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Calculation of Subdifferentials for the Difference of Two Convex Functions*

E. S. Polovinkin
Department of Higher Mathematics, Moscow Institute of Physics and Technology, Institutskii pereulok 9, Dolgoprudny, Moscow region, Russia 141700
polovinkin@mail.mipt.ru

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It is shown that for some classes of functions all epiderivatives and subdifferentials of F. H. Clarke type and P. Michel – J.-P. Penot type and others coincide. Several rules of calculation of subdifferentials for the difference of two convex functions are obtained. Some examples are considered.

Keywords: Set-valued map, tangent cone, convex function, semiregular function, Lipschitz continuity, epiderivatives, subdifferentials

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1. Introduction

In the study of non-smooth continuous functions $f : X \to \mathbb{R}^1$ at a local minimum at some point $x_0 \in X$ the necessary extremum condition takes the form of inclusion $0 \in \partial_C f(x_0)$, where $\partial_C f(x_0)$ is the Clarke subdifferential ([3, 4]). In more complicated problems of mathematical control theory sometimes it is necessary to calculate subdifferentials of Clarke's type [6]. However, the calculation of these subdifferentials for non-smooth and non-convex function is not a very simple task. In this paper we obtain fairly simple formulas for calculation of various directional derivatives, and consequently subdifferentials, including Clarke subdifferential for nonsmooth functions which can be represented as the difference of two locally Lipschitz continuous convex functions.

Note that the differential calculus for functions, which include the class of functions under study, were created by V. F. Demyanov and A. M. Rubinov [5]. They introduced the concept of the quasidifferential of the function, which can be represented as a pair of convex compact sets, the first of which is the subdifferential (in the sense of convex analysis) of the first convex function,

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and the second is the hypodifferential of the second concave function. For more detailed information about the quasidifferential calculus see [5]. However, we need to calculate Clarke subdifferential for the difference of convex functions.

Note also that the brightest representative of the study in this paper class of functions is the class of *weakly convex functions*, that is the class of functions which can be represented in the following form $f(x) = f_1(x) - \alpha\|x\|^2$, where the deductible function is the norm in the second degree with a coefficient $\alpha > 0$. The subdifferentials of such functions can be easily calculated. For the given class of weakly convex functions computation of subdifferentials and directional epiderivatives reduces to calculation of the subdifferential and classical directional derivative of the first function (see, for example, [10], Appendix. Weakly and strongly convex function).

2. Set-valued maps

Let us recall some definitions of the tangent cones and derivatives of set-valued maps, based on which it will be convenient to define and explore various subdifferentials of non-convex functions.

Let X, Y be real Banach spaces. We denote by $B_r(x_0)$ the open ball of the radius $r > 0$ with the centre x_0 in the space X. We denote by $\overline{A}$ the closure of the subset $A \subset X$. Recall the notion of Minkowski sum of two sets A, B and the multiplication of a set A by a number λ.

$$A + B \doteq \{c \in X \mid c = a + b, a \in A,\, b \in B\},$$
$$\lambda A \doteq \{c \in X \mid c = \lambda a,\, a \in A\}.$$

The Minkowski difference of two sets A, B from the space X is the following set

$$A \overset{*}{-} B \doteq \{x \in X \mid x + B \subset A\}.$$

The distance function from the point $x \in X$ to the subset $A \subset X$ is given by the formula $\varrho(x, A) \doteq \inf\{\|x - a\| \mid a \in A\}$. We denote by $\mathcal{P}(Y)$ ($\mathcal{F}(Y)$) the set of all nonempty (closed) subsets of Y. For a map F: $X \to \mathcal{P}(Y)$ define its graph as follows $\operatorname{graph} F \doteq \{(x, y) \in X \times Y \mid y \in F(x)\}$. Following [5], [10], for a map F: $X \to \mathcal{P}(Y)$ at a point of its graph $z_0 \doteq (x_0, y_0) \in \overline{\operatorname{graph} F}$ we define its derivative as the map $DF(z_0)$: $X \to \mathcal{P}(Y)$, which graph is some tangent cone to the graph of the given map at the point (x_0, y_0). There are several types of tangent cones for non-convex sets. Among them should be emphasized *the upper tangent cone (otherwise referred to as a contingent cone)* $T_U(A; a)$ (see [1]), *the lower tangent cone (also called: the simple tangent cone)* $T_L(A; a)$ (see [10, 11, 9]) and *the Clarke tangent cone* $T_C(A; a)$ (see [3, 4]).

Definition 2.1. The upper tangent cone to the subset $A \subset X$ at the point $a \in \overline{A}$ is the set defined by the following formula

$$T_U(A; a) \doteq \{v \in X \mid \liminf_{\lambda \downarrow 0} \varrho(v,\ \lambda^{-1}(A - a)) = 0\}.$$

Definition 2.2. The lower tangent cone to the subset $A \subset X$ at the point $a \in \overline{A}$ is the set defined by

$$T_L(A; a) \doteq \{v \in X \mid \lim_{\lambda \downarrow 0} \varrho(v,\ \lambda^{-1}(A - a)) = 0\}.$$

Definition 2.3. The Clarke tangent cone to the subset $A \subset X$ at the point $a \in \overline{A}$ is the set defined by

$$T_C(A; a) \doteq \{v \in X \mid \lim_{\lambda \downarrow 0,\, x \to a} \varrho(v,\ \lambda^{-1}(A - x)) = 0\},$$

where x tends to a in the set A, that is $x \in A$.

Also we will need some more tangent cones which are defined, for example, in [10, 7].

Definition 2.4. The asymptotic lower tangent cone to the subset $A \subset X$ at the point $a \in \overline{A}$ is the set defined by

$$T_{AL}(A; a) \doteq T_L(A; a) \overset{*}{-} T_L(A; a).$$

Similarly, through the Minkowski difference of the upper tangent cone with itself, the asymptotic upper tangent cone (as some other tangent cones) can be determined (see, for example, [10]). However, in this paper, these cones will coincide with one of the above, so we omit them.

We recall some basic properties of these tangent cones. First of all, note that all of the above tangent cones to the convex sets coincide. For the case of non-convex sets following statement is true (see [10, 9]).

