for almost all $z\in\Omega$.

Remark 2.1. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $[0,+\infty)$, without any loss of generality we may (and will) assume that $f(z,\zeta)=0$ for almost all $z\in\Omega$ and all $\zeta\leqslant 0$. Evidently, the p-logistic reaction

$$f(\zeta) = \zeta^{\tau-1} - \zeta^{r-1}, \quad \zeta\geqslant 0,$$

where $1 < \tau \leqslant p < r < p^*$, satisfies above hypotheses. If $\tau < p$ we have a subdiffusive p-logistic equation and if $\tau = p$ we have an equidiffusive p-logistic equation.

In the sequel, in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space

$$C_0^1(\Omega) = \big\{u \in C^1(\overline{\Omega}) :\ u|_{\partial\Omega} = 0\big\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \big\{u \in C_0^1(\overline{\Omega}) :\ u(z) \geqslant 0 \ \text{ for all } z \in \overline{\Omega}\big\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ :\ u(z) > 0 \ \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\}.$$

Here by $n(\cdot)$ we denote the outward unit normal on $\partial\Omega$. As we already mentioned in the Introduction, all monotonicity properties are referred to the order structure introduced by C_+ and $\operatorname{int} C_+$ (for strict monotonicity). In what follows by $\|\cdot\|_{1,p}$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\|_{1,p} = \|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

By $\|\cdot\|$ we denote the norm of $\mathbb{R}^N$.

By $\widehat{\lambda}_1 > 0$ (respectively $\widehat{\lambda}_2 > \widehat{\lambda}_1$) we will denote the first (respectively second) eigenvalue of the Dirichlet p-Laplacian. We know that $\widehat{\lambda}_1 > 0$ is simple and by $\widehat{u}_1$ we denote the corresponding L^p-normalized (i.e., $\|\widehat{u}_1\|_p = 1$) positive eigenfunction. The nonlinear regularity theory and the nonlinear maximum principle, imply that $\widehat{u}_1 \in \operatorname{int} C_+$. It is worth mentioning that $\widehat{\lambda}_1 > 0$ is the only eigenvalue with eigenfunctions of constant sign. All other eigenvalues have nodal eigenfunctions. For more details on the spectral properties of the Dirichlet p-Laplacian, we refer to Gasiński-Papageorgiou [11].

Also, in what follows $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ denotes the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega \|\nabla u\|^{p-2}(\nabla u, \nabla y)_{\mathbb{R}^N}\, dz \quad \forall u, y \in W_0^{1,p}(\Omega).$$

We know (see Gasiński-Papageorgiou [11]) that A is bounded (i.e., maps bounded sets to bounded ones), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, i.e., if $u_n \longrightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and $\limsup_{r\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$ (see e.g., Gasiński-Papageorgiou [10, Lemma 3.2, p. 5752]).

Proposition 2.2. *If hypotheses H_1 hold and $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, then problem (P_λ) admits at least one positive solution $u_\lambda \in \operatorname{int} C_+$.*

Proof. Let $\varphi_\lambda \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega F(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega),$$

where

$$F(z, \zeta) = \int_0^\zeta f(x, s)\, ds.$$

Evidently $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$. Hypotheses H_1(i) and (ii) imply that for a given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z, \zeta) \leqslant \frac{\varepsilon}{p}\zeta^p + c_\varepsilon \quad \text{for a.e. } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{1}$$

Hence, we have

$$\begin{aligned} \varphi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega F(z, u)\, dz \\ &\geqslant \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda\varepsilon}{p}\|u\|_p^p - \lambda c_\varepsilon |\Omega|_N \\ &\geqslant \frac{1}{p}\left(1 - \frac{\lambda\varepsilon}{\widehat{\lambda}_1}\right)\|\nabla u\|_p^p - \lambda c_\varepsilon |\Omega|_N \end{aligned} \tag{2}$$

(see (1)). Here by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Choosing $\varepsilon \in (0, \frac{\widehat{\lambda}_1}{\lambda})$, from (2), we see that φ_λ is coercive. Also, using the Sobolev embedding theorem, we see that φ_λ is sequentially weakly lower semicontinuous. Hence by the Weierstrass theorem, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \inf_{u \in W_0^{1,p}(\Omega)} \varphi_\lambda(u). \tag{3}$$

By virtue of hypothesis H_1(iii), for a given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{\eta_0 - \varepsilon}{p}\zeta^p \leqslant F(z, \zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in [0, \delta). \tag{4}$$

Let $t \in (0, 1)$ be small, such that $t\widehat{u}_1(z) \in [0, \delta]$ for all $z \in \overline{\Omega}$ (recall that $\widehat{u}_1 \in \operatorname{int} C_+$). We have

$$\begin{aligned} \varphi_\lambda(t\widehat{u}_1) &= \frac{t^p}{p}\|\nabla \widehat{u}_1\|_p^p - \lambda \int_\Omega F(z, t\widehat{u}_1)\, dz \leqslant \frac{t^p}{p}\widehat{\lambda}_1 - \lambda\frac{t^p}{p}(\eta_0 - \varepsilon) \\ &= \frac{t^p}{p}(\widehat{\lambda}_1 - \lambda(\eta_0 - \varepsilon)) \end{aligned} \tag{5}$$

(see (4) and recall that $\|\widehat{u}_1\|_p = 1$). Since $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, choosing $\varepsilon \in (0, \frac{\lambda\eta_0 - \widehat{\lambda}_1}{\lambda})$, from (5), it follows that

$$\varphi_\lambda(t\widehat{u}_1) < 0,$$

so

$$\varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0)$$

(see (3)) and hence $u_\lambda \neq 0$.

From (3), we have

$$A(u_\lambda) = \lambda N_f(u_\lambda), \tag{6}$$

where $N_f\colon L^r(\Omega) \longrightarrow L^{r'}(\Omega)$ ($\frac{1}{r} + \frac{1}{r'} = 1$) is the Nemytski map corresponding to f, i.e.,

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \quad \forall u \in L^r(\Omega).$$

On (6) we act with $-u_\lambda^- \in W_0^{1,p}(\Omega)$ and obtain

$$\|\nabla u_\lambda^-\|_p^p = 0,$$

hence $u_\lambda \geqslant 0$, $u_\lambda \neq 0$. Hypotheses H_1 imply that for a given $\varepsilon \in (0, \eta_0)$ and $q \in (r, p^*)$, we can find $c_1 = c_1(\varepsilon, q) > 0$ such that

$$f(z,\zeta) \geqslant (\eta_0 - \varepsilon)\zeta^{p-1} - c_1\zeta^{q-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{7}$$

From (6) and the nonlinear regularity theory (see Gasiński-Papageorgiou [11]), we have

$$-\Delta_p u_\lambda(z) = \lambda f(z, u_\lambda(z)) \quad \text{almost everywhere in } \Omega, \quad u_\lambda \in C_+ \setminus \{0\}.$$

Then using (7), we obtain

$$-\Delta_p u_\lambda(z) \geqslant \lambda(\eta_0 - \varepsilon)u_\lambda(z)^{p-1} - \lambda c_1 u_\lambda(z)^{q-1} \quad \text{almost everywhere in } \Omega,$$

so

$$\Delta_p u_\lambda(z) \leqslant \lambda c_1 \|u_\lambda\|_\infty^{q-p} u_\lambda(z) \quad \text{almost everywhere in } \Omega$$

(recall that $q > p$), so

$$u_\lambda \in \operatorname{int} C_+$$

(see Vázquez [14]). $\square$

Motivated from the unilateral growth estimate (7), we consider the following auxiliary Dirichlet problem

$$(A_\lambda) \qquad \begin{cases} -\Delta_p u(z) = \lambda(\eta_0 - \varepsilon)u(z)^{p-1} - \lambda c_1 u(z)^{q-1} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0. \end{cases}$$

Proposition 2.3. *If $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, then problem (A_λ) admits a unique positive solution $\widetilde{u}_\lambda \in \operatorname{int} C_+$.*

Proof. Let $\psi_\lambda \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\psi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda(\eta_0 - \varepsilon)}{p}\|u^+\|_p^p + \frac{\lambda c_1}{q}\|u^+\|_q^q \quad \forall u \in W_0^{1,p}(\Omega).$$

Since $q > p$, it is clear that ψ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W_0^{1,p}(\lambda)$ such that

$$\psi_\lambda(\widetilde{u}_\lambda) = \inf_{u \in W_0^{1,p}(\Omega)} \psi_\lambda(u). \tag{8}$$

For $t > 0$, we have

$$\begin{aligned} \psi_\lambda(t\widehat{u}_1) &= \frac{t^p}{p}\widehat{\lambda}_1 - \frac{\lambda(\eta_0 - \varepsilon)}{p}t^p + \frac{\lambda c_1}{q}t^q\|\widehat{u}_1\|_q^q \\ &= \frac{t^p}{p}(\widehat{\lambda}_1 - \lambda(\eta_0 - \varepsilon)) + \frac{t^q}{q}\lambda c_1\|\widehat{u}_1\|_q^q \end{aligned} \tag{9}$$

(recall that $\widehat{u}_1 \in \operatorname{int} C_+$ and $\|\widehat{u}_1\|_p = 1$).

