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A Note on n-Subhomogeneity of Periodic Extension of Convex Functions

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We prove that the T-periodic extension of a convex function $f_1 : [0;T[\to [0;+\infty[$, is n-subhomogeneous if and only if $A = \lim\limits_{x\to 0^+} f_1(x) \leq n f_1(k\frac{T}{n})$ and $B = \lim\limits_{x\to T^-} f_1(x) \leq n f_1(k\frac{T}{n})$ for every $k = 1, 2, ..., n-1, (n \geq 2)$.

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1. Introduction

The notion of subhomogeneous functions is known at least with Losonczi in [5] and used since then frequently by many authors to prove different results in both branches of mathematics and economics. In mathematics, for example, Matkowski, in [6], examines conditions under which the functional P_Φ, which serves to generalize the Minkowski inequality, is subhomogeneous; Losonczi in [4] proves that the integral f-mean of a function is positive subhomogeneous (Theorem 7, page 615); Buffoni in [3], proves that "It is standart in the concentration-compactness theory that strict subadditivity is a consequence of strict subhomogenity" ; in statistics and probability, the notion of log subhomogenity is used in [7, p. 197] etc.

In particular, much recent work has been directed towards extending the theory of linear time-invariant (LTI) positive systems to broader and more realistic system classes, as for instance to classes of nonlinear positive systems. In this context, in [1] the authors consider a particular class of nonlinear positive systems, subhomogeneous cooperative systems, and derive results on their stability that echo the properties of positive LTI systems.

The n-subhomogenity is an integral part of subhomogenity.

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We recall some definitions:

Definition 1.1. A function $f : I \to R$, (I an interval of R) is said to be *convex* if $\forall x \in I, \forall y \in I$ and $\forall \lambda \in [0;1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Definition 1.2. A function $f : [0;+\infty[\to [0;+\infty[$ is said to be *n-subhomogeneous* if $\forall x \in [0;+\infty[$,

$$f(nx) \leq nf(x).$$

Definition 1.3. A function $f : [0;+\infty[\to [0;+\infty[$ is said to be *subadditive* if $\forall x \in [0;+\infty[$, $\forall y \in [0;+\infty[$,

$$f(x+y) \leq f(x) + f(y).$$

2. Main Result

Theorem. *Let $n \geq 2$ be an integer. A necessary and sufficient condition that the T-periodic extension, $F : [0;+\infty[\to [0;+\infty[$, of a convex function $f_1 : [0;T[\to [0;+\infty[$, be n-subhomogeneous is that*

$$A \leq nf_1\left(k\frac{T}{n}\right) \quad \textit{and} \quad B \leq nf_1\left(k\frac{T}{n}\right)$$

for every $k = 1, 2, ..., n-1$, where $A = \lim_{x\to 0^+} f_1(x)$ and $B = \lim_{x \to T^-} f_1(x)$.

Proof. For the necessity, if F is n-subhomogeneous, a passage to the limit in $F(nx) \leq nF(x)$, respectively if $x \to (k\frac{T}{n})^+$ and if $x \to (k\frac{T}{n})^-$, for every fixed $k = 1, 2, ..., n-1$, yields:

$$A \leq nF\left(k\frac{T}{n}\right) = nf_1\left(k\frac{T}{n}\right) \quad \textit{and} \quad B \leq nF\left(k\frac{T}{n}\right) = nf_1\left(k\frac{T}{n}\right).$$

For the sufficiency, denote by $f : [0;T] \to [0;+\infty[$ the function defined by

$$f(x) = \begin{cases} A & \text{if } x = 0 \\ f_1(x) & \text{if } x \in]0;T[\\ B & \text{if } x = T. \end{cases}$$

Due to the convexity of f_1, f is continuous, convex and its right derivative, that exists in $[0;T[$, is increasing.

For every fixed $0 \leq k \leq n-1$, we denote by I_k the interval $[k\frac{T}{n}; (k+1)\frac{T}{n}]$. Let $g_k : I_k \to R$, be the function defined by $g_k(x) = f(nx - kT) - nf(x)$.

Since for $x \in I_k$, $0 \leq nx - kT \leq T$, g_k is continuous in I_k, right differentiable on I_k, and its right derivative is $g'_{kr}(x) = n[f'_r(nx-kT) - f'_r(x)]$.

For $1 \le k \le n-2$, $k\frac{T}{n} < k\frac{T}{n-1} < (k+1)\frac{T}{n}$. Since $nx-kT \le x$ for $x \in [k\frac{T}{n}; k\frac{T}{n-1}]$, and $nx - kT \ge x$ for $x \in [k\frac{T}{n-1}; (k+1)\frac{T}{n}]$, we have $g'_{kr}(x) \le 0$ in $[k\frac{T}{n}; k\frac{T}{n-1}]$ and $g'_{kr}(x) \ge 0$ in $[k\frac{T}{n-1}; (k+1)\frac{T}{n}]$.

