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On the Rockafellar Function Associated to a Cyclically Monotone Mapping

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We give some properties of the Rockafellar function used in integration theory of cyclically monotone mappings. This function is proper only in the case of cyclically monotone mappings. But, by a slight modification of its definition we can obtain a proper function even if the mapping is not cyclically monotone. Finally, two dual properties are established.

Keywords: Rockafellar function, cyclically monotone mapping, support function, subdifferential, Darboux integral, conjugate function

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1. Introduction and preliminaries

Let X be a linear normed space and let X^* be its dual space. Given a *convex function* $f : X \to \overline{\mathbb{R}}$, where $\overline{R} = [-\infty, +\infty]$, its conjugate $f^* : X^* \to \overline{\mathbb{R}}$ is defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x); x \in X\}, \quad x^* \in X^*. \tag{1}$$

If $A \subset X$, its *support function* $s_A : X^* \to \overline{\mathbb{R}}$ is defined by

$$s_A(x^*) = \sup\{x^*(x); x \in A\}, \quad x^* \in X^*. \tag{2}$$

Similarly, is defined the support function associated to a subset in X^*. We observe that $s_A = I_A^*$, where I_A is the *indicator function* of the set A. We recall that $I_A(x) = 0$ if $x \in A$ and $I_A(x) = \infty$ if $x \in X \backslash A$.

Also, the *subdifferential* is the multivalued operator $\partial f : X \to X^*$ defined as follows:

$$\partial f(x) = \{x^* \in X^*; x^*(u - x) \le f(u) - f(x), \text{ for all } u \in X\}, \quad x \in X. \tag{3}$$

Here, we suppose that f is a *proper function*, that is, its *domain*

$$D(f) = \{x \in X; f(x) < \infty\} \tag{4}$$

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is a nonvoid set and $f(u) > -\infty$ for any $u \in X$.

If a proper convex function f is lower-semiconitnuous, then the subdifferential of its conjugate f^* has the following property

$$\partial f^*(x^*) = (\partial f)^{-1}(x^*), \quad x^* \in X^*. \tag{5}$$

A detailed treatment of these concepts and special results in convex analysis can be found, for instance, in the book [2].

We recall that a multivalued mapping $T : X \to X^*$ is said to be *cyclically monotone* if

$$\sum_{i=0}^{n} x_i^*(x_i - x_{i+1}) \geq 0 \text{ for any } (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0, n}, \tag{6}$$

where $x_{n+1} = x_0$.

If this property is fulfilled for $n = 1$, then the mapping T is called *monotone.*

The mapping T is said to be *maximal cyclically monotone* if it is cyclically monotone and has no cyclically monotone extension in $X \times X^*$. A typically example of a maximal cyclically monotone mapping is the subdifferential of a proper lower semicontinuous convex function (see, for instance, [2], [12]). Also, by the famous result of Rockafellar established in [12] *any maximal cyclically monotone mapping can be represented as the subdifferential of a proper lower semicontinuous convex function.* In this line, we recall the *Rockafellar integration formula*, which can be associated to any *proper multivalued mapping* $T : X \to X^*$ and a fixed element $x_0 \in \operatorname{Dom} T$, defined by

$$f_{x_0;T}(x) = \sup \left\{ \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i); (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0, n}, x_{n+1} = x \right\}, \tag{7}$$

for any $x \in X$. But, this function is proper if and only if T is a cyclically monotone mapping (see Theorem 2.10).

Obviously, the function

$$x \to f_{x_0;T}(x), x \in X \text{ is a proper convex lower semicontinuous function} \tag{8}$$

and according to (6), we have

$$f_{x_0;T}(x_0) = 0, \tag{9}$$

for any cyclically monotone mapping T and $x_0 \in \operatorname{Dom} T$. Also,

$$\operatorname{Graph} T \subset \operatorname{Graph} \partial f_{x_0;T}, \quad \text{for any } x_0 \in \operatorname{Dom} T. \tag{10}$$

Indeed, if $x \in \operatorname{Dom} T$ taking an arbitrary finite system $x_i \in \operatorname{Dom} T$, $i = \overline{1, n}$, it follows that $\{x_1, x_2, \ldots, x_n, x\}$ is also a finite system in $\operatorname{Dom} T$. Therefore, by (7) we obtain

$$f_{x_0;T}(y) \geq x_0^*(x_1 - x_0) + \ldots + x_{n-1}^*(x_n - x_{n-1}) + x_n^*(x - x_n) + x^*(y - x),$$

for any $y \in X$ and $x^* \in Tx$. Thus, coming back to (7) we get

$$f_{x_0;T}(y) \geq f_{x_0;T}(x) + x^*(y - x), \quad \forall y \in X,$$

i.e. $x^* \in \partial f_{x_0;T}(x)$ as claimed.