Since $q > p$ and $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, choosing $t \in (0,1)$ and $\varepsilon > 0$ suitably small from (9), we see that

$$\psi_\lambda(t\widehat{u}_1) < 0 = \psi_\lambda(0),$$

so

$$\psi_\lambda(\widetilde{u}_\lambda) < 0 = \psi_\lambda(0)$$

(see (8)) and hence $\widetilde{u}_\lambda \neq 0$.

From (8), we have

$$A(\widetilde{u}_\lambda) = \lambda(\eta_0 - \varepsilon)(\widetilde{u}_\lambda^+)^{p-1} - \lambda c_1(\widetilde{u}_\lambda^+)^{q-1}. \tag{10}$$

On (10) we act with $-\widetilde{u}_\lambda^- \in W_0^{1,p}(\Omega)$ an obtain $\widetilde{u}_\lambda \geqslant 0$, $\widetilde{u}_\lambda \neq 0$. Hence

$$A(\widetilde{u}_\lambda) = \lambda(\eta_0 - \varepsilon)\widetilde{u}_\lambda^{p-1} - \lambda c_1\widetilde{u}_\lambda^{q-1}, \tag{11}$$

so $\widetilde{u}_\lambda$ is a positive solution of problem (A_λ).

The nonlinear regularity theory (see e.g., Gasiński-Papageorgiou [11, pp. 737–738]) implies that $\widetilde{u}_\lambda \in C_+ \setminus \{0\}$. We have (see (11))

$$\Delta_p\widetilde{u}_\lambda(z) \leqslant \lambda c_1\|\widetilde{u}_\lambda\|_\infty^{q-p}\widetilde{u}_\lambda(z)^{p-1} \quad \text{almost everywhere in } \Omega,$$

so

$$\widetilde{u}_\lambda \in \operatorname{int} C_+$$

(see Vázquez [14]).

The uniqueness of $\widetilde{u}_\lambda \in \operatorname{int} C_+$ follows from Theorem 1 of Diaz-Saa [4]. $\square$

Let $S_+(\lambda)$ denote the set of positive solutions of (P_λ). If $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, then from Proposition 2.2 and its proof, we have

$$\emptyset \neq S_+(\lambda) \subseteq C_+.$$

Proposition 2.4. *If hypotheses H_1 hold, $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$ and $u_\lambda \in S_+(\lambda)$, then $\widetilde{u}_\lambda \leqslant u_\lambda$.*

Proof. We consider the Carathéodory function $g_\lambda \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g_\lambda(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ \lambda(\eta_0-\varepsilon)\zeta^{p-1} - \lambda c_1 \zeta^{q-1} & \text{if } 0 \leqslant \zeta \leqslant u_\lambda(z), \\ \lambda(\eta_0-\varepsilon)u_\lambda(z)^{p-1} - \lambda c_1 u_\lambda(z)^{q-1} & \text{if } u_\lambda(z) < \zeta. \end{cases} \tag{12}$$

Let

$$G_\lambda(z,\zeta) = \int_0^\zeta g_\lambda(z,s)\,ds$$

and consider the C^1-functional $\sigma_\lambda \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega G_\lambda(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

From (12) it is clear that σ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda^* \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\lambda(\widetilde{u}_\lambda^*) = \inf_{u \in W_0^{1,p}(\Omega)} \sigma_\lambda(u). \tag{13}$$

Let $t \in (0,1)$ be small such that $t\widehat{u}_1 \leqslant u_\lambda$ (recall that $u_\lambda \in \operatorname{int} C_+$ and see Lemma 3.3 of Filippakis-Kristaly-Papageorgiou [7]). Then as in the proof of Proposition 2.2, we show that $\sigma_\lambda(t\widehat{u}_1) < 0 = \sigma_\lambda(0)$, hence

$$\sigma_\lambda(\widetilde{u}_\lambda^*) < 0 = \sigma_\lambda(0)$$

(see (13)) and so $\widetilde{u}_\lambda^* \neq 0$. From (13), we have

$$A(\widetilde{u}_\lambda^*) = N_{g_\lambda}(\widetilde{u}_\lambda^*) \tag{14}$$

(recall that for all $u \in W_0^{1,p}(\Omega)$, $N_{g_\lambda}(u)(\cdot) = g_\lambda(\cdot,u(\cdot))$). On (14) first we act with $-(\widetilde{u}_\lambda^*)^- \in W_0^{1,p}(\Omega)$ and obtain $\widetilde{u}_\lambda^* \geqslant 0$, $\widetilde{u}_\lambda^* \neq 0$. Then on (14) we act with $(\widetilde{u}_\lambda^* - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} \langle A(\widetilde{u}_\lambda^*), (\widetilde{u}_\lambda^* - u_\lambda)^+\rangle &= \int_\Omega g_\lambda(z,\widetilde{u}_\lambda^*)(\widetilde{u}_\lambda^* - u_\lambda)^+\,dz \\ &= \int_\Omega \left(\lambda(\eta_0-\varepsilon)u_\lambda^{p-1} - \lambda c_1 u_\lambda^{q-1}\right)(\widetilde{u}_\lambda^* - u_\lambda)^+\,dz \\ &\leqslant \int_\Omega f(z,u_\lambda)(\widetilde{u}_\lambda^* - u_\lambda)^+\,dz \\ &= \langle A(u_\lambda), (\widetilde{u}_\lambda^* - u_\lambda)^+\rangle \end{aligned}$$

(see (12), (7) and recall that $u_\lambda \in S_+(\lambda)$), so

$$\int_{\{\widetilde{u}_\lambda^* > u_\lambda\}} \big(\|\nabla \widetilde{u}_\lambda^*\|^{p-2}\nabla \widetilde{u}_\lambda^* - \|\nabla u_\lambda\|^{p-2}\nabla u_\lambda,\ \nabla \widetilde{u}_\lambda^* - \nabla u_\lambda\big)_{\mathbb{R}^N}\, dz \leqslant 0,$$

so

$$|\{\widetilde{u}_\lambda^* > u_\lambda\}|_N = 0,$$

hence $\widetilde{u}_\lambda^* \leqslant u_\lambda$.

So, we have proved that

$$\widetilde{u}_\lambda^* \in [0, u_\lambda],$$

where $[0, u_\lambda] = \{u \in W_0^{1,p}(\Omega) :\ 0 \leqslant u(z) \leqslant u_\lambda(z) \text{ a.e. in } \Omega\}$, thus

$$A(\widetilde{u}_\lambda^*) = \lambda(\eta_0 - \varepsilon)(\widetilde{u}_\lambda^*)^{p-1} - \lambda c_1(\widetilde{u}_\lambda)^{q-1}.$$

Hence $\widetilde{u}_\lambda^*$ is a positive solution of (Au_λ) and so by virtue of Proposition 2.3, we have $\widetilde{u}_\lambda^* = \widetilde{u}_\lambda$. This shows that

$$\widetilde{u}_\lambda \leqslant u \quad \forall u \in S_+(\lambda). \qquad \square$$

Now we can produce the extremal positive solution of (P_λ), $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$ (i.e., the smallest positive solution).

Proposition 2.5. *If hypotheses H_1 hold and $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, then problem (P_λ) admits a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$.*

Proof. From Filippakis-Kristaly-Papageorgiou [7], we know that the set $S_+(\lambda)$ is downward directed, i.e., if $u_1, u_2 \in S_+(\lambda)$, then we can find $u \in S_+(\lambda)$ such that $u \leqslant u_1$, $u \leqslant u_2$.

Let $C \subseteq S_+(\lambda)$ be a chain (i.e., a totally ordered subset of $S_+(\lambda)$). From Dunford-Schwartz [6, p. 336], we know that there exists a sequence $\{u_n\}_{n\geqslant 1} \subseteq C$ such that $\inf C = \inf_{n\geqslant 1} u_n$. Since $S_+(\lambda)$ is downward directed, without any loss of generality we may assume that

$$\|u_n\|_\infty \leqslant M_1 \quad \forall n \geqslant 1,$$

for some $M_1 > 0$. We have

$$A(u_n) = \lambda N_f(u_n) \quad \forall n \geqslant 1, \tag{15}$$

so the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Hence, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{16}$$

$$u_n \longrightarrow u \quad \text{in } L^r(\Omega). \tag{17}$$

On (15) we act with $u_n - u \in W_0^{1,p}(\Omega)$. Passing to the limit as $n \to +\infty$ and using (16), we obtain

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0,$$

so

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p}(\Omega) \tag{18}$$

(from the $(S)_+$-property of A).

