Journal of Convex Analysis
Volume 24 (2017), No. 1, 309–317

Convex Integration and Legendrian Approximation of Curves

Norbert Hungerbühler
Department of Mathematics, ETH Zürich,
Rämistrasse 101, 8092 Zürich, Switzerland
norbert.hungerbuehler@math.ethz.ch

Thomas Mettler
Institut für Mathematik, Goethe Universität Frankfurt,
Robert-Mayer-Str. 10, 60325 Frankfurt am Main, Germany
mettler@math.uni-frankfurt.de

Micha Wasem
Department of Mathematics, ETH Zürich,
Rämistrasse 101, 8092 Zürich, Switzerland
micha.wasem@math.ethz.ch

Received: August 11, 2015
Accepted: July 18, 2016

Using convex integration we give a constructive proof of the well-known fact that every continuous curve in a contact 3-manifold can be approximated by a Legendrian curve.

1. Introduction

A *contact structure* on a 3-manifold M is a maximally non-integrable rank 2 subbundle ξ of the tangent bundle of M. If α is a 1-form on M whose kernel is ξ, then ξ is a contact structure if and only if $\alpha \wedge \mathrm{d}\alpha \neq 0$. A curve η in a contact 3-manifold (M, ξ) is called *Legendrian*, whenever $\eta^*\alpha = 0$ for some (local) 1-form α defining ξ.

The purpose of this note is to give a detailed proof of the following statement which is often used in contact geometry and Legendrian knot theory.

Theorem 1.1. *Any continuous map from a compact* 1*-manifold to a contact* 3*-manifold can be approximated by a Legendrian curve in the* C^0*-Whitney topology.*

Whereas this theorem is a special case of Gromov's h-principle for Legendrian immersions [5], the curve-case can be treated by more elementary techniques. Sketches of proofs of Theorem 1.1 have already appeared in the literature, see for example [1, p. 6–7], [4, p. 40] or [3, p. 102]. Exploiting the fact that every contact 3-manifold is locally contactomorphic to $\mathbb{R}^3$ equipped with the standard

ISSN 0944-6532 / $ 2.50

contact structure defined by $\alpha = \mathrm{d}z - y\mathrm{d}x$, Etnyre and Geiges indicate that either the *front-projection* (x, z) of a given curve (x, y, z) can be approximated by a zig-zag-curve whose slope approximates the y-component of the curve or the *Lagrangian projection* (x, y) can be approximated by a curve whose area integral approximates the z component of the curve, which can be achieved by adding small negatively or positively oriented loops.

Here, we give a different and analytically rigorous proof of Theorem 1.1 by using convex integration. Our proof has the advantage of providing a constructive approximation. In particular, in the case of a continuous curve in $\mathbb{R}^3$ equipped with the standard contact structure, we obtain an explicit Legendrian curve given in terms of an elementary integral. For instance, we obtain a solution, even integral-free, to the "parallel parking problem" in Example 3.1. Example 3.2 shows how our technique recovers the zig-zag-curves and the small loops in the front - respectively Lagrangian projections.

2. Proof of the Theorem

We start by first treating the case where the contact manifold is $\mathbb{R}^3$ equipped with the standard contact structure, that is, we aim to prove the following:

Proposition 2.1. *Let $v \in C^0([0, 2\pi], \mathbb{R}^3)$. For every $\varepsilon > 0$ there exists a Legendrian curve $\eta \in C^\infty([0, 2\pi], \mathbb{R}^3)$ such that $\|v - \eta\|_{C^0([0,2\pi])} \leqslant \varepsilon$.*

Remark 2.2. Here, as usual, $\|\gamma\|_{C^0(I)} := \sup_{t\in I} |\gamma(t)|$ and $\|\gamma\|_{C^1(I)} := \|\gamma\|_{C^0(I)} + \|\mathrm{d}\gamma\|_{C^0(I)}$.