Rockafellar function $f_{x_0;T}$ was extended by Bachir, Daniilidis and Penot [1] to the quasiconvex case using Plastria subdifferential [10]. Also, in [4] the authors prove that, under special hypothesis in the definition of $f_{x_0;T}$, we can consider in the right hand only strongly exposed points of $\operatorname{Dom} T$. Other properties in the case of nonconvex functions are given in [8]. Moreover, in this paper the authors prove some integration results using different concepts of subdifferential (see also [11], [13]). Many interesting results concerning the integration with respect to Fenchel subdifferentials are given in [3], [4], [5]. It is clear that for any element $x_0 \in \operatorname{Dom} T$ we shall obtain corresponding associated functions $f_{x_0;T}$. On the other hand, the cyclically monotone mapping has many maximal cyclically monotone extensions which are not subdifferentials of certain functions of the type $f_{x_0;T}$, where $x_0 \in \operatorname{Dom} T$ (see, for instance, Example 2.6). But, if there exists an unique maximal cyclically monotone extension of T, then the all functions $f_{x_0;T}$, $x \in \operatorname{Dom} T$, coincide up to an additive constant function [12]. Thus, an interesting problem is to illustrate special cases in which two functions of the type $f_{x_0;T}$ associated to an arbitrary cyclically monotone mapping are equal up to an additive constant function.

In the usual case of real continuous functions T, defined on a real interval, it is obvious that the right hand of equality (7) becomes the *lower Darboux integral* of an increasing function. Consequently, the associated function $f_{x_0;T}$, $x \in \operatorname{Dom} T$ has the following additivity property with respect to the interval

$$f_{x_0;T}(x) = f_{x_0;T}(x_1) + f_{x_1;T}(x), \quad \text{for all } x, x_1 \in \operatorname{Dom} T. \tag{11}$$

On the other hand, if in the sums of equality (7) we consider only the sums which contain a certain element $(\overline{x}, \overline{x}^*) \in \operatorname{Graph} T$, then we obtain the same function $f_{x_0;T}$. But, these simple properties are not true for an arbitrary cyclically monotone mapping.

In this paper, taking into account the properties of antiderivatives in the case of real line with respect to the integration interval we investigate some properties of the function $f_{x_0;T}$, $x_0 \in \operatorname{Dom} T$. We prove that the convex function $f_{x_0;T}$ is proper only in the case of cyclically monotone mappings. But, by a slight modification of its definition we obtain a function of Rockafellar type which may be proper for non cyclically monotone mappings. Finally, we establish a duality property of type (5) among the associated functions $f_{x_0;T}$ and $f_{x_0^*;T^{-1}}$, $(x_0, x_0^*) \in \operatorname{Graph} T$.

2. Some properties of the Rockafellar function

We begin this section by proving that in the case of an arbitrary cyclically monotone mapping, the equality (11) becomes an inequality.

Proposition 2.1. *Let T be a cyclically monotone mapping. If x_0, x_1 are two elements in $\operatorname{Dom} T$, then*

$$f_{x_0;T}(x) \geq f_{x_0;T}(x_1) + f_{x_1;T}(x), \quad \textit{for all } x \in X. \tag{12}$$

Proof. Let us consider the finite set $\{u_1, u_2, \ldots, u_n, x_1, v_1, \ldots, v_m\}$ in $\operatorname{Dom} T$. By (7) we have

$$\begin{aligned} f_{x_0;T}(x) \geq & \left[x_0^*(u_1 - x_0) + \sum_{i=1}^{n-1} u_i^*(u_{i+1} - u_i) + u_n^*(x_1 - u_n) \right] \\ & + \left[x_1^*(v_1 - x_1) + \sum_{j=1}^{m-1} v_j^*(v_{j+1} - v_j) + v_m^*(x - v_m) \right], \text{ for all } x \in X, \end{aligned}$$

where $u_i^* \in T(u_i)$, $v_j^* \in T(v_j)$, $i = \overline{1,n}$, $j = \overline{1,m}$. □

Using again the definition (7) we obtain the inequality (12) as claimed.