Proposition 2.5. *Cones $T_C(A; a)$, $T_{AL}(A; a)$ are convex and closed and the following inclusions hold*

$$T_C(A; a) \subset T_{AL}(A; a) \subset T_L(A; a) \subset T_U(A; a),$$

where inclusions can be strict.

Each tangent cone generates a derivative of a set-valued map as follows (see [10, 7, 2, 8]).

Definition 2.6. The M-derivative (where $M \in \{U, L, C, AL, \dots\}$) of the map $F\colon X \to \mathcal{P}(Y)$ at the point $z_0 \in \overline{\operatorname{graph} F} \subset Z \doteq X \times Y$ is the map $D_M F(z_0) : X \to \mathcal{F}(Y)$ defined by the following formula

$$D_M F(z_0)(u) \doteq \{v \in Y \mid (u, v) \in T_M(\operatorname{graph} F; z_0)\}, \quad \forall u \in X.$$

The next statement easily follows from the Proposition 2.5 (see [10, 9]).

Proposition 2.7. *For M-derivatives of the map $F\colon X \to \mathcal{P}(Y)$ at the point of its graph $z_0 \in \overline{\operatorname{graph} F} \subset Z \doteq X \times Y$ the following inclusions hold*

$$D_C F(z_0)(u) \subset D_{AL} F(z_0)(u) \subset D_L F(z_0)(u) \subset D_U F(z_0)(u),$$

where inclusions can be strict.

3. Concepts of epiderivatives and subdifferentials of function

For a function $f\colon X \to \overline{\mathbb{R}^1}$ denote by $\operatorname{dom} f \doteq \{x \in X \mid f(x) \neq \pm\infty\}$ and $\operatorname{epi} f \doteq \{(x,\alpha) \in X \times \mathbb{R} \mid \alpha \geqslant f(x),\, x \in \operatorname{dom} f\}$ its *domain* and *epigraph* respectively.

Based on the above definition of M-derivative for a set-valued map $F(x) \doteq \{\alpha \in \mathbb{R}^1 \mid \alpha \geqslant f(x)\}$ (where $x \in \operatorname{dom} f$) at any point $(x_0, f(x_0))$ we obtain the following definitions of epiderivatives (see [10]).

Definition 3.1. The M-epiderivative (where $M \in \{U,\, L,\, C,\, AL, \dots\}$) of a function $f\colon X \to \overline{\mathbb{R}^1}$ at the point $x_0 \in \operatorname{dom} f$ in the direction $u \in X$ is defined as follows

$$D^+_M f(x_0)(u) \doteq \inf\{\alpha \in \mathbb{R}^1 \mid (u,\alpha) \in T_M(\operatorname{epi} f; (x_0, f(x_0)))\}.$$

Proposition 3.2 ([10, 8]). *Let the function $f\colon X \to \mathbb{R}^1$ be Lipschitz continuous in the neighborhood of the point x_0. Then for all $u \in X$ the following formulas are true:*

$$D^+_U f(x_0)(u) = \liminf_{\lambda\downarrow 0} \lambda^{-1}(f(x_0+\lambda u) - f(x_0)), \tag{1}$$

$$D^+_L f(x_0)(u) = \limsup_{\lambda\downarrow 0} \lambda^{-1}(f(x_0+\lambda u) - f(x_0)), \tag{2}$$

$$D^+_C f(x_0)(u) = \limsup_{\substack{\lambda,x:\\ \lambda\downarrow 0, x\to x_0}} \lambda^{-1}(f(x+\lambda u) - f(x)), \tag{3}$$

$$D^+_{AL} f(x_0)(u) = \sup_{w\in X}(D^+_L f(x_0)(u+w) - D^+_L f(x_0)(w)). \tag{4}$$

Often formulas (1)–(4) are considered as definitions of the corresponding derivatives.

Additionally there is another well-known (see [13]) epiderivative of P. Michel and J.-P. Penot.

Definition 3.3. The MP-epiderivative of a Lipschitz continuous function f: $X \to \overline{\mathbb{R}^1}$ at the point $x_0 \in \operatorname{dom} f$ in the direction $u \in X$ is defined by the formula

$$D^+_{MP} f(x_0)(u) \doteq \sup_{w\in X}\{\limsup_{\lambda\downarrow 0}(\lambda^{-1}(f(x_0+\lambda(u+w)) - f(x_0+\lambda w)))\}. \tag{5}$$

The following statement is easily follows from Proposition 3.2 and Definition 3.3 (see [10, 8]).

Proposition 3.4. *For a Lipschitz continuous function the following inequalities hold*

$$+\infty \geqslant D^+_C f(x_0)(u) \geqslant D^+_{MP} f(x_0)(u) \geqslant D^+_{AL} f(x_0)(u) \geqslant -\infty,$$

where inequalities can be strict.

Definition 3.5 ([10]). The M- subdifferential of a function $f\colon X \to \overline{\mathbb{R}^1}$ at the point $x_0 \in \operatorname{dom} f$ (for any $M \in \{C, MP, AL\}$) is the subset of the dual space X^* defined by the formula

$$\partial^+_M f(x_0) \doteq \{p \in X^* \mid \langle p, x\rangle \leqslant D^+_M f(x_0)(x) \ \forall\, x \in X\}$$

The following statement obviously follows from Proposition 3.4 and Definition 3.5 (see [10, 8]).

Corollary 3.6. *For a Lipschitz continuous function the following inclusions hold*

$$\partial^+_C f(x_0) \supset \partial^+_{MP} f(x_0) \supset \partial^+_{AL} f(x_0),$$

where inclusions can be strict.