Let the curve we wish to approximate be given by $(x, y, z) \in C^\infty([0, 2\pi], \mathbb{R}^3)$. The regularity is no restriction due to a standard approximation argument using convolution. Let $\eta = (a, b, c) \in C^\infty([0, 2\pi], \mathbb{R}^3)$ denote the approximating Legendrian curve. For every choice of smooth functions $(a, c) \in C^\infty([0, 2\pi], \mathbb{R}^2)$ satisfying $\dot{a} \neq 0$, we obtain a Legendrian curve by defining $b = \dot{c}/\dot{a}$. Therefore, if $(\dot{a}(t), \dot{c}(t))$ lies in the set

$$\mathcal{R}_{t,\varepsilon} := \left\{(u, v) \in \mathbb{R}^2, |v - y(t)u| \leqslant \varepsilon \min\{|u|, |u|^2\}\right\},$$

for every $t \in [0, 2\pi]$, then $\|b - y\|_{C^0([0,2\pi])} \leqslant \varepsilon$. This condition can be achieved by defining

$$(a(t), c(t)) := (x(0), z(0)) + \int_0^t \gamma(u, nu)\, \mathrm{d}u,$$

with $\gamma \in C^\infty([0, 2\pi] \times S^1, \mathbb{R}^2)$ and $n \in \mathbb{N}$, provided that $\gamma(t, \cdot) \in \mathcal{R}_{t,\varepsilon}$. Furthermore, if γ additionally satisfies

$$\frac{1}{2\pi} \oint_{S^1} \gamma(t, s)\, \mathrm{d}s = (\dot{x}(t), \dot{z}(t)),$$

for all $t \in [0, 2\pi]$, then – as we will show below – $(a(t), c(t))$ approaches $(x(t), z(t))$ as n gets sufficiently large.

The set $\mathcal{R}_{t,\varepsilon}$ is *ample*, i.e., the interior of its convex hull is all of $\mathbb{R}^2$. For any given point $(\dot{x}(t), \dot{z}(t)) \in \mathbb{R}^2$ we will thus be able to find a loop in $\mathcal{R}_{t,\varepsilon}$ having $(\dot{x}(t), \dot{z}(t))$ as its barycenter. This fact is sometimes referred to as the fundamental lemma of convex integration (see for instance [6, Prop. 2.11, p. 28]). In the particular case studied here we find an explicit formula for γ:

Lemma 2.3. *There exists a family of loops $\gamma \in C^\infty([0, 2\pi] \times S^1, \mathbb{R}^2)$ satisfying $\gamma(t, \cdot) \in \mathcal{R}_{t,\varepsilon}$ and such that*

$$\frac{1}{2\pi} \oint_{S^1} \gamma(t, s)\, \mathrm{d}s = (\dot{x}(t), \dot{z}(t)), \tag{1}$$

for all $t \in [0, 2\pi]$.

Proof. The map $\gamma := (\gamma_1, \gamma_2)$, where

$$\gamma_1(t, s) := r \cos s + \dot{x}(t)$$

and

$$\gamma_2(t, s) := \gamma_1(t, s) \left(y(t) + \frac{2(\dot{z}(t) - y(t)\dot{x}(t))}{r^2 + 2\dot{x}(t)^2} \gamma_1(t, s) \right)$$

satisfies (1) for every $r > 0$. If r is large enough one obtains $\gamma(t, \cdot) \in \mathcal{R}_{t,\varepsilon}$, where r can be chosen independently of t by compactness of $[0, 2\pi]$. $\square$

We now have:

Proof of Proposition 2.1. With the definitions above we obtain

$$b(t) := \frac{\dot{c}(t)}{\dot{a}(t)} = y(t) + \frac{2(\dot{z}(t) - y(t)\dot{x}(t))}{r^2 + 2\dot{x}(t)^2} \gamma_1(t, nt).$$