Now, we can establish a characterization of property (11).

Theorem 2.2. *If T is a cyclically monotone mapping and $x_0, x_1 \in \operatorname{Dom} T$, then the following statements are equivalent:*

(*i*) *the associated functions $f_{x_0;T}, f_{x_1;T}$ coincide up to an additive constant function;*

(*ii*) *in the right hand of the equality* (7) *we can consider only finite sets in* $\operatorname{Dom} T$ *containing the element x_1;*

(*iii*) $f_{x_0;T}(x_1) + f_{x_1;T}(x_0) = 0$.

Proof. If $f_{x_1;T}(x) = f_{x_0;T}(x) + c$, $x \in X$, where c is a constant function, taking $x = x_0$, by (9) we get $f_{x_1;T}(x_0) = c$. Similarly we obtain $c = -f_{x_0;T}(x_1)$, and so (*iii*) holds. Now, let us denote by $f_{x_0,x_1;T}(x)$ the right hand of (7) considering only the finite sets in $\operatorname{Dom} T$ which contain the element x_1, that is the set of the type $\{x_0, u_1, u_2, \ldots, u_n\} \cup \{x_1, v_1, v_2, \ldots, v_m\}$. Thus we have

$$\begin{aligned} f_{x_0,x_1;T}(x) \geq x_0^*(u_1 - x_0) + \sum_{i=1}^{n-1} u_i^*(u_{i+1} - u_i) + u_n^*(x_1 - u_n) \\ + x_1^*(v_1 - x_1) + \sum_{j=1}^{m-1} v_j^*(v_{j+1} - v_j) + v_m^*(x - v_m), \end{aligned}$$

and so, by (7) we get

$$f_{x_0,x_1;T}(x) \geq f_{x_0;T}(x_1) + f_{x_1;T}(x), \quad \text{for all } x \in X.$$

Using the inequality (12) we obtain

$$f_{x_0;x_1;T}(x) \geq f_{x_0;T}(x_1) + f_{x_1;T}(x_0) + f_{x_0;T}(x), \quad x \in X.$$

If we suppose (iii) it follows $f_{x_0;x_1;T}(x) \geq f_{x_0;T}(x)$, $x \in X$. But, the reverse inequality is obvious. Hence $(iii) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (i)$ follows using again (12) and (7). Indeed

$$\begin{aligned} f_{x_0;T}(x) &\geq f_{x_0;T}(x_1) + f_{x_1;T}(x) \\ &\geq x_0^*(u_1 - x_0) + \sum_{i=1}^{n-1} u_i^*(u_{i+1} - u_i) + u_n^*(x_1 - u_n) \\ &\quad + x_1^*(v_1 - x_1) + \sum_{j=1}^{m-1} v_j^*(v_{j+1} - v_j) + v_m^*(x - v_m), \end{aligned}$$

for all $x \in X$, $x_0^* \in Tx_0$, $x_1^* \in Tx_1$, $(u_i, u_i^*) \in \operatorname{Graph} T$, $i = \overline{1,n}$, $(v_j, v_j^*) \in \operatorname{Graph} T$, $j = \overline{1,m}$, and so, $f_{x_0;T}(x_1) + f_{x_1;T}(x) \geq f_{x_0,x_1;T}(x) = f_{x_0;T}(x)$, $x \in X$. Therefore the equality (11) holds, that is the properties (i) is true. □

Corollary 2.3. *If T is a cyclically monotone mapping and $x_0, x_1 \in \operatorname{Dom} T$. Then the associated functions $f_{x_0;T}$, $f_{x_1;T}$ are equal if and only if $f_{x_0;T}(x_1) = f_{x_1;T}(x_0) = 0$.*

Remark 2.4. According to the proof of Theorem 2.2 it follows that if x_0 is a given element in $\operatorname{Dom} T$, then the equality (11) and the property (i) for any $x_1 \in \operatorname{Dom} T$ are equivalent.