Proposition 3.7 ([8]). *Let $f\colon X \to \mathbb{R}^1$ be a Lipschitz continuous function in the neighborhood $B_r(x_0)$ with a Lipschitz constant $l > 0$. Then, for any $M \in \{C,\, MP, AL\}$, the function $u \to D^+_M f(x_0)(u)$ is positively homogeneous, convex and finite for all $u \in X$ and it also is Lipschitz continuous on the space X with a Lipschitz constant $l > 0$.*

As usual, the classical directional derivative of a function f at the point x_0 in the direction $u \in X$ is defined by the following expression

$$f'(x_0, u) = \lim_{\lambda \downarrow 0} \lambda^{-1}(f(x_0 + \lambda u) - f(x_0)).$$

For a convex function $f : X \to \mathbb{R}^1$, its subdifferential (in the sense of convex analysis) at the point x_0 is the following set

$$\partial f(x_0) \doteq \{p \in X^* \mid f(x) - f(x_0) \geqslant \langle p, x - x_0\rangle \ \forall x \in X\}.$$

Note that if $f : B_r(x_0) \to \mathbb{R}^1$ is a convex and continuous function, then its epigraph is a convex closed subset and all defined above tangent cones to the epigraph are convex and they coincide. Therefore the Clarke subdifferential of the function at the point x_0 coincides with the subdifferential in the sense of convex analysis, and its Clarke epiderivative $D^+_C f(x_0)(\cdot)$ coincides with the classical directional derivative $f\,'(x_0, \cdot)$ (see, for example, [4]).

4. The class of functions representable as a difference of two convex functions

Lemma 4.1. *Let $f\colon X \to \mathbb{R}^1$ be a locally Lipschitz continuous at the point $x_0 \in \operatorname{dom} f$ function. Suppose that for all vectors $u \in X$ there exist finite classical directional derivatives at the point x_0, that is*

$$f'(x_0, u) = D^+_L\ f(x_0)(u) = D^+_U\ f(x_0)(u), \quad \forall u \in X.$$

Then the epiderivatives $D^+_M\ f(x_0)(u)$ *for* $M \in \{MP, AL\}$ *coincide for all* $u \in X$*, and the following formula holds*

$$D^+_M f(x_0)(u) = \sup_{w\in X}(f'(x_0, u+w) - f'(x_0, w)). \tag{6}$$

Proof. By Proposition 3.2 and conditions of the lemma we obtain

$$D^+_{AL} f(x_0)(u) = \sup_{w\in X}(f'(x_0, u+w) - f'(x_0, w)), \quad \forall u \in X.$$

Using the properties of limit we have

$$\begin{aligned} f'(x_0, u+w) - f'(x_0, w) = & \lim_{\lambda\downarrow 0} \lambda^{-1}(f(x_0 + \lambda(u+w)) - f(x_0)) \\ & - \lim_{\lambda\downarrow 0} \lambda^{-1}(f(x_0 + \lambda w) - f(x_0)) \\ = & \lim_{\lambda\downarrow 0} \lambda^{-1}(f(x_0 + \lambda(u+w)) - f(x_0 + \lambda w)). \end{aligned}$$

Calculating supremum over $w \in X$ in the last equality, by definition of the MP-epiderivative (5) we obtain the following equality $D^+_{AL} f(x_0)(u) = D^+_{MP} f(x_0)(u)$.
□

Theorem 4.2. *Let* $f\colon B_r(x_0) \to \mathbb{R}^1$ *be a function such that there exist two continuous convex functions* $f_1, f_2 : B_r(x_0) \to \mathbb{R}^1$ *that* $f(x) = f_1(x) - f_2(x)$ *for all* $x \in B_r(x_0)$ *and for which, for some* $u \in X$*, one of the functions* f_k*,* $k \in \overline{1,2}$*, satisfies the following condition* $f'_k(x_0, u) + f'_k(x_0, -u) = 0$ *(for example this function is Gateaux differentiable at the point* x_0*). Then, for such* $u \in X$ *and for any* $M \in \{C, MP, AL\}$*, the epiderivatives* $D^+_M\ f(x_0)(u)$ *coincide and the following formula is true*

$$D^+_M f(x_0)(u) = f'_1(x_0, u) + f'_2(x_0, -u). \tag{7}$$

Proof. From the properties of convex function it follows that functions f_1, f_2 have finite classical directional derivatives and for all $u \in X$ equalities hold $f'(x_0, u) = f'_1(x_0, u) - f'_2(x_0, u)$. By Lemma 4.1 for all $M \in \{AL, MP\}$ the following formula holds

$$D^+_M f(x_0)(u) = \sup_{w\in X}(f'_1(x_0, u+w) - f'_2(x_0, u+w) - f'_1(x_0, w) + f'_2(x_0, w)).$$

By the convexity and positive homogeneity of functions $f'_k(x_0, \cdot)$, $k \in \overline{1,2}$, we obtain two inequalities

$$f'_1(x_0, u+w) \leqslant f'_1(x_0, u) + f'_1(x_0, w), \tag{8}$$

$$f'_2(x_0, w) \leqslant f'_2(x_0, u+w) + f'_2(x_0, -u), \tag{9}$$

from the last inequalities we have

$$D^+_{AL}f(x_0)(u) \leqslant f_1'(x_0,u) + f_2'(x_0,-u).$$

To show the last inequality is an equality, it is enough that for some w two inequalities (8)–(9) become equalities. If condition $f_1'(x_0,u) + f_1'(x_0,-u) = 0$ holds we choose $w = -u$, and if condition $f_2'(x_0,u) + f_2'(x_0,-u) = 0$ holds we choose $w = 0$.