We are left to show that $|(a, c) - (x, z)| \leqslant \varepsilon$ provided n is large enough. This follows from the following estimate

$$\|(a, c) - (x, z)\|_{C^0([0,2\pi])} \leqslant \frac{4\pi^2}{n} \|\gamma\|_{C^1([0,2\pi] \times S^1)}. \tag{2}$$

The estimate is in fact a geometric property of the derivative and can be interpreted as follows: Since $(\dot{a}, \dot{c})$ and $(\dot{x}, \dot{z})$ coincide "in average" on shorter and shorter intervals when n gets bigger and bigger, (a, c) and (x, z) tend to become close: Let

$$I_k := \left[\frac{2\pi k}{n}, \frac{2\pi(k+1)}{n} \right] \quad \text{for } k = 0, \ldots, \left\lfloor \frac{nt}{2\pi} \right\rfloor - 1 \quad \text{and} \quad J := \left[\left\lfloor \frac{nt}{2\pi} \right\rfloor \frac{2\pi}{n}, t \right].$$

Then we can estimate $D = |(a(t), c(t)) - (x(t), z(t))|$:

$$
\begin{aligned}
D &= \left| \int_0^t \gamma(u, nu)\,\mathrm{d}u - \int_0^t (\dot{x}, \dot{z})(u)\,\mathrm{d}u \right| \\
&\leqslant \sum_{k=0}^{\lfloor \frac{nt}{2\pi} \rfloor - 1} \left| \int_{I_k} \gamma(u, nu)\,\mathrm{d}u - \int_{I_k} \frac{1}{2\pi} \int_0^{2\pi} \gamma(u, v)\,\mathrm{d}v\,\mathrm{d}u \right| \\
&\qquad + \int_J \left(|\gamma(u, nu)| + \|\gamma\|_{C^0([0,2\pi]\times S^1)} \right)\,\mathrm{d}u \\
&\leqslant \sum_{k=0}^{\lfloor \frac{nt}{2\pi} \rfloor - 1} \left| \frac{1}{n} \int_0^{2\pi} \gamma\left(\frac{v + 2k\pi}{n}, v \right)\,\mathrm{d}v - \int_{I_k} \frac{1}{2\pi} \int_0^{2\pi} \gamma(u, v)\,\mathrm{d}v\,\mathrm{d}u \right| \\
&\qquad + \frac{4\pi}{n} \|\gamma\|_{C^0([0,2\pi]\times S^1)} \\
&\leqslant \sum_{k=0}^{\lfloor \frac{nt}{2\pi} \rfloor - 1} \left| \frac{1}{2\pi} \int_{I_k} \int_0^{2\pi} \left(\gamma\left(\frac{v + 2k\pi}{n}, v \right) - \gamma(u, v) \right)\,\mathrm{d}v\,\mathrm{d}u \right| \\
&\qquad + \frac{4\pi}{n} \|\gamma\|_{C^0([0,2\pi]\times S^1)} \\
&\leqslant \left\lfloor \frac{nt}{2\pi} \right\rfloor \frac{4\pi^2}{n^2} \|\partial_t \gamma\|_{C^0([0,2\pi]\times S^1)} + \frac{4\pi}{n} \|\gamma\|_{C^0([0,2\pi]\times S^1)} \\
&\leqslant \frac{4\pi}{n} \left(\pi \|\partial_t \gamma\|_{C^0([0,2\pi]\times S^1)} + \|\gamma\|_{C^0([0,2\pi]\times S^1)} \right).
\end{aligned}
$$

By construction, the curve (a, b, c) is Legendrian and the desired approximation of (x, y, z). $\square$

Next we show that we can approximate closed curves by closed Legendrian curves.