Remark 2.5. The inequality (12) can be extended to a finite number of elements $x_i \in \operatorname{Dom} T$, $i = \overline{1,k}$; namely we have

$$f_{x_0;T}(x) \geq \sum_{i=0}^{k-1} f_{x_i;T}(x_{i+1}) + f_{x_k;T}(x), \quad x \in X. \tag{13}$$

Example 2.6. Let us consider the mapping T defined by $T(x) = 2x$, if $x \in [0,1]$ and $T(2) = 3$. Then, for any $n_0 \in [0,1]$ we get

$$f_{x_0;T}(x) = \begin{cases} -x_0^2, & \text{if } x < 0 \\ x^2 - x_0^2, & \text{if } 0 \leq x \leq 1 \\ 2x - x_0^2 - 1, & \text{if } 1 < x \leq 2 \\ 3x - x_0^2 - 3, & \text{if } x > 2. \end{cases}$$

Now, if $x_0 = 2$ we have

$$f_{2,T}(x) = \begin{cases} -4, & \text{if } x < 0 \\ x^2 - 4, & \text{if } 0 \leq x \leq 1 \\ 3(x-2), & \text{if } x > 1. \end{cases}$$

Therefore, the equality (iii) is true if and only if $x_0, x_1 \in [0,1]$. Now, if $x_1 = 2$, then $f_{2;T}(x_0) + f_{x_0;T}(2) \neq 0$, for any $x_0 \in [0,1]$.

Also, if $x_0 \in [0,1]$, then the subdifferential is as follows: $\partial f_{x_0;T}(x) = 0$ if $x < 0$, $\partial f_{x_0;T}(x) = 2x$ if $x \in [0,1]$, $\partial f_{x_0;T}(x) = 2$ if $x \in (1,2)$, $\partial f_{x_0}(2) = [2,3]$ and $\partial f_{x_0;T}(x) = 3$ if $x > 2$. In the case $x_0 = 2$ we have: $\partial f_{2;T}(x) = 0$ if $x < 0$, $\partial f_{2;T}(x) = 2x$ if $x \in [0,1)$, $\partial f_{2;T}(x) = [2,3]$ if $x = 1$ and $\partial f_{x_0;T}(x) = 3$ if $x > 1$. Thus, $\partial f_{x_0;T}(x) = Tx$ only for $x \in [0,1)$, $x_0 \in \operatorname{Dom} T$ but obviously, Graph $T \subset$ Graph $\partial f_{x_0;T}$. Moreover, $\partial f_{x_0;T}(x) = \partial f_{2;T}(x)$, $x_0 \in [0,1]$, if and only if $x \in (\infty, 1) \cup [2, \infty)$. On every of these intervals the functions coincide up to a constant function which can be different.

Finally, we remark that the mapping T have monotone extension to real line which are not of the form $\partial f_{x_0;T}$ for a certain $x_0 \in \operatorname{Dom} T$. Therefore, *there exist maximal ciclically monotone extensions of a given ciclically monotone mapping which are not subdifferential of a corresponding Rockafellar function for some* $x_0 \in \operatorname{Dom} T$.

Similarly to (7), we can consider the *upper Darboux integral* as follows

$$\begin{aligned} F_{x_0;T}(x) = \inf \Bigg\{ & \sum_{i=0}^{n} x_{i+1}^*(x_{i+1} - x_i); (x_i, x_i^*) \in \operatorname{Graph} T, \\ & i = \overline{1, n+1}, x_{n+1} = x \Bigg\}, \quad \text{for any } x \in \operatorname{Dom} T, \end{aligned} \tag{14}$$

where x_0 is a given element in $\operatorname{Dom} T$. Here we must consider only $x \in \operatorname{Dom} T$. But, this associated function is not essentially different from $f_{x_0;T}$ since by simple arguments it follows that

$$F_{x_0;T}(x) = -f_{x;T}(x_0), \quad \text{for all } x_0, x \in \operatorname{Dom} T. \tag{15}$$

Obviously, this equality can be regarded as an extended form of a well known property of Darboux integrals of real functions on an interval.

Remark 2.7. In the real case the cyclically monotone mapping T is an increasing function and so, if T contains a continuous increasing function on a closed real internal I, then we have even the equality $f_{x_0;T}(x) = F_{x_0;T}(x)$ for all $x, x_0 \in I$. For instance, if T is the operator in Example 2.6 we have $f_{x_0;T}(x) = F_{x_0;T}(x)$, whenever $x, x_0 \in [0,1]$. But, if $x = 2$ and $x_0 \in [0,1]$ then $f_{x_0,T}(2) = 3 - x_0^2$ and $F_{x_0;T}(2) = 4 - x_0^2$, and so, $f_{x_0;T} \neq F_{x_0;T}$.