Let us show that in this case Clarke epiderivative and Michel-Penot epiderivative coincide. From the definition of Clarke epiderivative we obtain that

$$\begin{aligned} D^+_C f(x_0)(u) = &\limsup_{\lambda\downarrow 0,\, x\to x_0} (\lambda^{-1}(f_1(x+\lambda u) - f_1(x) - f_2(x+\lambda u) + f_2(x))) \\ \leqslant &\limsup_{\lambda\downarrow 0,\, x\to x_0} \left(\lambda^{-1}(f_1(x+\lambda u) - f_1(x))\right) \\ &\quad + \limsup_{\lambda\downarrow 0,\, x\to x_0} \left(\lambda^{-1}(f_2(x) - f_2(x+\lambda u))\right). \end{aligned}$$

By the properties of Clarke epiderivative for a Lipschitz continuous convex function the following equality holds $D^+_C f_1(x_0)(u) = f_1'(x_0,u)$. Similarly, making a replacement $y = x + \lambda u$, we obtain

$$\begin{aligned} &\limsup_{\lambda\downarrow 0,\, x\to x_0} \lambda^{-1}(f_2(x) - f_2(x+\lambda u)) \\ &= \lim_{\delta\downarrow 0} \sup_{\|x-x_0\|<\delta} \sup_{\lambda\in(0,\delta)} \lambda^{-1}(f_2(x) - f_2(x+\lambda u)) \\ &= \lim_{\delta\downarrow 0} \sup_{\|y-x_0\|<\delta(1+\|u\|)} \sup_{\lambda\in(0,\delta)} \lambda^{-1}(f_2(y-\lambda u) - f_2(y)) \\ &= D^+_C f_2(x_0)(-u) = f_2'(x_0,-u). \end{aligned}$$

Finally we have the inequality

$$D^+_C f(x_0)(u) \leqslant f_1'(x_0,u) + f_2'(x_0,-u) = D^+_{AL} f(x_0)(u),$$

which together with the opposite inequality, that always holds (see Proposition 3.4), gives the required equality. □

Here is one more criterion for coincidence of all epiderivatives of a function which can be represented as a difference of two convex functions.

Recall that a function $f : X \to \mathbb{R}^1$ is called *positively homogeneous*, if for any $x \in X$ and any number $\lambda \geqslant 0$ the equality holds $f(\lambda x) = \lambda f(x)$. Obviously such a function has classical directional derivative at zero and the following formula is true

$$f'(0,u) = f(u), \quad \forall u \in X. \tag{10}$$

Theorem 4.3. *Let $f : X \to \mathbb{R}^1$ be a Lipschitz continuous and positively homogeneous (not necessarily convex) function. Then, for all $M \in \{C, MP, AL\}$, the epiderivatives $D^+_M\, f(0)(u)$ coincide and next formula holds true*

$$D^+_M f(0)(u) = \sup_{w\in X}(f(u+w) - f(w)), \quad \forall u \in X. \tag{11}$$

Proof. By the definition of Clarke derivative, from the positive homogeneity of the function f and equalities (3), (10) and (6) we obtain that

$$\begin{aligned}
&D^+_C f(0)(u)\\
&= \lim_{\delta\downarrow 0} \sup_{\lambda\in(0,\delta)} \sup_{\|x\|\leqslant\delta} \lambda^{-1}(f(x+\lambda u) - f(x))\\
&= \lim_{\delta\downarrow 0} \sup_{\lambda\in(0,\delta)} \sup_{\|w\|\leqslant\delta/\lambda} (f(u+w) - f(w))\\
&\leqslant \sup_{w\in X}(f(u+w) - f(w)) = \sup_{w\in X}(f'(0,u+w) - f'(0,w)) = D^+_{AL} f(0)(u).
\end{aligned}$$

Together with the opposite inequality (see Proposition 3.4) it gives equality (11). $\square$

Corollary 4.4. *Let $f_k : B_r(x_0) \to \mathbb{R}^1$, $k \in \overline{1,2}$ be convex bounded functions. Define a function $f(x) \doteq f_1(x) - f_2(x)$ for $x \in B_r(x_0)$ and functions $g_k(x) \doteq f_k(x_0) + f'_k(x_0, x - x_0)$ and $g(x) \doteq g_1(x) - g_2(x)$ for $x \in X$. Then all M-epiderivatives (where $M \in \{C, MP, AL\}$) of the function g at the point x_0 coincide, and the formula holds*

$$D^+_M g(x_0)(u) = \sup_{w\in X}(f'(x_0, u+w) - f'(x_0, w)), \quad \forall u \in X.$$

Proof. The corollary is obvious by Theorem 4.3, the positive homogeneity of the function $h(x) \doteq g(x_0 + x) - f_1(x_0) + f_2(x_0)$, and equality $g'(x_0, u) = f'(x_0, u) = h(u)$ for all $u \in X$. $\square$

In the Euclidean space $\mathbb{R}^n$ formulas (6) and (11) can be simplified which allow us to consider various examples of finding epiderivatives of the difference of two convex functions.

Recall that by Rademacher's theorem (see [12, Chapter XI, §4]) for any Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^1$ there exists a measurable Lebesgue full set $B_f \subset \mathbb{R}^n$, where the function f is differentiable. Furthermore as a consequence of Clarke's theorem about representation of Clarke subdifferential as a convex hull of limit points of gradients (see [4, Theorem 2.5.1 and Corollary]) we have an equality

$$D^+_C f(x)(u) = \limsup_{y\to x,\, y\in B_f} \langle f'(y), u\rangle, \quad \forall x, u \in \mathbb{R}^n. \tag{12}$$

Based on this equation we obtain

Theorem 4.5. *Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be a Lipschitz continuous and positively homogeneous function. Let $B_f \subset \mathbb{R}^n$ be a measurable Lebesgue full set where the function f is differentiable. Then, for all $M \in \{C, MP, AL\}$, the following formula is true*

$$D^+_M f(0)(u) = \sup_{z\in B_f} \langle f'(z), u\rangle, \quad \forall u \in \mathbb{R}^n. \tag{13}$$

Proof. By positive homogeneity of the function f it follows that $\lambda y \in B_f$ for all $y \in B_f$ and any number $\lambda > 0$ (as equality $f'(y) = f'(\lambda y)$ holds). That is $\overline{B_f \cap \partial B_1(0)} = \partial B_1(0)$. By Theorem 4.3, equality (12) and equality (11) we obtain

$$\begin{aligned} D^+_M f(0)(u) = D^+_C f(0)(u) &= \limsup_{y\to 0,\, y\in B_f} \left\langle f'\left(\frac{y}{\|y\|}\right), u\right\rangle \\ &\leqslant \sup_{z\in\partial B_1(0)\cap B_f} \langle f'(z), u\rangle = \sup_{z\in B_f} \langle f'(z), u\rangle. \end{aligned}$$