Proposition 2.4. *Let $v \in C^0(S^1, \mathbb{R}^3)$. For every $\varepsilon > 0$ there exists a Legendrian curve $\eta \in C^\infty(S^1, \mathbb{R}^3)$ such that $\|v - \eta\|_{C^0(S^1)} \leqslant \varepsilon$.*

Proof. Using standard regularization, let the curve we wish to approximate be given by $(x, y, z) \in C^\infty([0, 2\pi], \mathbb{R}^3)$, where the values of (x, y, z) in 0 and 2π agree to all orders. Define $g(t) := \gamma_1^2(t, nt)$. Since $\|g\|_{L^1([0,2\pi])} = O(r^2)$ as $r \to \infty$, we can choose $r > 0$ large enough such that $f := g/\|g\|_{L^1([0,2\pi])}$ is well-defined. With the notation

$$
\mathrm{I}_2 := \int_0^{2\pi} \gamma_2(u, nu)\,\mathrm{d}u,
$$

we define $\eta = (a, b, c)$ as follows:

$$(a(t), c(t)) := (x(0), z(0)) + \int_0^t \Big[\gamma(u, nu) - (0, \mathrm{I}_2 f(u))\Big]\,\mathrm{d}u, \tag{3}$$

$$b(t) := \frac{\dot{c}(t)}{\dot{a}(t)} = y(t) + \gamma_1(t, nt)\left(\frac{2(\dot{z}(t) - y(t)\dot{x}(t))}{r^2 + 2\dot{x}(t)^2} - \frac{\mathrm{I}_2}{\|g\|_{L^1([0,2\pi])}}\right). \tag{4}$$

A straightforward computation shows that the values of (a, b, c) in 0 and 2π agree to all orders, hence $\eta \in C^\infty(S^1, \mathbb{R}^3)$ and it is Legendre by construction. Using (2) we obtain $|\mathrm{I}_2| \leqslant \frac{4\pi^2}{n}\|\gamma_2\|_{C^1([0,2\pi]\times S^1)}$, hence we find using (4) as $r \to \infty$:

$$\|b - y\|_{C^0([0,2\pi])} \leqslant \|\gamma_1\|_{C^0([0,2\pi]\times S^1)}\left(1 + \frac{1}{n}\|\gamma\|_{C^1([0,2\pi]\times S^1)}\right) O(r^{-2}).$$

For the remaining components we find find using (2) and (3) the uniform bound

$$\begin{aligned} |(a(t), c(t)) - (x(t), z(t))| &\leqslant \frac{4\pi^2}{n}\|\gamma\|_{C^1([0,2\pi]\times S^1)} + \frac{|\mathrm{I}_2|}{\|g\|_{L^1([0,2\pi])}}\int_0^t g(u)\,\mathrm{d}u \\ &\leqslant \frac{8\pi^2}{n}\|\gamma\|_{C^1([0,2\pi]\times S^1)}. \end{aligned}$$

Choosing r large enough and $n \sim r^2$ concludes the proof. □

We show now how to glue together two local approximations of a curve Γ in M on two intersecting coordinate neighborhoods. Let therefore U_σ and U_τ in M be coordinate patches such that $U = U_\sigma \cap U_\tau \neq \emptyset$. Let I_σ and I_τ be compact intervals such that $I = I_\sigma \cap I_\tau$ contains an open neighborhood of $t = 0$ (after shifting the variable t if necessary) and such that $\Gamma(I_\sigma) \subset U_\sigma$, $\Gamma(I_\tau) \subset U_\tau$. Assume without restriction that Γ is smooth and let (x, y, z) represent Γ on U. Suppose that (x, y, z) is approximated by Legendrian curves $\sigma : I_\sigma \to \mathbb{R}^3$ and $\tau : I_\tau \to \mathbb{R}^3$ such that

$$\|\sigma - (x, y, z)\|_{C^0(I)} < \varepsilon^2, \qquad \|\tau - (x, y, z)\|_{C^0(I)} < \varepsilon^2 \tag{5}$$

for some fixed $0 < \varepsilon < \frac{1}{2}$. For $r > 0$, define $R(r)$ to be the smallest number such that $\bar{B}_r(0) \subset \operatorname{conv}\big(\mathcal{R}_{0,\varepsilon} \cap \bar{B}_R(0)\big)$. Note that R depends continuously on r and if $r > r_0 := \frac{\varepsilon}{\sqrt{1+y(0)^2}}$, then

$$R(r) = \frac{r}{\varepsilon}\sqrt{(1 + y(0)^2)\,(1 + (|y(0)| + \varepsilon)^2)} =: \frac{r}{\varepsilon}w(y(0), \varepsilon). \tag{6}$$