So, if in (15) we pass to the limit as $n \to +\infty$ and use (18), then

$$A(u) = \lambda N_f(u).$$

Also, from Proposition 2.4, we have $\widetilde{u}_\lambda \leqslant u_n$ for all $n \geqslant 1$, hence $\widetilde{u}_\lambda \leqslant u$. Therefore $u \neq 0$ and so

$$u \in S_+(\lambda) \subseteq \operatorname{int} C_+, \quad u = \inf C.$$

Since $C \subseteq S_+(\lambda)$ is an arbitrary chain, invoking the Kuratowski-Zorn lemma, we infer that $S_+(\lambda)$ has a minimal element $u_\lambda^* \in S_+(\lambda) \subseteq \operatorname{int} C_+$. If $u \in S_+(\lambda)$, then since $S_+(\lambda)$ is downward directed, we can find $u_\lambda \in S_+(\lambda)$ such that

$$u_\lambda \leqslant u_\lambda^* \quad \text{and} \quad u_\lambda \leqslant u,$$

so

$$u_\lambda = u_\lambda^* \leqslant u$$

(due to the minimality of u_λ^*), so $u_\lambda^* \in \operatorname{int} C_+$ is the smallest positive solution of (P_λ), $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$. □

3. Monotonicity properties of the map $\lambda \longmapsto u_\lambda^*$

Let $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$ and let $u_\lambda^* \in \operatorname{int} C_+$ be the extremal positive solution of problem (P_λ). In this section we study the monotonicity properties of the map $\lambda \longmapsto u_\lambda^*$ from $\left(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty\right)$ into $C_0^1(\overline{\Omega})$. To this end we need to strengthen our hypotheses on $f(z,\cdot)$.

$\underline{H_2}$ $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and $f(z,\zeta) \geqslant 0$ for almost all $z \in \Omega$ and all $\zeta \geqslant 0$ and

(i) $|f(z,\zeta)| \leqslant a(z)(1+\zeta^{p-1})$ for almost all $z \in \Omega$, all $\zeta \geqslant 0$ with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{\zeta\to+\infty} \frac{f(z,\zeta)}{\zeta^{p-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iii) $\liminf_{\zeta\to 0^+} \frac{f(z,\zeta)}{\zeta^{p-1}} \geqslant \eta_0 > 0$ uniformly for almost all $z \in \Omega$.

Remark 3.1. The main extra condition (compared to hypotheses H_1 used in Section 2) is the positivity condition, namely that $f(z,\zeta) \geqslant 0$ for almost all $z \in \Omega$ and all $\zeta \geqslant 0$. This hypothesis has a consequence on the asymptotic condition as $\zeta \longrightarrow +\infty$ (see hypothesis H_1(ii)). Now the quotient $\frac{f(z,\zeta)}{\zeta^{p-1}}$ cannot reach a strictly negative value as $\zeta \to +\infty$ and so asymptotically it must be zero (see hypothesis H_2(ii)). This then implies that for almost all $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-sublinear (see hypothesis H_2(i) and compare it with hypothesis H_1(i)). So, we see that the positivity condition on $f(z,\cdot)$ has a "chain effect" on the structural properties of the reaction. The fact that our equation is driven by the p-Laplacian (a nonlinear differential operator), makes comparison results difficult. The positivity condition implies that $u_\lambda^* \in \operatorname{int} C_+$ (the extremal solution of the problem (P_λ)) can serve as an upper solution for problem (P_μ) with $\mu \in (0,\lambda)$. Of course $u \equiv 0$ is always a solution of (P_μ) (recall that $f(z,0) = 0$ for almost all $z \in \Omega$). So, we have an ordered pair of upper-lower solutions for problem (P_μ) (namely the pair $(u_\lambda^*, 0)$) and this fact permits the use of truncation techniques which via the direct method lead to a nontrivial solution $u_\mu \in [0, u_\lambda^*] \cap C_+$. This implies the monotonicity of $\lambda \longmapsto u_\lambda^*$. In the last step of this argument, we need to establish the nontriviality of the solution u_μ produced via the direct method (a minimizer of the truncated energy functional). The tool to achieve this is hypothesis H_2(iii). Finally we mention that the positivity condition excludes from consideration the p-logistic equation.

Proposition 3.2. *If hypotheses H_2 hold and $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, then the map $\lambda \longmapsto u_\lambda^*$ is nondecreasing from $\left(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty\right)$ into $C_0^1(\overline{\Omega})$, i.e.,*

$$u_\lambda^* - u_\mu^* \in C_+ \quad \forall \mu < \lambda.$$

Proof. Let $\frac{\widehat{\lambda}_1}{\eta_0} < \mu < \lambda$. We have

$$-\Delta_p u_\lambda^*(z) = \lambda f(z, u_\lambda^*(z)) \geqslant \mu f(z, u_\lambda^*(z)) \quad \text{a.e. in } \Omega. \tag{19}$$

We introduce the following Carathéodory function $g\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ f(z,\zeta) & \text{if } 0 \leqslant \zeta \leqslant u_\lambda^*(z), \\ f(z, u_\lambda^*(z)) & \text{if } u_\lambda^*(z) < \zeta. \end{cases} \tag{20}$$

We set

$$G(z,\zeta) = \int_0^\zeta g(z,s)\, ds$$

and consider the C^1-functional $\psi\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|\nabla u\|_p^p - \mu \int_\Omega G(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Clearly ψ is coercive (see (20)) and sequentially weakly lower semicontinuous. So, we can find $\overline{u}_\mu \in W_0^{1,p}(\Omega)$ such that

$$\psi(\overline{u}_\mu) = \inf_{u \in W_0^{1,p}(\Omega)} \psi(u) < 0 = \psi(0)$$

(see hypothesis H_2(iii) and the proof of Proposition 2.2). Hence $\overline{u}_\mu \neq 0$ and as in the proof of Proposition 2.4, we can check that

$$\overline{u}_\mu \in [0, u_\lambda^*],$$

where $[0, u_\lambda^*] = \big\{u \in W_0^{1,p}(\Omega) :\ 0 \leqslant u(z) \leqslant u_\lambda^*(z) \text{ a.e. in } \Omega\big\}$, so

$$\overline{u}_\mu \in S_+(\mu)$$

(see (20)) and so

$$u_\mu^* \leqslant \overline{u}_\mu \leqslant u_\lambda^*. \qquad \square$$

Strengthening further the conditions on $f(z,\cdot)$, we can improve the conclusion of the above proposition. So, the new hypotheses on the reaction $f(z,\zeta)$ are the following.

$\underline{H_3}$ $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for almost all $z \in \Omega$ and $f(z,\zeta) \geqslant 0$ for almost all $z \in \Omega$ and all $\zeta \geqslant 0$ and hypotheses H_3(i), (ii) and (iii) are the same as the corresponding hypotheses H_2(i), (ii) and (iii) and

(iv) for every $\varrho, \vartheta > 0$, there exists $\xi_\varrho, \gamma_\vartheta > 0$ such that for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta) + \xi_\varrho \zeta^{p-1}$ is nondecreasing on $[0,\varrho]$ and $f(z,\zeta) \geqslant \gamma_\vartheta$ for all $\zeta \geqslant \vartheta$.

Proposition 3.3. *If hypotheses H_3 hold and $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, then the map $\lambda \longmapsto u_\lambda^*$ is increasing from $(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty)$ into $C_0^1(\overline{\Omega})$ (i.e., if $\frac{\widehat{\lambda}_1}{\eta_0} < \mu < \lambda$, then $u_\lambda^* - u_\mu^* \in \operatorname{int} C_+$) and $\|u_\lambda^*\|_\infty \longrightarrow +\infty$ as $\lambda \to +\infty$.*

Proof. Let $\frac{\widehat{\lambda}_1}{\eta_0} < \mu < \lambda$. From Proposition 3.2 we know that $u_\mu^* \leqslant u_\lambda^*$. Let $\varrho = \|u_\lambda^*\|_\infty$ and let $\xi_\varrho > 0$ be as postulated by hypothesis H_3(iv). Then

$$\begin{aligned}
&-\Delta_p u_\mu^*(z) + \mu\xi_\varrho u_\mu^*(z)^{p-1} = \mu f(z, u_\mu^*(z)) + \mu\xi_\varrho u_\mu^*(z)^{p-1}\\
&\leqslant \mu f(z, u_\lambda^*(z)) + \mu\xi_\varrho u_\lambda^*(z)^{p-1} \leqslant \lambda f(z, u_\lambda^*(z)) + \mu\xi_\varrho u_\lambda^*(z)^{p-1}\\
&= -\Delta_p u_\lambda^*(z) + \mu\xi_\varrho u_\lambda^*(z)^{p-1} \quad \text{in } W^{-1,p'}(\Omega)
\end{aligned} \tag{21}$$