To prove the opposite inequality denote $\alpha \doteq \sup_{z\in B_f}\langle f'(z), u\rangle$. It is obvious that $\alpha \leqslant L\|u\|$ (where L is a lipschitz constant for the function f) and by definition of supremum there exists such a sequence $\{x_k\} \subset \partial B_1(0) \cap B_f$ that $\lim_{k\to\infty}\langle f'(x_k), u\rangle = \alpha$. For a sequence $y_k \doteq \frac{x_k}{k}$, which converges to zero, we have $\lim_{y_k\to 0}\langle f'(y_k), u\rangle = \alpha$, that is $D^+_C f(0)(u) = \limsup_{y\to 0,\, y\in B_f}\langle f'(y), u\rangle \geqslant \alpha$. □

5. Semiregular functions

This section continues the investigation of directional epiderivatives and subdifferentials of functions defined on a Banach space X. For a continuous function $f : B_r(x_0) \to \mathbb{R}^1$, which has finite directional derivatives at the point $x_0 \in X$, we define functions g and φ by formulas

$$g(x) \doteq f(x_0) + f'(x_0, x - x_0), \qquad \varphi(x) \doteq f(x) - g(x), \tag{14}$$

that is the function f in the neighborhood $B_r(x_0)$ of the point x_0 is represented as a sum of its quasilinear part g and a remainder φ. It is obvious that the functions g and φ at the point x_0 have classical directional derivatives, and besides

$$g'(x_0, u) = f'(x_0, u), \qquad \varphi'(x_0, u) = 0 \quad \forall u \in X.$$

hence by Proposition 3.4, Lemma 4.1 and Theorem 4.3 for all $u \in X$ the following inequlities hold

$$D^+_C f(x_0)(u) \geqslant D^+_{AL} f(x_0)(u) = D^+_{AL} g(x_0)(u) = D^+_C g(x_0)(u) \geqslant f'(x_0, u),$$

$$D^+_{MP}\varphi(x_0)(u) = 0,$$

from which, in particular, the inequality follows $0 \leqslant D^+_C\varphi(x_0)(u)$ for all $u \in X$, which is equivalent to the inclusion $0 \in \partial^+_C\varphi(x_0)$.

Definition 5.1. Let X be a real Banach space. A continuous function $f: B_r(x_0) \to \mathbb{R}^1$ is called *semiregular at the point* $x_0 \in X$, if it has finite directional derivatives along any vector at the point x_0, corresponding function φ defined by (14) is Lipschitz continuous in some neighborhood of the point x_0 and the equality is true $\partial_C^+\varphi(x_0) = \{0\}$, that is $D_C^+\varphi(x_0)(u) = 0$ for all $u \in X$.

Here are some classes of semiregular functions.

Any positively homogeneous Lipschitz continuous function is semiregular at the point $0 \in X$, as $\varphi(u) = 0$.

A function which has finite directional derivatives along any vector at the point x_0, and for which corresponding function φ from (14) is convex and bounded in some neighborhood of the point x_0, is semiregular.

A convex function f is semiregular at the point x_0, if the equalities holds $f'(x_0, u) + f'(x_0, -u) = 0$ for all $u \in X$, since in this case we have an estimation

$$\begin{aligned} 0 &\leqslant D_C^+\varphi(x_0)(u) \leqslant D_C^+ f(x_0)(u) + D_C^+(-g)(x_0)(u) \\ &= f'(x_0, u) + g'(x_0, -u) = f'(x_0, u) + f'(x_0, -u) = 0. \end{aligned}$$

In particular, every convex Gateaux differentiable at the point x_0 function f is semiregular at this point.

At the same time non-convex differentiable function can be not semiregular. For example, function $f: \mathbb{R}^1 \to \mathbb{R}^1$ of the form $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ is differentiable, but it is not semiregular at zero. Here $f'(0, u) = 0$, $\varphi(x) = f(x)$ and $D_C^+\varphi(0)(u) = |u|$ for all $u \in \mathbb{R}^1$.

For non-convex function $f: X \to \mathbb{R}^1$ to be semiregular at the point x_0 it is sufficient that it is strictly differentiable at the point $x_0 \in X$. Let us show it. From the strict differentiability of the function $f: X \to \mathbb{R}^1$ at the point x_0 it follows that f is Lipschitz continuous in some neighborhood of the point x_0 and there exists such $\zeta \in X^*$ that $D_C^+ f(x_0)(u) = \langle \zeta, u \rangle$ for all $u \in X$ (see [4, Proposition 2.2.1]). The last equality implies that the corresponding function φ from (14) is Lipschitz continuous in some neighborhood of the point x_0 (as a difference of two Lipschitz continuous functions), and $D_C^+\varphi(x_0)(u) = 0$ for all $u \in X$. Indeed, since f is strictly differentiable then by Corollary 1 to Proposition 2.3.3 from [4] the following equality holds $D_C^+\varphi(x_0)(u) = D_C^+ f(x_0)(u) + D_C^+(-g)(x_0)(u) = \langle \zeta, u \rangle + \langle -\zeta, u \rangle = 0$.

Further in Example 6.3 we show that not every Lipschitz continuous function, which is regular by Clarke at some point, is semiregular at this point.

The following theorem is a generalization of Theorem 4.2 and Theorem 4.3.

Theorem 5.2. *Let a function $f\colon X \to \mathbb{R}^1$ be a difference of two convex semiregular at the point x_0 functions f_1 and f_2, that is $f = f_1 - f_2$. Then* *all epiderivatives coincide, that is* $\forall M \in \{C, MP, AL\}$

$$D_M^+ f(x_0)(u) = \sup_{w \in X}(f'(x_0, u + w) - f'(x_0, w)), \quad \forall u \in X,$$

which is equivalent to coincidence of all subdifferentials $\partial_C^+ f(x_0) = ... = \partial_{AL}^+ f(x_0)$.