Choose $0 < \delta < \varepsilon^2$ such that $[-\delta, \delta] \subset I$ and such that $\delta\|(x, y, z)\|_{C^1(I)} \leqslant \varepsilon^2$ and define

$$\begin{aligned} p_1 &:= (\sigma_1(-\delta), \sigma_3(-\delta)), \\ \dot{p}_1 &:= (\dot{\sigma}_1(-\delta), \dot{\sigma}_3(-\delta)), \\ p_2 &:= (\tau_1(\delta), \tau_3(\delta)), \\ \dot{p}_2 &:= (\dot{\tau}_1(\delta), \dot{\tau}_3(\delta)). \end{aligned}$$

From (5) and the choice of δ we obtain $\dot p_1, \dot p_2 \in \mathcal{C}_\varepsilon := \big\{(u,v) \in \mathbb{R}^2, |v - y(0)u| \leqslant \varepsilon |u|\big\}$ and

$$\frac{p_2 - p_1}{2\delta} =: p \in B_{\bar r}(0), \quad \text{where } \bar r = \frac{2\varepsilon^2}{\delta}.$$

Since $3\bar r > r_0$, we can express $R(3\bar r)$ by means of formula (6). This will be used in computation (9). We construct a path $\gamma = (\gamma_1, \gamma_2) : [-\delta, \delta] \to \mathcal{C}_\varepsilon$ as follows: For $\rho < \delta/2$, let $\gamma|_{[-\delta, -\delta+\rho]}$ be a continuous path from $\dot p_1$ to 0 and let $\gamma|_{[\delta-\rho,\delta]}$ be a continuous path from 0 to $\dot p_2$. We construct γ such that the quotient γ_2/γ_1 is well-defined on $[-\delta, -\delta+\rho] \cup [\delta-\rho, \delta]$ and equals $y(0)$ in $t = -\delta + \rho$ and $t = \delta - \rho$. Moreover, we require that

$$\int_{-\delta}^{-\delta+\rho} |\gamma(t)| \, \mathrm{d}t < \frac{\delta\varepsilon}{2} \quad \text{and} \quad \int_{\delta-\rho}^{\delta} |\gamma(t)| \, \mathrm{d}t < \frac{\delta\varepsilon}{2}. \tag{7}$$

On $[-\delta, -\delta+\rho]$, such a path is for example given by

$$t \mapsto \left(1 - \frac{\delta+t}{\rho}\right)^k \begin{pmatrix} \dot\sigma_1(-\delta) \\ y(0)\dot\sigma_1(-\delta) + (\dot\sigma_3(-\delta) - y(0)\dot\sigma_1(-\delta))\left(1 - \frac{\delta+t}{\rho}\right)^k \end{pmatrix}$$

provided $k \in \mathbb{N}$ is sufficiently large. We obtain

$$\frac{1}{2(\delta-\rho)}\left(2\delta p - \int_{-\delta}^{-\delta+\rho} \gamma(t)\,\mathrm{d}t - \int_{\delta-\rho}^{\delta} \gamma(t)\,\mathrm{d}t\right) =: \bar p \in B_{3\bar r}(0)$$

and hence $\bar p \in \operatorname{int}\operatorname{conv}(B_{R(3\bar r)}(0) \cap \mathcal{R}_{0,\varepsilon})$. Using the fundamental lemma of convex integration we let $\gamma|_{[-\delta+\rho,\delta-\rho]}$ be a continuous closed loop in $B_{R(3\bar r)}(0) \cap \mathcal{R}_{0,\varepsilon}$ based at 0 such that

$$\frac{1}{2(\delta-\rho)} \int_{-\delta+\rho}^{\delta-\rho} \gamma(t)\,\mathrm{d}t = \bar p.$$