(where $\frac{1}{p} + \frac{1}{p'} = 1$). Let K be a compact subset of Ω. Since $u_\mu^* \in \operatorname{int} C_+$, we have

$$u_\lambda^*(z) \geqslant u_\mu^*(z) \geqslant m_K > 0 \quad \forall z \in K. \tag{22}$$

Then, we have

$$\begin{aligned}
&\lambda f(z, u_\lambda^*(z)) + \mu\xi_\varrho u_\lambda^*(z)^{p-1} - \mu f(z, u_\mu^*(z)) - \mu\xi_\varrho u_\mu^*(z)^{p-1}\\
&= \mu\left(f(z, u_\lambda^*(z)) + \xi_\varrho u_\lambda^*(z)^{p-1} - f(z, u_\mu^*(z)) - \xi_\varrho u_\mu^*(z)^{p-1}\right)\\
&\qquad + (\lambda-\mu) f(z, u_\lambda^*(z))\\
&\geqslant (\lambda-\mu) f(z, u_\lambda^*(z)) \geqslant (\lambda-\mu)\gamma_K > 0 \quad \text{for a.e. } z \in K
\end{aligned}$$

(see (22) and hypothesis H_3(iv)). This fact in conjunction with (21) permit the use of Proposition 2.6 of Arcoya-Ruiz [1] and so we infer that

$$u_\lambda^* - u_\mu^* \in \operatorname{int} C_+.$$

Next, suppose that $\lambda_n \longrightarrow +\infty$ and assume that the sequence $\{u_{\lambda_n}^*\}_{n\geqslant 1} \subseteq L^\infty(\Omega)$ is bounded. Then from the nonlinear regularity theory (see, e.g., Gasiński-Papageorgiou [11, p. 738]), we know that we can find $\beta \in (0,1)$ and $M_2 > 0$ such that

$$u_{\lambda_n}^* \in C_0^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad \|u_{\lambda_n}^*\|_{C_0^{1,\beta}(\overline{\Omega})} \leqslant M_2 \quad \forall n \geqslant 1.$$

Exploiting the compact embedding of $C_0^{1,\beta}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, by passing to a suitable subsequence if necessary, we may assume that

$$u_{\lambda_n}^* \longrightarrow u^* \quad \text{in } C_0^1(\overline{\Omega}). \tag{23}$$

For every $n \geqslant 1$, we have

$$A(u_{\lambda_n}^*) = \lambda_n N_f(u_{\lambda_n}^*).$$

Let $v \in C_c^1(\Omega)$, $v \geqslant 0$, $v \neq 0$ and let $K = \operatorname{supp} v$. We have

$$\begin{aligned} \langle A(u_{\lambda_n}^*), v\rangle &= \lambda_n \int_\Omega f(z, u_{\lambda_n}^*) v\,dz \geqslant \lambda_n \int_K f(z, u_{\lambda_n}^*) v\,dz \\ &\geqslant \lambda_n m_K \int_K v\,dz \; > \; 0. \end{aligned} \tag{24}$$

Note that

$$|\langle A(u_{\lambda_n}^*), v\rangle| \leqslant M_3 \|v\|_{1,p} \quad \forall n \geqslant 1,$$

for some $M_3 > 0$ (see (23)) and

$$\lambda_n m_K \int_K v\,dz \longrightarrow +\infty \quad \text{as } n \to +\infty.$$

These facts and (24) lead to a contradiction. This means that

$$\|u_\lambda^*\|_\infty \longrightarrow +\infty \quad \text{as } \lambda \to +\infty. \qquad \square$$

4. Continuity properties of the map $\lambda \longmapsto u_\lambda^*$

The question of the continuity of the map $\lambda \longmapsto u_\lambda^*$ from $(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty)$ into $C_0^1(\overline{\Omega})$ is more delicate and we have only partial results valid under stronger conditions on the reaction $f(z,\zeta)$.

For this first continuity result, we impose the following conditions of f.

$\underline{H_4}$ $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and hypotheses H_4(i), (ii) and (iii) are the same as the corresponding hypotheses H_1(i), (ii) and (iii) and

(iv) for almost all $z \in \Omega$, the function $\zeta \longmapsto \frac{f(z,\zeta)}{\zeta^{p-1}}$ is strictly decreasing on $[0, +\infty)$.

Remark 4.1. Note that we have dropped the sign condition, which was crucial for our monotonicity results in Section 3 (see hypotheses H_2 and H_3).

Proposition 4.2. *If hypotheses H_4 hold and $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, then problem (P_λ) admits a unique positive solution $u_\lambda^* \in \operatorname{int} C_+$ and the map $\lambda \longmapsto u_\lambda^*$ is continuous from $(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty)$ into $C_0^1(\overline{\Omega})$.*

Proof. The existence of a positive solution $u_\lambda^* \in \operatorname{int} C_+$ follows from Proposition 2.2. The uniqueness of this positive solution is a consequence of Theorem 1 of Diaz-Saa [4].

Next, let $\{\lambda_n\}_{n\geqslant 1} \subseteq (\frac{\widehat{\lambda}_1}{\eta_0}, +\infty)$ be a sequence such that $\lambda_n \longrightarrow \lambda$ with $\lambda \in (\frac{\widehat{\lambda}_1}{\eta_0}, +\infty)$. Let $u_n^* = u_{\lambda_n}^* \in \operatorname{int} C_+$ be the unique positive solution of problem (P_{λ_n}), $n \geqslant 1$. We have

$$A(u_n^*) = \lambda_n N_f(u_n^*) \quad \text{and} \quad \frac{\widehat{\lambda}_1}{\eta_0} < \lambda_n \leqslant c^* \quad \forall n \geqslant 1, \tag{25}$$

for some $c^* > 0$. Hypotheses H_4(i) and (ii) imply that for a given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$f(z, \zeta) \leqslant \varepsilon \zeta^{p-1} + c_\varepsilon \quad \text{for a.e. } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{26}$$

On (25) we act with $u_n^* \in \operatorname{int} C_+$ and obtain

$$\begin{aligned} \|\nabla u_n^*\|_p^p = \lambda_n \int_\Omega f(z, u_n^*) u_n^* \, dz &\leqslant \lambda_n \varepsilon \|u_n^*\|_p^p + \lambda_n \widehat{c}_\varepsilon \|\nabla u_n^*\|_p \\ &\leqslant \frac{c^* \varepsilon}{\widehat{\lambda}_1} \|\nabla u_n^*\|_p^p + \lambda_n \widehat{c}_\varepsilon \|\nabla u_n^*\|_p, \end{aligned}$$

for some $\widehat{c}_\varepsilon > 0$ (see (26)), so

$$\left(1 - \frac{c^* \varepsilon}{\widehat{\lambda}_1}\right) \|\nabla u_n^*\|_p^p \leqslant \lambda_n \widehat{c}_\varepsilon \|\nabla u_n^*\|_p. \tag{27}$$

Choosing $\varepsilon \in (0, \frac{\widehat{\lambda}_1}{c^*})$ and since $p > 1$, from (27) we infer that the sequence $\{u_n^*\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

As before, the nonlinear regularity theory (see Gasiński-Papageorgiou [11]) implies that we can find $\beta \in (0,1)$ such that the sequence $\{u_n^*\}_{n\geqslant 1} \subseteq C_0^{1,\beta}(\overline{\Omega})$ is bounded. Hence, passing to a subsequence if necessary, we may assume that

$$u_n^* \longrightarrow u^* \quad \text{in } C_0^1(\overline{\Omega}),$$

so

$$A(u^*) = \lambda N_f(u^*)$$

(see (25)).