Proof. For any function f_k we define functions g_k and φ_k by formulas (14). Also we define functions $g \doteq g_1 - g_2$ and $\varphi \doteq \varphi_1 - \varphi_2$. Then the following equality holds true $f = g + \varphi$, besides $f'(x_0, u) = g'(x_0, u)$ and $\varphi'(x_0, u) = 0$ for all $u \in X$. By Corollary 4.4 for all $u \in X$ the expressions $D_{AL}^+ g(x_0)(u) = D_{AL}^+ f(x_0)(u)$ and $D_{AL}^+ \varphi(x_0)(u) = 0 \leqslant D_C^+ \varphi(x_0)(u)$ hold true. By the properties of Clarke derivative for the function $\varphi = \varphi_1 + (-\varphi_2)$ and by semiregularity of functions f_1 and f_2 we obtain

$$\begin{aligned} D_C^+ \varphi(x_0)(u) &\leqslant D_C^+ \varphi_1(x_0)(u) + D_C^+(-\varphi_2)(x_0)(u) \\ &= D_C^+ \varphi_1(x_0)(u) + D_C^+ \varphi_2(x_0)(-u) = 0, \end{aligned}$$

that is $D_C^+ \varphi(x_0)(u) = 0$ for all $u \in X$. Hence by the properties of Clarke derivative for a sum of functions we obtain

$$\begin{aligned} D_C^+ f(x_0)(u) &\leqslant D_C^+ g(x_0)(u) + D_C^+ \varphi(x_0)(u) \\ &= D_{AL}^+ g(x_0)(u) + 0 = D_{AL}^+ f(x_0)(u) \leqslant D_C^+ f(x_0)(u), \end{aligned}$$

which together with Lemma 4.1, applied to the function f, completes the proof. □

Note that Theorem 5.2 is a consequence of the following theorem.

Theorem 5.3. *Let* $f : B_r(x_0) \to \mathbb{R}^1$ *be a semiregular at the point* x_0 *function. Then all epiderivatives coincide, that is* $\forall M \in \{C, MP, AL\}$

$$D_M^+ f(x_0)(u) = \sup_{w \in X}(f'(x_0, u + w) - f'(x_0, w)), \quad \forall u \in X,$$

which is equivalent to coincidence of all subdifferentials $\partial_C^+ f(x_0) = \partial_{MP}^+ f(x_0) = \partial_{AL}^+ f(x_0)$.

Proof. For the function $f : B_r(x_0) \to \mathbb{R}^1$ by the definition 9 we define functions g and φ by formulas (14). By [4, Proposition 2.2.4] the function φ is strictly differentiable at the point x_0. By the Corollary 1 to Proposition 2.3.3 from [4] the following equality holds

$$D_C^+ f(x_0)(u) = D_C^+ g(x_0)(u) + D_C^+ \varphi(x_0)(u) = D_C^+ g(x_0)(u), \quad \forall u \in X.$$

By Theorem 4.3 we have $D_C^+ g(x_0)(u) = D_{AL}^+ g(x_0)(u)$. In turn, since $g'(x_0, u) = f'(x_0, u)$, then $D_M^+ g(x_0)(u) = D_M^+ f(x_0)(u)$ for any $M \in \{MP, AL\}$, which implies the required equality. □

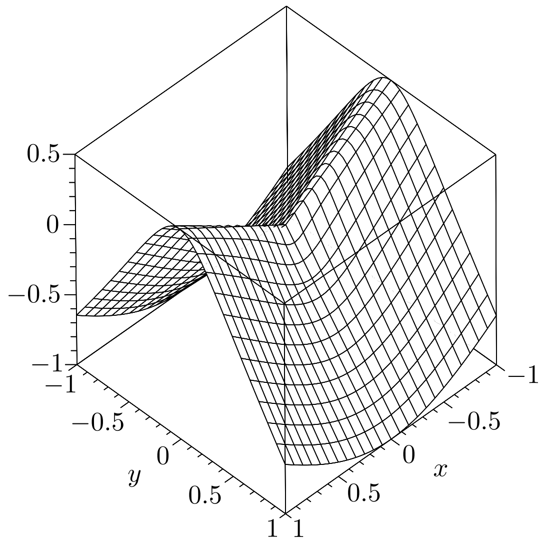

Figure 6.1: Graph of the function $f(x)$ in Example 6.1.

6. Examples of the calculation of subdifferentials

Example 6.1. Let a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ be defined by the formula

$$f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} - \sqrt{(ax_1)^2 + (bx_2)^2},$$

where numbers a, b are such that $0 < a \leqslant 1$, $b \geqslant 1$, $a \neq b$. Our purpose is to find directional epiderivatives and subdifferentials of the function f at the point 0.

This function is positively homogeneous and non-convex. To calculate directional epiderivatives we will use formula (13), for which we calculate the gradient $f'(x)$ of the function f at the point $x \neq 0$. We obtain that $f'(x) = (f_1'(x), f_2'(x))$, where

$$f_1'(x_1, x_2) = -\frac{x_1}{\sqrt{x_1^2 + x_2^2}} + \frac{a^2 x_1}{\sqrt{(ax_1)^2 + (bx_2)^2}},$$

$$f_2'(x_1, x_2) = -\frac{x_2}{\sqrt{x_1^2 + x_2^2}} + \frac{b^2 x_2}{\sqrt{(ax_1)^2 + (bx_2)^2}}.$$

From positive homogeneity of functions $f_k'(x_1, x_2)$ and by formula (13) we get

$$D_M^+ f(0)(u) = \sup_{z \in \mathbb{R}^1_+} (\varphi_1(z)|u_1| + \varphi_2(z)|u_2|),$$

where $\varphi_k(z) \doteq |f_k'(1, \sqrt{z})|$, and $z \doteq (\frac{x_2}{x_1})^2$. To calculate the supremum we find extreme points of functions $\varphi_1(z)$ and $\varphi_2(z)$. The equality

$$\varphi_1'(z) = -\frac{1}{(\sqrt{1+z})^3} + \frac{a^2 b^2}{(\sqrt{a^2 + b^2 z})^3} = 0$$