With these definitions we obtain

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} \gamma(t) = p.$$

Now we define $\eta = (a, b, c) : [-\delta, \delta] \to \mathbb{R}^3$ by letting $b(t) := \dot c(t)/\dot a(t)$, where

$$(a, c)(t) := p_1 + \int_{-\delta}^{t} \gamma(u)\mathrm{d}u.$$

The curve η is well-defined and Legendrian by construction. It satisfies $\eta(-\delta) = \sigma(-\delta)$ and $\eta(\delta) = \tau(\delta)$. Moreover, (a, c) and (σ_1, σ_3) agree to first order in

$t = -\delta$ and so do (a, c) and (τ_1, τ_3) in $t = \delta$. From $\gamma([-\delta, \delta]) \in \mathcal{C}_\varepsilon$ and the choice of δ we find

$$|b(t) - y(t)| \leqslant |b(t) - y(0)| + |y(t) - y(0)| \leqslant \varepsilon + \delta \|y\|_{C^1(I)} < 2\varepsilon. \tag{8}$$

Using (5), (6), (7) and the choice of δ we obtain for the remaining components the uniform bound

$$\begin{aligned} |(a,c)(t) - (x,z)(t)| &\leqslant |p_1 - (x,z)(-\delta)| + \int_{-\delta}^{t} (|\gamma(u)| + |(\dot{x},\dot{z})(u)|)\, \mathrm{d}u \\ &\leqslant \varepsilon^2 + \delta\varepsilon + \int_{-\delta+\rho}^{\delta-\rho} |\gamma(u)|\, \mathrm{d}u + 2\delta \|(x,z)\|_{C^1(I)} \\ &\leqslant 2\varepsilon + 2\delta R(3\bar{r}) \\ &\leqslant \varepsilon \left(14 + 12 \left(|y(0)| + \frac{1}{2} \right)^2 \right). \end{aligned} \tag{9}$$

Finally, suppose υ is a continuous curve from a compact 1-manifold N (that is, N is a compact interval or S^1) into a contact 3-manifold (M, ξ). We fix some Riemannian metric g on M. Then it follows with the bounds (8,9) and the compactness of the domain of υ that for every $\varepsilon > 0$ there exists a ξ-Legendrian curve η such that

$$\sup_{t \in N} d_g(\upsilon(t), \eta(t)) < \varepsilon,$$

where d_g denotes the metric on M induced by the Riemannian metric g. In particular, every open neighborhood of $\upsilon \in C^0(N, M)$ – equipped with the uniform topology – contains a Legendrian curve $N \to M$. Since N is assumed to be compact the uniform topology is the same as the Whitney C^0-topology, thus proving Theorem 1.1.

3. Examples

Example 3.1 (Parallel Parking). The trajectory of a car moving in the plane can be thought of as a curve $[0, 2\pi] \to S^1 \times \mathbb{R}^2$. Denoting by (φ, a, c) the natural coordinates on $S^1 \times \mathbb{R}^2$, the angle coordinate φ denotes the orientation of the car with respect to the a-axis and the coordinates (a, c) the position of the car in the plane. Admissible motions of the car are curves satisfying

$$\dot{a} \sin \varphi = \dot{c} \cos \varphi.$$

The manifold $S^1 \times \mathbb{R}^2$ together with the contact structure defined by the kernel of the 1-form $\theta := \sin \varphi\, \mathrm{d}a - \cos \varphi\, \mathrm{d}c$ is a contact 3-manifold. Indeed, we have

$$\theta \wedge \mathrm{d}\theta = -\cos^2\!\varphi\, \mathrm{d}\varphi \wedge \mathrm{d}a \wedge \mathrm{d}c - \sin^2\!\varphi\, \mathrm{d}\varphi \wedge \mathrm{d}a \wedge \mathrm{d}c = -\mathrm{d}\varphi \wedge \mathrm{d}a \wedge \mathrm{d}c \neq 0.$$