If we come back to the Theorem 2.2 (*iii*) it follows that the last equality is not true for an arbitrary cyclically monotone mapping. But we have a weaker property given by equality (15). On the other hand, according to the property (6) always holds the following inequality

$$f_{x_0;T}(x) \leq F_{x_0;T}(x), \quad \forall x, x_0 \in \operatorname{Dom} T. \tag{16}$$

Consequently, by (15) we have

$$f_{x_0;T}(x_1) + f_{x_1;T}(x_0) \leq 0, \quad \text{for all } x_0, x_1 \in \operatorname{Dom} T. \tag{17}$$

Thus, the property (*iii*) in Theorem 2.2 has the following weaker equivalent form

$$f_{x_0;T}(x_1) + f_{x_1;T}(x_0) \geq 0. \tag{18}$$

Concerning the inclusion (10) we remark that the functions $f_{x_0;T}, x_0 \in \operatorname{Dom} T$ have a property of minimality.

Proposition 2.8 ([3]). *Let $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous function such that* $\operatorname{Graph} T \subset \operatorname{Graph} \partial f$. *Then, for any $x_0 \in \operatorname{Dom} T$ we have*

$$f_{x_0;T}(x) \leq f(x) - f(x_0), \quad \textit{for all } x \in X. \tag{19}$$

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a finite system of elements in $\operatorname{Dom} T$. By hypothesis, using (3) we get

$$\sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) \leq \sum_{i=0}^{n} (f(x_{i+1}) - f(x_i)) = f(x) - f(x_0),$$

for any $(x_i, x_i^*) \in \operatorname{Graph} T$, $i = \overline{0, n+1}$, where $x_{n+1} = x$.

Therefore, by (7) we obtain the inequality (19) as claimed. □

Remark 2.9. In [3] the authors give sufficient conditions to have just equality.

Now, we prove that the functions $f_{x_0;T}, x_0 \in \operatorname{Dom} T$ are specific only to cyclically monotone mappings.

Theorem 2.10. *The following properties are equivalent:*

(*i*) *T is a cyclically monotone mapping;*

(*ii*) $f_{x_0;T}(x_0) = 0$, *for any* $x_0 \in \operatorname{Dom} T$;
(*iii*) $\operatorname{Dom}(f_{x_0;T}) \neq \emptyset$, *for any* $x_0 \in \operatorname{Dom} T$.

Proof. Obviously $(i) \Rightarrow (ii) \Rightarrow (iii)$. Now, if we suppose that T is not cyclically monotone, then the exists a system of elements $(\overline{x}_i, \overline{x}_i^*) \in \operatorname{Graph} T$, $i = \overline{1,k}$, such that

$$\sum_{i=1}^{k-1} \overline{x}_i^*(\overline{x}_{i+1} - \overline{x}_i) = a > 0, \quad \text{where } \overline{x}_k = \overline{x}_1.$$

For a given element $x_0 \in \operatorname{Dom} T$, we consider the system $(x_i, x_i^*) \in \operatorname{Graph} T$, $i = \overline{0, nk}$, where $x_i = \overline{x}_i$, $i = \overline{1,k}$, $\overline{x}_j = x_{j+lk}$, $j = \overline{1,k}$, $l = \overline{1, n-1}$. Then, we have

$$\sum_{i=0}^{nk} x_i^*(x_{i+1} - x_i) + x_{nk}^*(x - x_{nk}) = x_0^*(x_0 - \overline{x}_1) + na + \overline{x}_1^*(x - \overline{x}_1).$$

Now, using (7) we get

$$f_{x_0;T}(x) \geq \lim_{n\to\infty} [na + x_0^*(x_0 - \overline{x}_1) + \overline{x}_1^*(x - \overline{x}_1)] = \infty, \text{ for all } x \in X.$$

Consequently, the implications $(iii) \rightarrow (i)$ holds. □

Therefore, the Rockfellar functions $f_{x_0;T}$, $x_0 \in \operatorname{Dom} T$ can be considered only for cyclically monotone mappings. Thus it is natural to find versions of definitions (7) for arbitrary mappings. A possibility is generated by the next result.