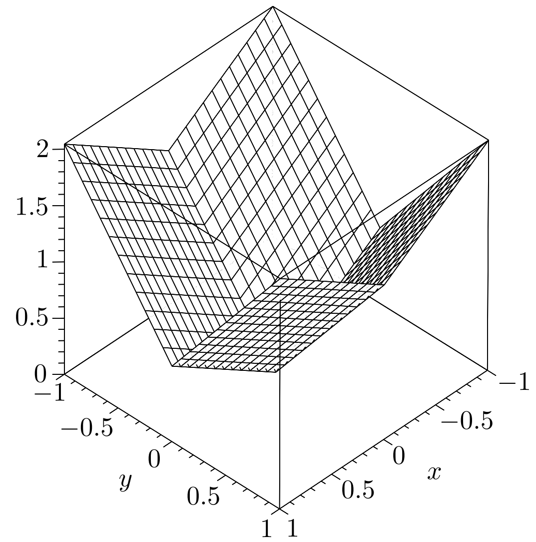

Figure 6.2: Graph of the derivative $D^+_M f(0)(u)$ in Example 6.1.

holds at the unique point $z_0 = \frac{(ab)^{\frac{4}{3}}-a^2}{b^2-(ab)^{\frac{4}{3}}}$. Similarly we obtain that the equality $\varphi_2'(z) = 0$ holds at the same point z_0. Finally we get $D^+_M f(0)(u) = \varphi_1(z_0)|u_1| + \varphi_2(z_0)|u_2|$, that is

$$D^+_M f(0)(u) = c|u_1| + d|u_2|, \quad \text{where } c = \frac{(b^{\frac{2}{3}} - a^{\frac{4}{3}})^{\frac{3}{2}}}{(b^2-a^2)^{\frac{1}{2}}},\ d = \frac{(b^{\frac{4}{3}} - a^{\frac{2}{3}})^{\frac{3}{2}}}{(b^2-a^2)^{\frac{1}{2}}}.$$

Hence by definition of subdifferentials we obtain that the M-subdifferential of the function f at zero is a quadrangle of the following form

$$\partial^+_M f(0) = \mathrm{co}\{(c,d); (-c,d); (c,-d); (-c,-d)\} \quad \forall M \in \{C, MP, AL\}.$$

Example 6.2. Let a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ be defined by the formula

$$f(x_1, x_2) = 6 \cdot \max\{|x_1|; |x_2|\} - 5 \cdot \sqrt{x_1^2 + x_2^2}.$$

Our goal is to find directional epiderivatives and subdifferentials of the function f at the point 0.

This function is also positively homogeneous and non-convex. To calculate directional epiderivatives we will use formula (13). Let us calculate the gradient of the function f at the points where it exists, that is for $x \doteq (x_1, x_2) \neq 0$ and $|x_1| \neq |x_2|$. We obtain that

$$f'(x_1,x_2) = \left(6\,\mathrm{sign}\{x_1\} - \frac{5x_1}{\sqrt{x_1^2+x_2^2}}; \frac{-5x_2}{\sqrt{x_1^2+x_2^2}}\right), \quad \text{for } |x_1| > |x_2|,$$

$$f'(x_1,x_2) = \left(\frac{-5x_1}{\sqrt{x_1^2+x_2^2}}; 6\,\mathrm{sign}\{x_2\} - \frac{5x_2}{\sqrt{x_1^2+x_2^2}}\right), \quad \text{for } |x_1| < |x_2|.$$

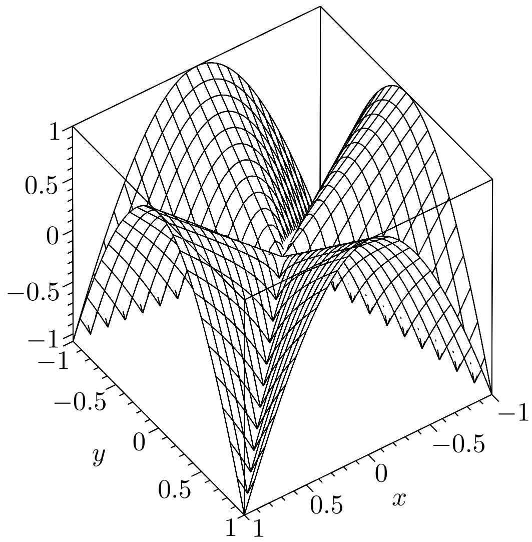

Figure 6.3: Graph of the function $f(x)$ in Example 6.2.

Let $z \doteq \frac{|x_2|}{|x_1|}$. By formula (13) and the fact that the function f is even we obtain that for all $u \neq 0$

$$D_M^+ f(0)(u) = \max\left(\sup_{0<z<1} h_1(z);\ \sup_{1<z<+\infty} h_2(z)\right), \tag{15}$$

where

$$h_1(z) \doteq \left(6 - \frac{5}{\sqrt{1+z^2}}\right)|u_1| + \frac{5z}{\sqrt{1+z^2}}|u_2|, \quad \text{for } 0 \leqslant z \leqslant 1,$$

$$h_2(z) \doteq \frac{5}{\sqrt{1+z^2}}|u_1| + \left(6 - \frac{5z}{\sqrt{1+z^2}}\right)|u_2|, \quad \text{for } z \geqslant 1.$$

It is easy to see that $h_1'(z) > 0$ and $h_2'(z) < 0$ for all admissible z, from which we obtain that

$$\sup_{0<z<1} h_1(z) = h_1(1) = \left(6 - \frac{5}{\sqrt{2}}\right)|u_1| + \frac{5}{\sqrt{2}}|u_2|,$$

$$\sup_{z>1} h_2(z) = h_2(1) = \frac{5}{\sqrt{2}}|u_1| + \left(6 - \frac{5}{\sqrt{2}}\right)|u_2|.$$