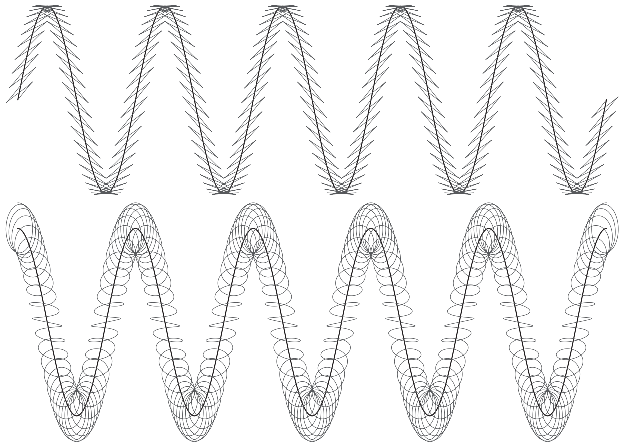

Figure 3.1: The front (top) and the Lagrangian projection (bottom) of the Legendrian approximation of η.

Applying Theorem 1.1 with $b = \tan\varphi$ gives an explicit approximation of the curve

$$t \mapsto (x(t), y(t), z(t)) = (0, 0, t).$$

Lemma 2.3 gives the loop

$$\gamma(t, s) = 2(r\cos s, \cos^2 s),$$

and hence the desired Legendrian curve

$$(\operatorname{arccot}(r\sec(nt)), 2rt\operatorname{sinc}(nt), t + t\operatorname{sinc}(2nt)),$$

provided r is large enough and $n \sim r^2$.

Example 3.2 (Legendrian Helix). The Legendrian approximation of the helix

$$\upsilon : [0, 2\pi] \to \mathbb{R}^3, \qquad t \mapsto (t, \cos(5t), \sin(5t)),$$

with $n = \frac{2}{9}r^2$ and $r = 30$ is given by

$$\begin{aligned} a(t) &= t + \frac{3}{20}\sin(200t) \\ b(t) &= \frac{455}{451}\cos(5t) + \frac{120}{451}\cos(5t)\cos(200t) \\ c(t) &= \sin(5t) + \frac{459}{5863}\sin(195t) + \frac{1377}{18491}\sin(205t) \\ &\quad + \frac{180}{35629}\sin(395t) + \frac{20}{4059}\sin(405t). \end{aligned}$$

and produces the zig-zags and the small loops in its front and Lagrangian projections (see Figure 3.1).

References

[1] J. B. Etnyre: Legendrian and transversal knots, in: Handbook of Knot Theory, W. Menasco et al. (ed.), Elsevier, Amsterdam (2005) 105–185.

[2] D. Fuchs, S. Tabachnikov: Invariants of Legendrian and transverse knots in the standard contact space, Topology 36(5) (1997) 1025–1053.

[3] H. Geiges: An Introduction to Contact Topology, Cambridge Studies in Advanced Mathematics 109, Cambridge University Press, Cambridge (2008).

[4] H. Geiges: Contact geometry, in: Handbook of Differential Geometry. Vol. II, F. J. E. Dillen et al. (ed.), Elsevier, Amsterdam (2006) 315–382.

[5] M. Gromov: Partial Differential Relations, Ergebnisse der Mathematik und ihrer Grenzgebiete 9(3), Springer, Berlin (1986).

[6] D. Spring: Convex Integration Theory. Solutions to the h-Principle in Geometry and Topology, Modern Birkhäuser Classics, Birkhäuser, Basel (2010).