Proposition 2.11. *If T is a cyclically monotone mapping then*

$$f_{x_0;T} = \widetilde{f}_{x_0;T}, \quad \textit{for all } x_0 \in \operatorname{Dom} T, \tag{20}$$

where $\widetilde{f}_{x_0;T}$ is defined by an equality of type (7) taking only systems $x_i \in \operatorname{Dom} T$, $i = \overline{1,n}$, of distinct elements, $x_i \neq x_j$, $i \neq j$, $i,j = \overline{1,n}$.

Proof. Let us consider a system $(x_i, x_i^*) \in \operatorname{Graph} T$, $i = \overline{0,n}$ which contains two equal elements, $x_k = x_{k+l}$, $k, k+l = \overline{1,n}$, $l > 0$. Using (6) we obtain

$$\begin{aligned}
&\sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) + x_n^*(x - x_n) \\
&\leq \sum_{i=0}^{k-1} x_i^*(x_{i+1} - x_i) + \sum_{i=k+l}^{n-1} x_i^*(x_{i+1} - x_i) + x_n^*(x - x_n).
\end{aligned}$$

Obviously, in the right hand of this inequality we also have a sum corresponding to a system in $\operatorname{Graph} T$, and so, in the definition (7) of the function $f_{x_0;T}$ all systems $x_i \in \operatorname{Dom} T$, $i = \overline{1,n}$, which contain equal elements can be omitted. Hence the equality (20) holds. □

If T is not a cyclically monotone mapping, then $f_{x_0;T}$ is identically ∞, while $\widetilde{f}_{x_0;T}$ can be a proper convex lower semicontinuous function. This is the case of mappings T which contain cyclically monotone mappings. Also, it is possible that $\partial\widetilde{f}_{x_0;T}(x) = Tx$ for some elements $x \in \operatorname{Dom} T$.

Remark 2.12. The converse of Proposition 2.9 is not true. For example, if T is the Dirichlet function, $Tx = 1$ if x is a rational number and $Tx = 0$ if x is an irrational number, then $f_{x_0;T} = \widetilde{f}_{x_0;T} = \infty$, for any $x_0 \in \mathbb{R}$.

3. Some dual properties

Firstly, we give the equivalent definitions of the functions $f_{x_0;T}$ and $F_{x_0;T}$ with respect to dual space X^* namely

$$f_{x_0;T}(x) = \sup\left\{ \sum_{i=1}^{n}(x^*_{i-1} - x^*_i)(x_i) + x^*_n(x) - x^*_0(x_0); \right. \\ \left. (x_i, x^*_i) \in \operatorname{Graph} T, i = \overline{0,n} \right\}, \quad x \in X, \tag{21}$$

respectively,

$$F_{x_0;T}(x) = \inf\left\{ \sum_{i=0}^{n}(x^*_i - x^*_{i+1})(x_i) + x^*(x) - x^*_0(x_0); \right. \tag{22}$$
$$\left. (x_i, x^*_i) \in \operatorname{Graph} T, i = \overline{0,n+1}, x_{n+1} = x \right\}, \quad x \in \operatorname{Dom} T.$$

Obviously, if T is a cyclically monotone mapping, then T^{-1} is also a cyclically monotone mapping from X^* into X, and so, according to (7) and (14), the corresponding associated functions are the following:

$$f_{x^*_0;T^{-1}}(x^*) = \sup\left\{ \sum_{i=0}^{n-1}(x^*_{i+1} - x^*_i)(x_i) + (x^* - x^*_n)(x_n); \right. \\ \left. (x_i, x^*_i) \in \operatorname{Graph} T, i = \overline{0,n} \right\}, \quad x^* \in X^*, \tag{23}$$

$$_{x^*_0;T^{-1}}(x^*) = \inf\left\{ \sum_{i=0}^{n-1}(x^*_{i+1} - x^*_i)(x_{i+1}) + (x^* - x^*_n)(x); \right. \tag{24}$$
$$\left. x \in T^{-1}(x^*), (x_i, x^*_i) \in \operatorname{Graph} T, i = \overline{0,n} \right\}, \quad x^* \in \operatorname{Dom} T^{-1}.$$

