Journal of Convex Analysis
Volume 24 (2017), No. 1, 333–347

A Note on the Extension of Continuous Convex Functions from Subspaces

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Received: June 6, 2014
Revised manuscript received: February 26, 2015
Accepted: July 22, 2015

Let Y be a subspace of a real normed space X. We say that the couple (X, Y) has the CE-*property* ("convex extension property") if each continuous convex function on Y admits a continuous convex extension defined on X.

By using techniques of Johnson and Zippin, we prove the following results about the CE-property: if X is the $c_0(\Gamma)$-sum or the $\ell_p(\Gamma)$-sum $(1 < p < \infty)$ of separable normed spaces, then the couple (X, Y) has the CE-property, for each subspace Y of X. Another similar result concerns weak*-closed subspaces Y of $X = \ell_1(\Gamma) = c_0(\Gamma)^*$.

Keywords: Convex function, extension, normed linear space

1991 Mathematics Subject Classification: Primary 52A41; Secondary 26B25, 46A99

Introduction

Let X be a real normed linear space and $Y \subset X$ a subspace. We say that the couple (X, Y) has the CE-*property* ("convex extension property") if each real-valued continuous convex function on Y admits a continuous convex extension defined on X.

The CE-property has been recently studied in [3, 4, 5, 10]. It is known [4, Theorem 5.5 and Lemma 4.2] that if X/Y is separable or Y is dense in X then the couple (X, Y) has the CE-property; moreover, it is an easy observation that if Y is complemented in X then the couple (X, Y) has the CE-property (see e.g. [4, Proposition 4.3]). However not all the couples (X, Y) have the CE-property, as the couple (ℓ_∞, c_0) shows (see e.g. [4, Corollary 4.6]).

The results cited above easily imply that, if X is separable or isomorphic with a Hilbert space, the couple (X, Y) has the CE-property, whenever Y is a subspace

*The author was supported in part by the fund FSE, Regione Lombardia, and by GNAMPA-INdAM.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

of X. In the present paper, we introduce a wider class of normed linear spaces satisfying the same property. For example, we show that, if X is an arbitrary $c_0(\Gamma)$-sum or $\ell_p(\Gamma)$-sum $(1 < p < \infty)$ of separable normed spaces, then the couple (X, Y) has the CE-property for each subspace Y of X. Another similar result is obtained in the case X is an arbitrary $\ell_1(\Gamma)$-sum of separable normed spaces and Y is closed in a certain weak-topology (cf. Corollary 3.4).

The main tools used in our proofs are the results concerning the CE-property, contained in [4, 5], and a slight variation of a *decomposition lemma* [8, Lemma 2], introduced by W. B. Johnson and M. Zippin to study extension properties of operators from subspaces of $c_0(\Gamma)$ into $C(K)$ spaces.

The paper is organized as follows. In Section 1 we recall some known results from [4, 5] about the characterization of the CE-property. Moreover, in the same section, we give some new sufficient conditions for the CE-property. In Section 2 we introduce some notations and basic results about certain sums of normed spaces. Section 3 contains the main results of the present paper.

1. Preliminaries and the CE-property

We consider normed linear spaces over $\mathbb{R}$. Let X be a real normed linear space, we denote by B_X and B_X^0 the closed unit ball and the open unit ball of X, respectively. Moreover, we denote by X^* the topological dual of X and by w^* the weak*-topology of X^*. If Y is a subspace of X then we denote by q the linear w^*-w^*-continuous map $q : X^* \to Y^*$ defined by $q(x^*) = x^*|_Y$, for each $x^* \in X^*$. All convex functions we consider are real-valued.

Notation 1.1.

(a) In the sequel, all nets of the form $\{t_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$, where Γ is a nonempty set, are directed by the following relation:

$$(n_1, \gamma_1) \le (n_2, \gamma_2) \ \text{ iff } n_1 \le n_2.$$

(b) Given sets $E, E_1, E_2, \ldots$, the notation $E_n \nearrow E$ means that the sequence $\{E_n\}$ is nondecreasing and its union is the set E. The notation $E_n \searrow E$ means that the sequence $\{E_n\}$ is nonincreasing and its intersection is the set E.

Now, we recall some results, concerning the CE-property, contained in [4] (stated here for normed spaces).