It follows from well-known properties of right differentiable continuous functions (Bourbaki [2]), that the function g_k is decreasing on $[k\frac{T}{n}; k\frac{T}{n-1}]$ and increasing on $[k\frac{T}{n-1}; (k+1)\frac{T}{n}]$.

Therefore for every $k = 1, 2, ..., n-2$, g_k reaches its maximal value at any of the endpoints of the interval $[k\frac{T}{n}; (k+1)\frac{T}{n}]$, which is included in $]0; T[$, where $f(x) = f_1(x)$, so, by hypothesis, $g_k(k\frac{T}{n}) = f(0) - nf(k\frac{T}{n}) = A - nf_1(k\frac{T}{n}) \le 0$ and $g_k((k+1)\frac{T}{n}) = f(T) - nf((k+1)\frac{T}{n}) = B - nf_1((k+1)\frac{T}{n}) \le 0$.

For $k = 0$, g_0 is increasing on the whole interval $[0; \frac{T}{n}]$, because in this interval $nx - kT = nx \ge x$. Consequently, $g'_{kr}(x) \ge 0$, and the greatest value of g_0 is $g_0(\frac{T}{n}) = f(T) - nf(\frac{T}{n}) = B - nf_1(\frac{T}{n}) \le 0$.

For $k = n-1$, g_{n-1} is decreasing on the whole interval $[(n-1)\frac{T}{n}; T]$, because in this interval, $nx - kT = nx - (n-1)T \le x$. Consequently, $g'_{kr}(x) \le 0$, and the greatest value of g_{n-1} is $g_{n-1}((n-1)\frac{T}{n}) = f(0) - nf((n-1)\frac{T}{n}) = A - nf_1((n-1)\frac{T}{n}) \le 0$. It follows that $g_k(x) \le 0$, that is $f(nx - kT) \le nf(x)$, for every fixed $0 \le k \le n-1$ and for every $x \in [k\frac{T}{n}; (k+1)\frac{T}{n}]$.

If $x \in]0; T[$, $f(x) = f_1(x)$ and there exists a $k \in \{0, 1, 2, ..., n-1\}$ such that $x \in]k\frac{T}{n}; (k+1)\frac{T}{n}[$; consequently $nx - kT \in]0; T[$ and

$$F(nx) = f_1(nx - kT) = f(nx - kT) \le nf(x) = nf_1(x) = nF(x).$$

For $x = 0$, obviously $F(n.0) = F(0) \le nF(0)$ and for $x = T$, $F(nT) = F(0) = f_1(0) \le nf_1(0) = nF(T)$. It follows that $F(nx) \le nF(x)$ for all $x \in [0; T]$.

Let now $x \in [0; +\infty[$ and $m \in N$, such that $mT \le x < (m+1)T$. Denote $s = x - mT$. Since $s \in [0; T[$, $F(ns) \le nF(s)$ and consequently, $F(nx) \le nF(x)$, because $F(x) = F(s)$ and $F(nx) = F(n(s + mT)) = F(ns)$.

This completes the proof.

A first application of this theorem for $n = 2$, gives another proof for a necessary and sufficient condition that the periodic extensions of convex functions be subadditive:

Corollary. *A necessary and sufficient condition that the T-periodic extension, $F : [0; +\infty[\to [0; +\infty[$, of a convex function $f_1 : [0; T[\to [0; +\infty[$, be subadditive is that $A \le 2f_1(\frac{T}{2})$ and $B \le 2f_1(\frac{T}{2})$ where $A = \lim_{x \to 0^+} f_1(x)$ and $B = \lim_{x \to T^-} f_1(x)$.*

Proof. Every subadditive function is $2-$subhomogeneous, so, by the part "necessity" of Theorem,

$A \leq 2F(\frac{T}{2}) = 2f_1(\frac{T}{2})$ and $B \leq 2F(\frac{T}{2}) = 2f_1(\frac{T}{2})$.

To prove the sufficiency, let $x, y \in [0; +\infty[$; there is $x_1, y_1 \in [0; T[$ such that $x = nT + x_1$ and $y = mT + y_1$; so $x + y = (n+m)T + x_1 + y_1$ and $F(x) = F(x_1)$, $F(y) = F(y_1)$, $F(x+y) = F(x_1 + y_1)$.

By Theorem, $F(x_1 + y_1) = F(2\frac{x_1+y_1}{2}) \leq 2F(\frac{x_1+y_1}{2})$ and by convexity of f_1,

$$F\left(\frac{x_1 + y_1}{2}\right) = f_1\left(\frac{x_1 + y_1}{2}\right) \leq \frac{1}{2}f_1(x_1) + \frac{1}{2}f_1(y_1) = \frac{1}{2}F(x_1) + \frac{1}{2}F(y_1).$$

It follows that

$$F(x+y) \leq F(2\frac{x_1 + y_1}{2}) \leq 2F(\frac{x_1 + y_1}{2}) \leq F(x_1) + F(y_1) = F(x) + F(y).$$

This completes the proof.

Note. Simple examples show that a function F that satisfies the conditions of the Corollary, need not satisfy in general, $\max F \leq 2 \min F$, in which case F is obviously subadditive.

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