Therefore u^* is a solution of (P_λ) and $u^* \in C_+$. We show that $u^* \neq 0$, hence $u^* \in \operatorname{int} C_+$. From Proposition 2.4 we know that $\widetilde{u}_{\lambda_n} \leqslant u_n^*$ for all $n \geqslant 1$. We have

$$A(\widetilde{u}_{\lambda_n}) = \lambda_n(\eta_0 - \varepsilon)\widetilde{u}_{\lambda_n}^{p-1} - \lambda_n c_1 \widetilde{u}_{\lambda_n}^{p-1}, \tag{28}$$

so

$$\lambda_n c_1 \|\widetilde{u}_{\lambda_n}\|_q^q \leqslant \lambda_n(\eta_0 - \varepsilon)\|\widetilde{u}_{\lambda_n}\|_p^p$$

and thus

$$c_2 \|\widetilde{u}_{\lambda_n}\|_p^q \leqslant (\eta_0 - \varepsilon)\|\widetilde{u}_{\lambda_n}\|_p^p \quad \forall n \geqslant 1,$$

for some $c_2 > 0$ (recall that $q > p$ and see (7)), hence

$$\text{the sequence } \{\widetilde{u}_{\lambda_n}\}_{n\geqslant 1} \subseteq L^p(\Omega) \text{ is bounded.} \tag{29}$$

From (28) and (29), it follows that

$$\|\nabla \widetilde{u}_{\lambda_n}\|_p^p \leqslant M_4 \quad \forall n \geqslant 1,$$

for some $M_4 > 0$, so the sequence $\{\widetilde{u}_{\lambda_n}\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Then, as we did earlier for the sequence $\{u_n^*\}_{n\geqslant 1}$, for at least a subsequence, we have

$$\widetilde{u}_{\lambda_n} \longrightarrow \widetilde{u} \quad \text{in } C_0^1(\overline{\Omega}).$$

If $\widetilde{u} = 0$, then setting

$$\widetilde{y}_n = \frac{\widetilde{u}_{\lambda_n}}{\|\widetilde{u}_{\lambda_n}\|_{1,p}} \quad \forall n \geqslant 1,$$

we have $\|\widetilde{y}_n\|_{1,p} = 1$, $\widetilde{y}_n \geqslant 0$ for all $n \geqslant 1$ and from (28) it follows that

$$A(\widetilde{y}_n) = \lambda_n(\eta_0 - \varepsilon)\widetilde{y}_n^{p-1} - \lambda_n c_1 \widetilde{u}_n^{q-p}\widetilde{y}_n^{p-1}. \tag{30}$$

We may assume that

$$\widetilde{y}_n \longrightarrow \widetilde{y} \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{31}$$
$$\widetilde{y}_n \longrightarrow \widetilde{y} \quad \text{in } L^q(\Omega) \tag{32}$$

(recall that $q < p^*$). On (30) we act with $\widetilde{y}_n - \widetilde{y} \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (31). We obtain

$$\lim_{n\to+\infty} \langle A(\widetilde{y}_n), \widetilde{y}_n - \widetilde{y}\rangle = 0,$$

so

$$\widetilde{y}_n \longrightarrow \widetilde{y} \quad \text{in } W_0^{1,p}(\Omega)$$

and so $\|\widetilde{y}\|_{1,p} = 1$.

So, if in (30) we pass to the limit as $n \to +\infty$ and since $\widetilde{u} = 0$, we have

$$A(\widetilde{y}) = \lambda(\eta_0 - \varepsilon)\widetilde{y}^{p-1},$$

so

$$\begin{cases} -\Delta_p \widetilde{y}(z) = \lambda(\eta_0 - \varepsilon)\widetilde{y}(z)^{p-1} & \text{a.e. in } \Omega, \\ \widetilde{y}|_{\partial\Omega} = 0. \end{cases}$$

But recall that we choose $\varepsilon > 0$ small so that $\lambda(\eta_0 - \varepsilon) > \widehat{\lambda}_1$. It follows that $\widetilde{y}$ must be nodal, a contradiction. This proves that $\widetilde{u} \neq 0$ and so $\widetilde{u} = \widetilde{u}_\lambda$ (see Proposition 2.4). We have

$$u^* \geqslant \widetilde{u}_\lambda,$$

so

$$u^* \neq 0$$

and thus $u^* = u_\lambda^* \in \operatorname{int} C_+$. This proves that the map $\lambda \longmapsto u_\lambda^*$ is continuous from $(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty)$ into $C_0^1(\overline{\Omega})$. $\square$

Remark 4.3. For $\lambda = \frac{\widehat{\lambda}_1}{\eta_0}$ the auxiliary problem (A_λ) fails to have a positive solution and for this reason we do not consider the case $\lambda \longrightarrow \frac{\widehat{\lambda}_1}{\eta_0}$ since then u^* is not guaranteed to be nontrivial.

As a special case of the above result, we consider the following p-logistic equation

$$(L_\lambda) \qquad \begin{cases} -\Delta_p u(z) = \lambda\left(u(z)^{\tau-1} - u(z)^{q-1}\right) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u \geqslant 0. \end{cases}$$

We assume that $1 < \tau < p < q$ (respectively $1 < \tau = p < q$) and so we have a subdiffusive (respectively equidiffusive) p-logistic equation.

As a consequence of Proposition 4.2, we have

Corollary 4.4. *For every $\lambda > 0$ in the subdiffusive case and for every $\lambda > \widehat{\lambda}_1$ in the equidiffusive case, problem (P_λ) admits a unique positive solution $u_\lambda^* \in \operatorname{int} C_+$ and the map $\lambda \longmapsto u_\lambda^*$ is continuous from $(0, +\infty)$ (respectively $(\widehat{\lambda}_1, +\infty)$) into $C_0^1(\overline{\Omega})$.*

Remark 4.5. Uniqueness fails in the superdiffusive case $(1 < p < \tau < q)$, which exhibits bifurcation phenomena. The superdiffusive case was investigated by Dong [5], Takeuchi [13] who has the classical logistic reaction $(f(\zeta) = \zeta^{\tau-1} - \zeta^{q-1}, \zeta \geqslant 0)$ and by Gasiński-Papageorgiou [9] who had a more general reaction.

It is natural to ask what can be said about the continuity of the map $\lambda \longmapsto u_\lambda^*$ in the absence of uniqueness. Then the situation turns out to be more complicated and we have a continuity result for the semilinear problem (i.e., $p = 2$) and under conditions on the reaction $f(z, \zeta)$ which exclude from consideration the model logistic equation $f(\zeta) = \zeta^{\tau-1} - \zeta^{r-1}$, $1 < \tau < r < 2^*$ for all $\zeta \geqslant 0$. However, our result appears to be the first such continuity result for nonlinear parametric problems under general conditions on the reaction.

So, we deal with the following Dirichlet parametric problem

$$(S_\lambda) \qquad \begin{cases} -\Delta u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u \geqslant 0. \end{cases}$$

We impose stronger conditions on the reaction f.

$\underline{H_5}$ $f \colon \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(0, +\infty)$, $f(z, \cdot)$ is concave, increasing and

(i) $f'_\zeta(z, \zeta) \leqslant a(z)$ for almost all $z \in \Omega$, all $\zeta > 0$ with $a \in L^\infty(\Omega)_+$;

(ii) $\lim_{\zeta \to +\infty} \frac{f(z,\zeta)}{\zeta} = 0$ uniformly for almost all $z \in \Omega$;

(iii) $0 < \eta_0 \leqslant (f'_\zeta)_+(z, 0) = \lim_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta} \leqslant \eta_1 < +\infty$ uniformly for almost all $z \in \Omega$.

To have also strict monotonicity, we need the following stronger version of hypotheses H_5.

$\underline{H_6}$ $f \colon \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(0, +\infty)$, $f(z, \cdot)$ is concave, increasing, hypotheses H_6(i), (ii) and (iii) are the same as the corresponding hypotheses H_5(i), (ii) and (iii) and

(iv) for every $\vartheta > 0$, there exists $\gamma_\vartheta > 0$ such that $f(z, \zeta) \geqslant \gamma_\vartheta$ for almost all $z \in \Omega$, all $\zeta \geqslant \vartheta$.

Remark 4.6. As before, we assume that $f(z, \zeta) = 0$ for almost all $z \in \Omega$, all $\zeta \leqslant 0$. Therefore $(f'_\zeta)_-(z, 0) = 0$ which need not be equal to $(f'_\zeta)_+(z, 0)$. Note that the differentiability of $f(z, \cdot)$ and hypothesis H_5(i), imply that for a given $\varrho > 0$, we can find $\xi_\varrho > 0$ such that for almost all $z \in \Omega$, the function $z \longmapsto f(z, \zeta) + \xi_\varrho \zeta$ is nondecreasing on $[0, \varrho]$. This fact and hypothesis H_6(iv) imply that when conditions H_6 are in effect, we can use Proposition 3.3 and so we can say that the map $\lambda \longmapsto u^*_\lambda$ is in fact increasing from $(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty)$ into $C^1_0(\overline{\Omega})$. The following function satisfies hypotheses H_6 (for the sake of simplicity, we drop the z-dependence):

$$f(\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ \zeta & \text{if } 0 \leqslant \zeta \leqslant 1, \\ \zeta^{\frac{1}{2}} + \frac{1}{2} \ln \zeta & \text{if } 1 < \zeta. \end{cases}$$

Let $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$ and let $u_\lambda^* \in \operatorname{int} C_+$ be the extremal solution of problem (S_λ) (see Proposition 2.5). We consider the following linear parametric problem

$$(E) \qquad \begin{cases} -\Delta v(z) - \lambda f'_\zeta(z, u_\lambda^*(z)) v(z) = \widetilde{\lambda} v(z) & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

It is well known that problem (E) has a smallest eigenvalue $\widetilde{\lambda}_1$ which is simple and has eigenfunctions of constant sign (see Gasiński-Papageorgiou [11]). In what follows by v we denote the positive and L^2-normalized (i.e., $\|v\|_2 = 1$) eigenfunction corresponding to $\widetilde{\lambda}_1$. We have $v \in \operatorname{int} C_+$. All the other eigenvalues have nodal eigenfunctions.