On the other hand, if in the right hand of (22) the system $\{x_1, x_2, \ldots, x_n\}$ is replaced by the system $\{x_n, x_{n-1}, \ldots, x_1\}$, then we get the following equivalent formulas:

$$f_{x_0^*;T^{-1}}(x^*) = \sup\left\{\sum_{i=1}^{n-1}(x_i^* - x_{i+1}^*)(x_{i+1}) + (x_n^* - x_0^*)(x_0) + (x^* - x_1^*)(x_1);\right.$$
$$\left.(x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0,n}\right\}, \quad x^* \in X^*, \tag{25}$$

$$F_{x_0^*;T^{-1}}(x^*) = \inf\left\{\sum_{i=1}^{n-1}(x_i^* - x_{i+1}^*)(x_i) + (x_n^* - x_0^*)(x_n)\right.$$
$$+ (x_{n+1}^* - x_1^*)(x_{n+1}) \tag{26}$$
$$\left.(x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0, n+1}, x_{n+1}^* = x^*\right\}, \quad x^* \in \operatorname{Dom} T^{-1}.$$

Indeed, from (23) we obtain

$$\begin{aligned} f_{x_0^*;T^{-1}}(x^*) = \sup\Big\{&(x_1^* - x_0^*)(x_0) + \sum_{i=1}^{n-1}(x_{i+1}^* - x_i^*)(x_i) \\ &+ (x^* - x_n^*)(x_n); (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0,n}\Big\} \\ = \sup\Big\{&(x_n^* - x_0^*)(x_0) + \sum_{i=1}^{n-1}(x_i^* - x_{i+1}^*)(x_{i+1}) \\ &+ (x^* - x_1^*)(x_1); (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0,n}\Big\}, \end{aligned}$$

that is (25) holds. Similarly we obtain (26).

In the next we extend $F_{x_0;T}$ to X by convexity taking $\overline{\operatorname{conv}}\, F_{x_0;T}$ on $\overline{\operatorname{conv}}\, \operatorname{Dom} T$ and $F_{x_0;T}(x) = \infty$ if $x \overline{\in} \overline{\operatorname{conv}}\, \operatorname{Dom} T$. Thus, the inequality (16) is conserved for all $x \in X$. Also,

$$F_{x_0;T}^* \le f_{x_0;T}^*. \tag{27}$$

Theorem 3.1. *Let T be a cyclically monotone mapping such that $T(x_0) = \{x_0^*\}$. Then we have the following inequalities:*

$$f_{x_0^*;T^{-1}}(x^*) \le f_{x_0;T}(x) + s_{\operatorname{Dom} T}(x^* - x_0^*) + s_{\operatorname{Dom} T^{-1}}(x_0 - x), \quad \textit{for all}\,(x, x^*) \in X \times X^*; \tag{28}$$

$$x_0^*(x_0) + f_{x_0^*;T^{-1}}(x^*) \le F_{x_0;T}^*(x^*), \quad \textit{for all } x^* \in X^*; \tag{29}$$

$$x_0^*(x_0) + F_{x_0^*;T^{-1}}(x^*) \ge f_{x_0;T}^*(x^*), \quad \textit{for all } x^* \in \operatorname{Dom} \partial f_{x_0;T}^*. \tag{30}$$

Proof. Using (2) and (25) we have

$$
\begin{aligned}
& f_{x_0^*;T^{-1}}(x^*) \\
= \ & \sup\Big\{ \sum_{i=1}^{n}(x_{i-1}^* - x_i^*)(x_i) + (x_n^* - x_0^*)(x_0) \\
& \quad + (x^* - x_0^*)(x_1); (x_i, x_i^*) \in \operatorname{Graph} T \Big\} \\
= \ & \sup\Big\{ \Big[\sum_{i=1}^{n}(x_{i-1}^* - x_i^*)(x_i) + x_n^*(x) - x_0^*(x_0) \Big] \\
& \quad + [x_n^*(x_0 - x) + (x^* - x_0^*)(x_1)]; (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0,n} \Big\} \\
\leq \ & \sup\Big\{ \sum_{i=1}^{n}(x_{i-1}^* - x_i^*)(x_i) + x_n^*(x) - x_0^*(x_0); (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0,n} \Big\} \\
& \quad + \sup\{x_n^*(x_0 - x); x_n^* \in \operatorname{Dom} T^{-1}\} + \sup\{(x^* - x_0^*)(x_1); x_1 \in \operatorname{Dom} T\} \\
= \ & f_{x_0;T}(x) + s_{\operatorname{Dom} T^{-1}}(x_0 - x) + s_{\operatorname{Dom} T}(x^* - x_0^*).
\end{aligned}
$$

Therefore, the inequality (28) holds.