Let us define numbers $a \doteq 6 - \frac{5}{\sqrt{2}}$ and $b \doteq \frac{5}{\sqrt{2}}$. Since $h_1(1) \geqslant h_2(1)$ if and only if $|u_1| \leqslant |u_2|$, from the last formulas and formula (15) we get

$$D_M^+ f(0)(u) = a \cdot \min(|u_1|; |u_2|) + b \cdot \max(|u_1|; |u_2|).$$

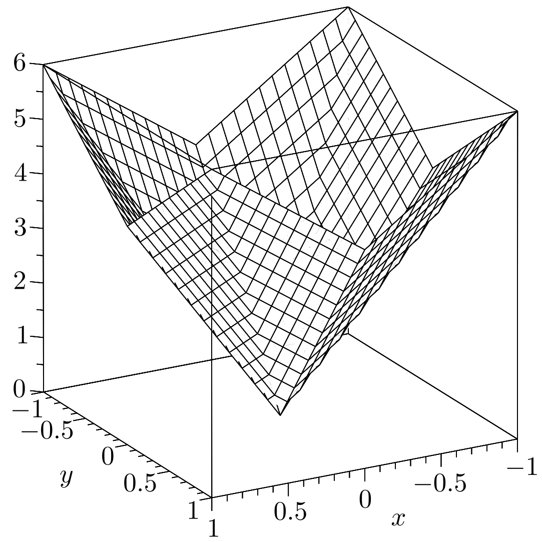

Figure 6.4: Graph of the derivative $D^+_M f(0)(u)$ in Example 6.2.

Hence by definition of subdifferentials we obtain that for all $M \in \{C, MP, AL\}$ subdifferential of the function f at zero is an octagon of the following type

$$\begin{aligned}&\partial^+_M f(0)\\ =\ &\mathrm{co}\{(a,b);(a,-b);(-a,b);(-a,-b);(b,a);(b,-a);(-b,a);(-b,-a)\}.\end{aligned}$$

Let us consider the example constructed by G. M. Ivanov which shows that even in the case of two-dimensional space $X = \mathbb{R}^2$ there exist convex bounded functions which are not semiregular at some points.

Example 6.3. Let a function $f : P \to \mathbb{R}^1$ be defined on the set $P \doteq \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_2| \leqslant 1\}$ by the following formula

$$f(x_1, x_2) \doteq \max\{|x_1| - \sqrt{1 - x_2^2}; 0\}.$$

The function is convex as a maximum of two convex functions.

Let $x_0 \doteq (1, 0)$. Let us consider the function $\varphi : P \to \mathbb{R}^1$, defined by formula (14), that is

$$\varphi(x) \doteq f(x) - f(x_0) - f'(x_0, x - x_0).$$

Let $v \doteq (1, 0)$. By construction it is obvious, that $\varphi'(x_0, v) = 0$, that is $D^+_{MP}\varphi(x_0)(v) = 0$. We will show that the Clarke epiderivative of the function φ at the point x_0 in the direction v is not zero, which implies that considered function f is not semiregular at the point x_0. Calculating the Clarke

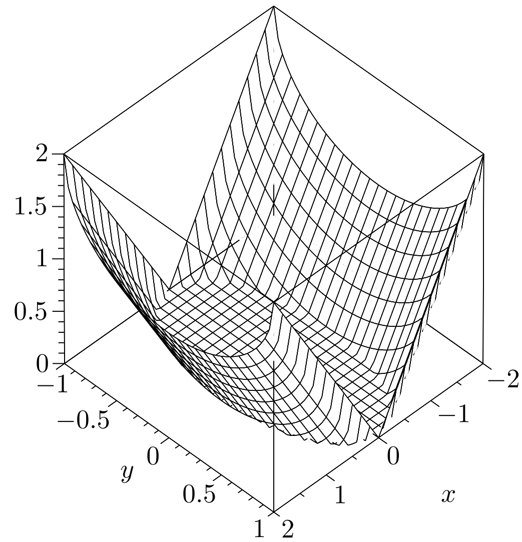

Figure 6.5: Graph of the function $f(x)$ in Example 6.3.

epiderivative we obtain

$$D_C^{+}\varphi(x_0)(v) = \limsup_{y\to x_0,\, t\downarrow 0}\left(\frac{\varphi(y+tv)-\varphi(y)}{t}\right)$$
$$= \limsup_{y\to x_0,\, t\downarrow 0}\left(\frac{f(y+tv)-f(y)}{t} - \frac{f'(x_0, y+tv-x_0)-f'(x_0, y-x_0)}{t}\right). \quad (16)$$

Note that $f(x)=0$ for all $x\in B_1(0)$. That is why for all u such that $\langle x_0, u-x_0\rangle < 0$ the following equality holds $f'(x_0, u-x_0)=0$. Define the sequence $\{y_k\}$ as follows

$$y_k \doteq \left(1-\frac{1}{2k^2},\ \sqrt{1-\left(1-\frac{1}{2k^2}\right)^2}\right).$$

It is obvious that $\|y_k\|=1$ and $\|y_k - x_0\| = \frac{1}{k}$ for all $k\in\mathbb{N}$. Also it is obvious that

for all $k\in\mathbb{N}$ for $t_k \doteq \frac{1}{4k^2}$ the following inequality holds true $\langle x_0, y_k + t_k v - x_0\rangle < 0$. Therefore for any $k\in\mathbb{N}$ we obtain the equalities

$$f'(x_0, y_k + t_k\, v - x_0) = 0; \qquad f'(x_0, y_k - x_0) = 0;$$
$$\frac{f(y_k + t_k\, v) - f(y_k)}{t_k} = 1.$$

From this and formula (16) we get

$$D_C^{+}\varphi(x_0)(v) \geqslant \lim_{k\to\infty}\frac{\varphi(y_k + t_k\, v)-\varphi(y_k)}{t_k} = 1 > D_{MP}^{+}\varphi(x_0)(v) = 0.$$

This proves that the function f is not semiregular at the point $x_0 = (1, 0)$.

It is easy to check that the function f is not semiregular at any point on the unit circle $\partial B_1(0)$, excluding the points $(0, 1)$ and $(0, -1)$.

Remark. Example 6.3 shows that the condition of semiregularity in Theorem 5.2 is important. To see that it suffices to consider two convex functions $f_1 \doteq f$ and $f_2(x) \doteq f(x_0) + f'(x_0, x - x_0)$ (where f is from Example 6.3). Then for the function $\varphi(x) = f_1(x) - f_2(x)$, which is the difference of two convex functions and one of which is not semiregular, its Clarke epiderivative (subdifferential) at the point x_0 does not coincide with Michel-Penot epiderivative (subdifferential) at this point.

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