Proposition 1.2 ([4, Corollary 3.7]). *Let Y be a subspace of a normed linear space X. Then the following assertions are equivalent.*

(*i*) *The couple (X, Y) has the* CE*-property.*

(*ii*) *For every sequence $\{C_n\}$ of open convex sets in Y such that $C_n \nearrow Y$, there exists a sequence $\{D_n\}$ of open convex sets in X such that $D_n \nearrow X$, and $D_n \cap Y = C_n$ for each $n \in \mathbb{N}$.*

(iii) *For every sequence $\{B_n\}$ of w^*-compact convex bounded sets in Y^* such that $B_n \searrow \{0\}$, there exists a sequence $\{E_n\}$ of w^*-compact convex bounded sets in X^* such that $q(E_n) = B_n$, for each $n \in \mathbb{N}$, and such that $E_n \searrow \{0\}$.*

(iv) *For every bounded w^*-null net $\{y^*_{n,\gamma}\} \subset Y^*$, there exists a bounded w^*-null net $\{x^*_{n,\gamma}\} \subset X^*$ such that $q(x^*_{n,\gamma}) = y^*_{n,\gamma}$ for each $(n,\gamma) \in \mathbb{N} \times \Gamma$.*

Remark 1.3. If X is a Banach space and B is a w^*-compact set in X^* then the uniform boundedness principle implies that B is bounded. Let us observe that the completeness hypothesis is necessary as the following example shows: let $X = c_{00}$, the space of all eventually null sequences endowed with the supremum norm, then if $\{e_n\}_{n\in\mathbb{N}}$ is the canonical basis of $\ell_1 = X^*$ we have that $B := \{ne_n;\, n \in \mathbb{N}\} \cup \{0\}$ is a w^*-compact unbounded set.

However, it is known (see e.g. [7, p. 17]) that, if X is a normed space, each w^*-compact convex set in X^* is bounded. Hence, in Proposition 1.2, assertion (iii), the word "bounded" can be omitted.

If X is a normed space, we denote by $c_0(X)$ the c_0-sum of countably many copy of the space X. In [5], L. Veselý and the author observed that a couple (X,Y) has the CE-property iff, whenever Γ is a nonempty set, each continuous linear operator $T_0 : Y \to c_0(\ell_\infty(\Gamma))$ admits a continuous linear extension $T : X \to c_0(\ell_\infty(\Gamma))$. In view of this fact, we give the following definition.

Definition 1.4. Let X be a normed space, let Y be a subspace of X and let $\lambda \geq 1$. We say that the couple (X,Y) has the λ-EP (λ-*extension property*) iff, whenever Γ is a nonempty set, each continuous linear operator $T_0 : Y \to c_0(\ell_\infty(\Gamma))$ admits a continuous linear extension $T : X \to c_0(\ell_\infty(\Gamma))$, with $\|T\| \leq \lambda \|T_0\|$.

We have the following quantitative version of Proposition 1.2, the proof of which follows easily by the proof of [4, Corollary 3.7] and [5, Proposition 5.4].

Proposition 1.5. *Let Y be a subspace of a normed linear space X and $\lambda \geq 1$. Then the following assertions are equivalent.*

(i) *The couple (X,Y) has the λ-EP.*

(ii) *For every sequence $\{C_n\}$ of open convex sets in Y such that $C_n \nearrow Y$ and such that $B^0_Y \subset C_1$, there exists a sequence $\{D_n\}$ of open convex sets in X such that $D_n \nearrow X$, $\frac{1}{\lambda}B^0_X \subset D_1$ and $D_n \cap Y = C_n$ for each $n \in \mathbb{N}$.*

(iii) *For every sequence $\{B_n\}$ of w^*-compact convex sets contained in B_{Y^*} such that $B_n \searrow \{0\}$, there exists a sequence $\{E_n\}$ of w^*-compact convex sets contained in λB_{X^*} such that $q(E_n) = B_n$, for each $n \in \mathbb{N}$, and such that $E_n \searrow \{0\}$.*

(iv) *For every w^*-null net $\{y^*_{n,\gamma}\} \subset B_{Y^*}$, there exists a w^*-null net $\{x^*_{n,\gamma}\} \subset \lambda B_{X^*}$ such that $q(x^*_{n,\gamma}) = y^*_{n,\gamma}$ for each $(n,\gamma) \in \mathbb{N} \times \Gamma$.*

Fact 1.6. *Let S be a Hausdorff topological space, let $\{E_n\}$ be a decreasing sequence of compact subsets of S. Suppose that V is an open set in S such that $V \supset \bigcap_n E_n$, then there exists n_0 such that $E_{n_0} \subset V$.*

Proposition 1.7. *Let Y be a subspace of a normed space X and let $\lambda \geq 1$. Let us consider the following assertions.*