Lemma 4.7. $\widetilde{\lambda}_1 > 0$.

Proof. We argue indirectly. So, suppose that $\widetilde{\lambda}_1 \leqslant 0$. We have

$$-\Delta v(z) = \left(\lambda f'_\zeta(z, u_\lambda^*(z)) + \widetilde{\lambda}_1\right) v(z) \quad \text{a.e. in } \Omega. \tag{33}$$

Also, recall that

$$-\Delta u_\lambda^*(z) = \lambda f(z, u_\lambda^*(z)) \quad \text{a.e. in } \Omega. \tag{34}$$

We multiply (34) with $v(z)$ and integrate over Ω. Using Green's identity, we obtain

$$\int_\Omega (\nabla v, \nabla u_\lambda^*)_{\mathbb{R}^N}\, dz = \int_\Omega \lambda f(z, u_\lambda^*) v\, dz. \tag{35}$$

Similarly, multiplying (33) by $u_\lambda^*(z)$ and integrating over Ω, we obtain

$$\int_\Omega (\nabla u_\lambda^*, \nabla v)_{\mathbb{R}^N}\, dz = \int_\Omega \left(\lambda f'_\zeta(z, u_\lambda^*) + \widetilde{\lambda}_1\right) v u_\lambda^*\, dz. \tag{36}$$

Subtracting (36) from (35), we have

$$\int_\Omega \left(\lambda f(z, u_\lambda^*) - \lambda f'_\zeta(z, u_\lambda^*) u_\lambda^*\right) v\, dz - \int_\Omega \widetilde{\lambda}_1 v u_\lambda^*\, dz = 0. \tag{37}$$

Let $\vartheta \in (\frac{\widehat{\lambda}_1}{\eta_0}, \lambda)$. In a similar way we obtain

$$\int_\Omega \left(\vartheta f(z, u_\vartheta^*) - \lambda f'_\zeta(z, u_\lambda^*) u_\vartheta^*\right) v\, dz - \int_\Omega \widetilde{\lambda}_1 v u_\vartheta^*\, dz = 0. \tag{38}$$

We subtract (37) from (38). Then

$$\int_\Omega \lambda \left(f(z, u_\vartheta^*) - f(z, u_\lambda^*) - (u_\vartheta^* - u_\lambda^*) f'_\zeta(z, u_\lambda^*)\right) v\, dz \\ + (\vartheta - \lambda) \int_\Omega f(z, u_\vartheta^*) v\, dz + \widetilde{\lambda}_1 \int_\Omega v(u_\lambda^* - u_\vartheta^*)\, dz = 0. \tag{39}$$

Since $f(z,\cdot)$ is concave for almost all $z \in \Omega$ and $v \in \operatorname{int} C_+$, we have

$$\int_\Omega \lambda \left(f(z,u_\vartheta^*) - f(z,u_\lambda^*) - (u_\vartheta^* - u_\lambda^*) f'_\zeta(z,u_\lambda^*) \right) v \, dz \leqslant 0. \tag{40}$$

Also since by hypothesis the function $f(z,\cdot)$ is increasing for almost all $z \in \Omega$, $u_\vartheta^*, v \in \operatorname{int} C_+$ and $\vartheta < \lambda$, it follows that

$$(\vartheta - \lambda) \int_\Omega f(z,u_\vartheta^*) v \, dz < 0. \tag{41}$$

Finally recall that since $\vartheta < \lambda$, we have $u_\vartheta^* \leqslant u_\lambda^*$ (respectively $u_\vartheta^* < u_\lambda^*$), see Proposition 3.2 (respectively Proposition 3.3). Also $\widetilde{\lambda}_1 \leqslant 0$. Therefore

$$\widetilde{\lambda}_1 \int_\Omega v(u_\lambda^* - u_\vartheta^*) \, dz \leqslant 0. \tag{42}$$

Returning to (39) and using (40), (41) and (42), we reach a contradiction. This proves that $\widetilde{\lambda}_1 > 0$. $\square$

Using this lemma and adapting some techniques of Cazenave-Escobedo-Pozio [2], we have the following result.

Proposition 4.8. *If hypotheses H_5 (respectively H_6) hold, then the map $\lambda \longmapsto u_\lambda^*$ is nondecreasing (respectively increasing) and continuous form $\left(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty\right)$ into $C_0^1(\overline{\Omega})$.*

Proof. From Proposition 2.5 we know that for all $\lambda > \frac{\widehat{\lambda}_1}{\eta_0}$, problem (S_λ) admits a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$. Moreover, from Proposition 3.2 (respectively 3.3), we know that the map $\lambda \longmapsto u_\lambda^*$ is nondecreasing (respectively increasing) from $\left(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty\right)$ into $C_0^1(\overline{\Omega})$.

Suppose that $\lambda_n \longrightarrow \lambda$ with $\lambda_n, \lambda > \frac{\widehat{\lambda}_1}{\eta_0}$ for all $n \geqslant 1$. First assume that $\lambda_n \nearrow \lambda$ and set

$$\overline{u}_\lambda = \lim_{n \to +\infty} u_{\lambda_n}^*.$$

This limit exists since the sequence $\{u_{\lambda_n}^*\} \subseteq C_+$ is a nondecreasing sequence. Evidently

$$\overline{u}_\lambda \leqslant u_\lambda^*. \tag{43}$$

Also, we have

$$A(u_{\lambda_n}^*) = \lambda_n N_f(u_{\lambda_n}^*) \quad \forall n \geqslant 1. \tag{44}$$

As before (see the proof of Proposition 4.2), using the nonlinear regularity theory (see Gasiński-Papageorgiou [11]), for some $\beta \in (0,1)$, we have that the sequence $\{u^*_{\lambda_n}\}_{n\geqslant 1} \subseteq C^{1,\beta}_0(\overline{\Omega})$ is bounded. Exploiting the compactness of the embedding of $C^{1,\beta}_0(\overline{\Omega})$ into $C^1_0(\overline{\Omega})$ and recalling that $u^*_{\lambda_n}(z) \longrightarrow \overline{u}_\lambda(z)$ for all $z \in \overline{\Omega}$, we have

$$u^*_{\lambda_n} \longrightarrow \overline{u}_\lambda \quad \text{in } C^1_0(\overline{\Omega}),$$

hence $\overline{u}_\lambda \in C^1_0(\overline{\Omega})$. So $\overline{u}_\lambda \in S_+(\lambda)$ (pass to the limit as $n \to +\infty$ in (44)) and thus

$$\overline{u}_\lambda = u^*_\lambda \in \operatorname{int} C_+$$

(see (43)).

Next assume that $\lambda_n \searrow \lambda$ and set

$$\underline{u}_\lambda = \lim_{n\to+\infty} u^*_{\lambda_n}.$$

Again the limit exists since the sequence $\{u^*_{\lambda_n}\}_{n\geqslant 1} \subseteq \operatorname{int} C_+$ is nonincreasing. Similarly as for $\overline{u}_\lambda$, we have that $\underline{u}_\lambda \in S_+(\lambda)$. We consider the following auxiliary Dirichlet problem

$$\begin{cases} -\Delta e(z) - \lambda f'_\zeta(z, u^*_\lambda(z))e(z) = 1 & \text{in } \Omega, \\ e|_{\partial\Omega} = 0, \ e \geqslant 0. \end{cases} \tag{45}$$

It is easily seen that the differential operator $u \longmapsto -\Delta u - \lambda f'_\zeta(\cdot, u^*_\lambda(\cdot))u$ is linear pseudomonotone from $H^1_0(\Omega)$ into $H^{-1}(\Omega) = H^1_0(\Omega)^*$ (see Gasiński-Papageorgiou [11]). Also, Lemma 4.7 implies that this operator is coercive. Therefore Theorem 3.2.52 of Gasiński-Papageorgiou [11, p. 336], implies that problem (45) admits a solution $e \in H^1_0(\Omega)$. Clearly $e \neq 0$. Multiplying (35) with $-e^- \in H^1_0(\Omega)$ and integrating over Ω, we obtain

$$\|\nabla e^-\|_2^2 + \lambda \int_\Omega f'_\zeta(z, u^*_\lambda)(e^-)^2\, dz \leqslant 0,$$

so

$$\|\nabla e^-\|_2^2 \leqslant 0$$

(recall that $f'_\zeta(z, u^*_\lambda) \geqslant 0$) and thus $e \geqslant 0$, $e \neq 0$.