Now, using (1), (22) and (23) we obtain

$$
\begin{aligned}
F_{x_0;T}^*(x^*) = \ & \sup\{x^*(u) - F_{x_0;T}(u); u \in X\} \\
= \ & \sup\{x^*(u) - F_{x_0;T}(u); u \in \operatorname{Dom} T\} \\
= \ & \sup\Big\{ x^*(u) + \sum_{i=0}^{n}(x_{i+1}^* - x_i^*)(x_i) - u^*(u) + x_0^*(x_0); (u, u^*), \\
& \quad (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0;n} \Big\} \\
\geq \ & x_0^*(x_0) + \sup\Big\{ \sum_{i=0}^{n-1}(x_{i+1}^* - x_i^*)(x_i) + (x^* - x_n^*)(x_n); \\
& \quad (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0,n} \Big\}, \quad x^* \in X^*.
\end{aligned}
$$

Hence, the inequality (29) is proved.

Let us consider an element $x^* \in Tx$, $x \in \operatorname{Dom} T$. Using (1) and (21) we get

$$
\begin{aligned}
f^*_{x_0;T}(x^*) &= \sup\{x^*(u) - f_{x_0;T}(u); u \in X\} \\
&= \sup\Bigg\{x^*(u) - \sup\Bigg\{\sum_{i=1}^{n}(x^*_{i-1} - x^*_i)(x_i) - x^*_n(u) + x^*_0(x_0); \\
&\qquad u \in X, (x_i, x^*_i) \in \operatorname{Graph} T, i = \overline{1,n}\Bigg\}\Bigg\} \\
&= x^*_0(x_0) + \sup\inf\Bigg\{(x^* - x^*_n)(u) + \sum_{i=0}^{n-1}(x^*_{i+1} - x^*_i)(x_{i+1}); \\
&\qquad u \in X, (x_i, x^*_i) \in \operatorname{Graph} T, i = \overline{1,n}\Bigg\}.
\end{aligned}
$$

Taking $(x_n, x^*_n) = (x, x^*)$, according to (26) it follows that

$$
\begin{aligned}
f^*_{x_0;T}(x^*) &\leq x^*_0(x_0) + \inf\Bigg\{\sum_{i=0}^{n-2}(x^*_{i+1} - x^*_i)(x_{i+1}) + (x^* - x^*_{n-1})(x); \\
&\qquad (x_i, x^*_i) \in \operatorname{Graph} T, i = \overline{0, n-1}\Bigg\} \\
&= x^*_0(x_0) + F_{x^*_0;T^{-1}}(x^*), \quad \text{for all } x^* \in \operatorname{Dom} T^{-1}.
\end{aligned}
$$

But, we remark that the Roackafellar function corresponding to mapping $\partial f_{x_0;T}$ is also $f_{x_0;T}$. Consequently, the previous inequality is true for any $x^* \in \operatorname{Dom}(\partial f_{x_0;T})^{-1} = \operatorname{Dom} f^*_{x_0;T}$ (see (5)), and so, the proof is complete. □

Corollary 3.2. *Let T be a cyclically monotone mapping such that $T(x_0) = \{x^*_0\}$. If $(x, x^*) \in \operatorname{Graph} \partial f^*_{x_0;T}$ and $f^*_{x_0;T}(x^*) = F_{x^*_0;T^{-1}}(x^*)$, then*

$$
x^*_0(x_0) + f_{x^*_0;T^{-1}}(x^*) = f^*_{x_0;T}(x^*). \tag{31}
$$

Proof. Apply (27), (29) and (30). □

Remark 3.3. By a standard calculus we get that $f^*_{x_0;T}(x) = \infty$, for all $x \in (-\infty, 0) \cup (3, \infty)$, where T is the mapping in Example 2.6. On the other hand, $f_{x^*_0;T^{-1}}(x^*)$ is a real number for any $x^* \in X^*$. Therefore, the equality (31) is not true on X^*. Obviously, if $x^* \in (0, 2)$ the equality (31) holds.

Remark 3.4. Under certain specific hypothesis the equality (31) can be obtained using a result of Burachik, Martínez-Legaz and Rocco [7] concerning the equality of two maximal monotone operators since $\partial f_{x^*_0;T^{-1}} = \partial f^*_{x_0;T}$ on the intersection of their domains.

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