Regularity theory and the maximum principle (see Vázquez [14]) imply that $e \in \operatorname{int} C_+$. Also, we claim that this solution is unique. Indeed, if $\widehat{e} \in \operatorname{int} C_+$ is another solution, then

$$-\Delta(e - \widehat{e}) = \lambda f'_\zeta(z, u^*_\lambda)(e - \widehat{e}).$$

Since $\widetilde{\lambda}_1 > 0$ (see Lemma 4.7), it follows that $e = \widehat{e}$.

We set $y = u_\lambda^* + \vartheta e$ with $\vartheta > 0$. For $\vartheta > 0$ small we will have $y \in \operatorname{int} C_+$. Then for $\delta > 0$, we have

$$\begin{aligned} & -\Delta y - (\lambda+\delta) f(z,y) \\ = & -\Delta u_\lambda^* - \vartheta \Delta e - (\lambda+\delta) f(z,y) \\ = & \lambda f(z,u_\lambda^*) + \vartheta\big(1 + \lambda f'_\zeta(z,u_\lambda^*)e\big) - (\lambda+\delta) f(z,y) \\ = & \vartheta - \delta f(z,y) - \lambda\big(f(z,y) - f(z,u_\lambda^*) - (y-u_\lambda^*) f'_\zeta(z,u_\lambda^*)\big) \end{aligned} \tag{46}$$

almost everywhere in Ω. Note that

$$f(z,y) - f(z,u_\lambda^*) - (y-u_\lambda^*) f'_\zeta(z,u_\lambda^*) = o(\vartheta)$$

(with $\frac{o(\vartheta)}{\vartheta} \longrightarrow 0$ as $\vartheta \searrow 0$). From this fact and hypothesis H_5(i), we see that we can find $\delta = \delta(\vartheta) > 0$ such that

$$-\Delta y \geqslant (\lambda+\delta) f(z,y) \quad \text{a.e. in } \Omega$$

(see (46)). Then truncating the reaction of problem $(S_{\lambda+\delta})$ at $y(z)$ and using the direct method (see for example the proof of Proposition 3.2), we obtain $u_{\lambda+\delta}^* \in S_+(\lambda+\delta) \subseteq \operatorname{int} C_+$ such that

$$u_{\lambda+\delta}^* \in [0,y],$$

where $[0,y] = \big\{u \in H_0^1(\Omega) :\ 0 \leqslant u(z) \leqslant y(z) \text{ a.e. in } \Omega\big\}$, so

$$u_{\lambda+\delta}^* \leqslant y,$$

thus

$$\underline{u}_\lambda \leqslant y$$

(setting $\delta = \delta_n = \lambda_n - \lambda > 0$ and letting $n \to +\infty$) and hence

$$\underline{u}_\lambda \leqslant u_\lambda^*$$

(letting $\vartheta \searrow 0$). Finally we get

$$\underline{u}_\lambda = u_\lambda^*$$

Therefore $u_{\lambda_n}^*(z) \longrightarrow u_\lambda^*(z)$ for all $z \in \overline{\Omega}$ and by Dini's theorem we have that $u_{\lambda_n}^* \longrightarrow u_\lambda^*$ in $C(\overline{\Omega})$. Then using the regularity theory, as before, we infer that

$$u_{\lambda_n}^* \longrightarrow u_\lambda^* \quad \text{in } C_0^1(\overline{\Omega}),$$

so the map $\lambda \longmapsto u_\lambda^*$ is continuous $\big(\frac{\widehat{\lambda}_1}{\eta_0}, +\infty\big)$ into $C_0^1(\overline{\Omega})$. □

Remark 4.9. It is an interesting open question whether we can have a similar continuity result for the nonlinear problem (P_λ). Evidently the above proof does not work in the case of the p-Laplacian. Another interesting open question is whether such a continuity result is possible for the superdiffusive logistic equation. Recall that such equations do not have unique solution and exhibit bifurcation phenomena (see Dong [5] and Gasiński-Papageorgiou [9]).

5. Nodal solutions

In this section we produce a nodal (sign changing) solution for problem (P_λ). To do this, we need to impose asymptotic conditions also on the negative semiaxis $(-\infty, 0]$ and to restrict further the range of the parameter $\lambda > 0$.

The new hypotheses on the reaction f are the following.

$\underline{H_7}$ $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for almost all $z\in\Omega$ and

(i) $|f(z,\zeta)| \leqslant a(z)(1+\zeta^{r-1})$ for almost all $z\in\Omega$, all $\zeta\in\mathbb{R}$ with $a\in L^\infty(\Omega)_+$, with $p\leqslant r<p^*$;
(ii) $\limsup_{\zeta\to\pm\infty} \frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta} \leqslant 0$ uniformly for almost all $z\in\Omega$;
(iii) $\liminf_{\zeta\to 0} \frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta} \geqslant \eta_0 > 0$ uniformly for almost all $z\in\Omega$.

In what follows, for any functional $\varphi\in C^1(W_0^{1,p}(\Omega))$, by K_φ we denote its critical set, i.e., $K_\varphi = \{u\in W_0^{1,p}(\Omega):\ \varphi'(u)=0\}$.

Proposition 5.1. *If hypotheses H_7 hold and $\lambda > \frac{\widehat{\lambda}_2}{\eta_0}$, then problem (P_λ) admits a nodal solution $y_\lambda \in C_0^1(\overline{\Omega})$.*

Proof. Since the hypotheses on $f(z,\cdot)$ are now valid on all of $\mathbb{R}$, in addition to the smallest positive solution $u_\lambda^*\in \operatorname{int} C_+$ of (P_λ), in a similar way we can produce the biggest negative solution $v_\lambda^* \in -\operatorname{int} C_+$ of (P_λ). Using these extremal constant sign solutions, we introduce the following truncation of the reaction $f(z,\cdot)$:

$$g(z,\zeta) = \begin{cases} f(z,v_\lambda^*(z)) & \text{if } \zeta < v_\lambda^*(z), \\ f(z,\zeta) & \text{if } v_\lambda^*(z)\leqslant \zeta \leqslant u_\lambda^*(z), \\ f(z,u_\lambda^*(z)) & \text{if } u_\lambda^*(z) < \zeta. \end{cases} \tag{47}$$

This is a Carathéodory function. We set

$$G(z,\zeta) = \int_0^\zeta g(z,s)\,ds.$$

Also, we consider the positive and negative truncations of $g(z,\cdot)$, namely the Carathéodory functions

$$g_\pm(z,\zeta) = g(z,\pm\zeta^\pm).$$

We set

$$G_\pm(z,\zeta) = \int_0^\zeta g_\pm(z,s)\,ds.$$

Then we introduce the C^1-functionals $\psi_\lambda, \psi_\lambda^\pm\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega G(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega),$$
$$\psi_\lambda^\pm(u) = \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega G_\pm(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Reasoning as in the proof of Proposition 2.4, we can show that

$$K_{\psi_\lambda} \subseteq [v_\lambda^*, u_\lambda^*], \qquad K_{\psi_\lambda^+} \subseteq [0, u_\lambda^*], \qquad K_{\psi_\lambda^-} \subseteq [v_\lambda^*, 0].$$

The extremality of $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ implies that

$$K_{\psi_\lambda} \subseteq [v_\lambda^*, u_\lambda^*], \qquad K_{\psi_\lambda^+} = \{0, u_\lambda^*\}, \qquad K_{\psi_\lambda^-} = \{v_\lambda^*, 0\}. \tag{48}$$

Claim. $u_\lambda^* \in \operatorname{int} C_+$ *and* $v_\lambda^* \in -\operatorname{int} C_+$ *are local minimizers of the functional* ψ_λ.

From (47) it is clear that the functional ψ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda^* \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda^+(\widetilde{u}_\lambda^*) = \inf_{u \in W_0^{1,p}(\Omega)} \psi_\lambda^+(u). \tag{49}$$

As in the proof of Proposition 2.2, using hypothesis H_7(iii) and since $u_\lambda^* \in \operatorname{int} C_+$, $v_\lambda^* \in -\operatorname{int} C_+$, we show that

$$\psi_\lambda^+(\widetilde{u}_\lambda^*) < 0 = \psi_\lambda^+(0),$$

hence $\widetilde{u}_\lambda^* \neq 0$. From (48) and (49), we have

$$\widetilde{u}_\lambda^* = u_\lambda^* \in \operatorname{int} C_+.$$

Since $\psi_\lambda^+\big|_{C_+} = \psi_\lambda\big|_{C_+}$, it follows that u_λ^* is a local $C_0^1(\overline{\Omega})$-minimizer of ψ_λ. Then Theorem 1 of García Azorero-Manfredi-Peral Alonso [8] implies that u_λ^* is a local $W_0^{1,p}(\Omega)$-minimizer of ψ_λ. Similarly for $v_\lambda^* \in -\operatorname{int} C_+$, using this time the functionals ψ_λ^-. This proves the Claim.

We may assume that $\psi_\lambda(v_\lambda^*) \leqslant \psi_\lambda(u_\lambda^*)$ (the analysis is similar if the opposite inequality holds). We assume that K_{ψ_λ} is finite (or otherwise we already have infinity of distinct nodal solutions; see (48)). By virtue of the Claim we can find $\varrho \in (0,1)$ small such that

$$\psi_\lambda(v_\lambda^*) \leqslant \psi_\lambda(u_\lambda^*) < \inf\{\psi_\lambda(u) :\ \|u - u_\lambda^*\|_{1,p} = \varrho\} = \eta_\varrho^\lambda \tag{50}$$

(see Gasiński-Papageorgiou [10]). From (47) it is clear that ψ_λ is coercive and so it satisfies the Palais-Smale condition. This fact together with (50) permit the use of the mountain pass theorem (see e.g., Gasiński-Papageorgiou [11, p. 648]). So, we can find $y_\lambda \in W_0^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\psi_\lambda} \quad \text{and} \quad \eta_\varrho^\lambda \leqslant \psi_\lambda(y_\lambda). \tag{51}$$

From (48), (50) and (51) it follows that

$$y_\lambda \in [v_\lambda^*, u_\lambda^*], \quad y_\lambda \notin \{v_\lambda^*, u_\lambda^*\}.$$

Therefore y_λ is a solution of (P_λ) (see (47)) and the nonlinear regularity theory implies that $y_\lambda \in C_0^1(\overline{\Omega})$ (see Gasiński-Papageorgiou [11]). If we show the nontriviality of y_λ, then by virtue of the extremality of v_λ^* and u_λ^*, we will have that $y_\lambda^* \in C_0^1(\overline{\Omega})$ is the desired nodal solution.

The mountain pass theorem says that

$$\psi_\lambda(y_\lambda) = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \psi_\lambda(\gamma(t)), \tag{52}$$

where $\Gamma = \{\gamma \in C([0,1]; W_0^{1,p}(\Omega)) :\ \gamma(0) = v_\lambda^*,\ \gamma(1) = u_\lambda^*\}$ (see Gasiński-Papageorgiou [11]). Then according to (52), in order to show the nontriviality of y_λ, it suffices to produce a path $\gamma_* \in \Gamma$ such that $\psi_\lambda\big|_{\gamma_*} < 0$. To this end, let

$$M = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p},$$

where

$$\partial B_1^{L^p} = \{u \in L^p(\Omega) :\ \|u\|_p = 1\} \quad \text{and} \quad M_C = M \cap C_0^1(\overline{\Omega}).$$

Evidently M_C is dense in M. Also, we introduce the following sets of paths:

$$\widehat{\Gamma} = \big\{\widehat{\gamma} \in C([-1,1]; M) :\ \widehat{\gamma}(-1) = -\widehat{u}_1,\ \widehat{\gamma}(1) = \widehat{u}_1\big\},$$
$$\widehat{\Gamma}_C = \big\{\widehat{\gamma} \in C([-1,1]; M_C) :\ \widehat{\gamma}(-1) = -\widehat{u}_1,\ \widehat{\gamma}(1) = \widehat{u}_1\big\}.$$

Then $\widehat{\Gamma}_C$ is dense in $\widehat{\Gamma}$. From Cuesta-de Figueiredo-Gossez [3] we know that

$$\widehat{\lambda}_2 = \inf_{\widehat{\gamma}\in\widehat{\Gamma}} \max_{-1\leqslant t\leqslant 1} \|\nabla\widehat{\gamma}(t)\|_p^p.$$

Exploiting the density of $\widehat{\Gamma}_C$ in $\widehat{\Gamma}$, for a given $\delta_0 > 0$, we can find $\widehat{\gamma}_0 \in \widehat{\Gamma}_C$ such that

$$\max_{-1\leqslant t\leqslant 1} \|\nabla\widehat{\gamma}_0(t)\|_p^p \leqslant \widehat{\lambda}_2 + \delta_0. \tag{53}$$

By virtue of hypothesis H_7(iii), for a given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,\zeta) \geqslant \frac{\eta_0 - \varepsilon}{p}|\zeta|^p \quad \text{for a.e. } z \in \Omega, \text{ all } |\zeta| \leqslant \delta. \tag{54}$$

Since $u_\lambda^* \in \operatorname{int} C_+$, $v_\lambda^* \in -\operatorname{int} C_+$ and $\widehat{\gamma}_0([-1,1]) \subseteq C_0^1(\overline{\Omega})$ is compact, we can find $\tau \in (0,1)$ small such that

$$\tau\widehat{\gamma}_0(t) \in [v_\lambda^*, u_\lambda^*] \quad \forall t \in [-1,1]. \tag{55}$$

Then we have

$$\begin{aligned} \psi_\lambda(\tau\widehat{\gamma}_0(t)) &= \frac{\tau^p}{p}\|\nabla\widehat{\gamma}_0(t)\|_p^p - \lambda\int_\Omega F(z,\tau\widehat{\gamma}_0(t))\,dz \\ &\leqslant \frac{\tau^p}{p}(\widehat{\lambda}_2+\delta_0) - \frac{\tau^p}{p}\lambda(\eta_0-\varepsilon) \\ &= \frac{\tau^p}{p}\left(\widehat{\lambda}_2+\delta_0+\lambda\varepsilon-\lambda\eta_0\right) \end{aligned} \tag{56}$$

(see (47), (55), (53), (54) and recall that $\|\widehat{\gamma}_0(t)\|_p = 1$ for all $t \in [-1,1]$). Since by hypothesis $\lambda > \frac{\widehat{\lambda}_2}{\eta_0}$, choosing $\varepsilon, \delta_0 > 0$ small, from (56) we have

$$\psi_\lambda(\tau\widehat{\gamma}_0(t)) < 0 \quad \forall t \in [-1,1].$$

We set $\gamma_0 = \tau\widehat{\gamma}_0$. Then γ_0 is a path in $W_0^{1,p}(\Omega$ connecting $-\tau\widehat{u}_1$ and $\tau\widehat{u}_1$ and such that

$$\psi_\lambda\big|_{\gamma_0} < 0. \tag{57}$$

Let $(\psi_\lambda^+)^0 = \{u \in W_0^{1,p}(\Omega) : \ \psi_\lambda^+(u) \leqslant 0\big\}$. From the second deformation theorem (see e.g., Gasiński-Papageorgiou [11, p. 628]), we can find a deformation $h\colon [0,1]\times((\psi_\lambda^+)^0\setminus\{0\}) \longrightarrow (\psi_\lambda^+)^0$ such that

$$h(0,\cdot) = \mathrm{id}\big|_{(\psi_\lambda^+)^0\setminus\{0\}}, \qquad h(1,\psi_\lambda^+\setminus\{0\}) = \{u_\lambda^*\}, \tag{58}$$

$$\psi_\lambda^+(h(t,u)) \leqslant \psi_\lambda^+(u) \quad \forall t \in [0,1],\ u \in (\psi_\lambda^+)\setminus\{0\} \tag{59}$$

(see (48)). Let $\gamma_+(t) = h(t,\tau\widehat{u}_1)^+$. Then γ_+ is a continuous path in $W_+ = \{u \in W_0^{1,p}(\Omega) : \ u(z) \geqslant 0 \text{ a.e. in } \Omega\}$ such that

$$\gamma_+(0) = \tau\widehat{u}_1, \qquad \gamma_+(1) = u_\lambda^*, \qquad \psi_\lambda^+\big|_{\gamma_+} < 0$$

(see (57), (58), (59)). But note that $\psi_\lambda^+\big|_{W_+} = \psi_\lambda\big|_{W_+}$. Hence

$$\psi_\lambda\big|_{\gamma_+} < 0. \tag{60}$$

In a similar fashion we produce γ_- a path in $W_0^{1,p}(\Omega)$ connecting $-\tau\widehat{u}_1$ and v_λ^* and such that

$$\psi_\lambda\big|_{\gamma_-} < 0. \tag{61}$$

We concatenate $\gamma_-, \gamma_0, \gamma_+$ and produce $\gamma_* \in \Gamma$ such that

$$\psi_\lambda\big|_{\gamma_*} < 0$$

(see (57), (60), (61)), so $y_\lambda \neq 0$ and thus $y_\lambda \in C_0^1(\overline{\Omega})$ is nodal. $\square$

Remark 5.2. It will be interesting to extend the result of the paper to equations with a reaction of the form $f(z,x,\lambda)$ with $\lambda \longmapsto f(z,x,\lambda)$ increasing. We believe that this is possible under suitable conditions on the dependence of $f(z,x,\cdot)$ on λ.

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