(*i*) *There exists a function $\varphi : B_{Y^*} \to \lambda B_{X^*}$ such that $\varphi(0) = 0$, such that φ is w^*-w^*-continuous at 0, and such that $q \circ \varphi = Id_{B_{Y^*}}$.*
(*ii*) *There exists a function $\varphi : B_{Y^*} \to \lambda B_{X^*}$ such that $\varphi(0) = 0$, such that φ is w^*-w^*-sequentially continuous at 0, and such that $q \circ \varphi = Id_{B_{Y^*}}$.*
(*ii′*) *There exists a function $\widetilde{\varphi} : B_{Y^*} \to X^*$ such that $\widetilde{\varphi}(0) = 0$, such that $\widetilde{\varphi}$ is w^*-w^*-sequentially continuous at 0, such that $q \circ \widetilde{\varphi} = Id_{B_{Y^*}}$, and such that $\|\widetilde{\varphi}(y^*)\| \leq \lambda \|y^*\|$, for each $y^* \in B_{Y^*}$.*
(*iii*) *The couple (X, Y) has the λ-EP.*
(*iv*) *The couple (X, Y) has the CE-property.*

Then the following implications hold $(i) \Rightarrow (ii) \Leftrightarrow (ii') \Rightarrow (iii) \Rightarrow (iv)$. Moreover, if Y is separable then we have $(iii) \Rightarrow (i)$.

Proof. The implications $(i) \Rightarrow (ii)$, $(ii') \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

Let us prove that $(ii) \Rightarrow (iii)$. Let λ and φ be as in (ii) and let us consider a w^*-null net $\{y^*_{n,\gamma}\} \subset B_{Y^*}$. For every $(n, \gamma) \in \mathbb{N} \times \Gamma$, put $x^*_{n,\gamma} = \varphi(y^*_{n,\gamma})$. Then $\{x^*_{n,\gamma}\} \subset \lambda B_{X^*}$ is such that $x^*_{n,\gamma}|_Y = y^*_{n,\gamma}$, for each $(n, \gamma) \in \mathbb{N} \times \Gamma$. Moreover, $\{x^*_{n,\gamma}\}$ is a w^*-null net. Indeed, if this is not the case, there exists $x_0 \in X$ such that $x^*_{n,\gamma}(x_0) \not\to 0$, that is, there exists a sequence $(\gamma_n) \subset \Gamma$ such that $x^*_{n,\gamma_n}(x_0) \not\to 0$. But $\{y^*_{n,\gamma_n}\}$ is w^*-null and hence, by the w^*-w^*-sequentially continuity of φ at 0, x^*_{n,γ_n} is w^*-null. A contradiction.

Let us prove that $(ii) \Rightarrow (ii')$. Let φ as in (ii). For each $y^* \in B_{Y^*} \setminus \{0\}$, define $\widetilde{\varphi}(y^*) = \|y^*\| \varphi(y^*/\|y^*\|)$ and put $\widetilde{\varphi}(0) = 0$. It is easy to see that $\widetilde{\varphi}$ satisfies all the conditions in (ii').

Now, assume that Y is separable and that (iii) holds. Let us prove that (i) holds. Suppose that $\{y_n\}_n \subset Y$ separates the points of B_{Y^*}, put $B_0 = B_{Y^*}$ and put, for every $n \in \mathbb{N}$,

$$B_n = \{y^* \in B_{Y^*};\ |y^* y_i| \leq 1/n,\ i = 1, \ldots, n\}.$$

Then $\{B_n\}_{n=0}^{\infty}$ is a decreasing sequence of w^*-compact convex sets in Y^* such that, for every $n \in \mathbb{N} \cup \{0\}$, B_n is a neighborhood of the origin in (B_{Y^*}, w^*). Moreover, it is easy to see that $B_n \searrow \{0\}$.

Since (X, Y) has the λ-EP, there exists a sequence $\{E_n\}_{n=0}^{\infty}$ of w^*-compact convex sets in X^* such that $E_n \searrow \{0\}$, such that $E_0 \subset \lambda B_{X^*}$ and such that $q(E_n) = B_n$, for each $n \in \mathbb{N} \cup \{0\}$. Let us define a function $\varphi : B_{Y^*} \to \lambda B_{X^*}$ as follows. Put $\varphi(0) = 0$ and, if $y^* \in B_{n-1} \setminus B_n$ $(n \in \mathbb{N})$, put $\varphi(y^*) = x^*$, where x^* is any element of E_{n-1} such that $q(x^*) = y^*$. The function φ is well defined, moreover, by our construction, $q \circ \varphi = Id_{B_{Y^*}}$. We claim that φ is w^*-w^*-continuous at 0. To prove our claim, let W be a neighborhood of the origin in (X^*, w^*). By Fact 1.6, there exists n_0 such that $E_{n_0} \subset W$. In particular, $\varphi(B_{n_0}) \subset E_{n_0} \subset W$. This concludes the proof of the claim, since B_{n_0} is a a neighborhood of the origin in (B_{Y^*}, w^*). □

Corollary 1.8. *Let X be a separable normed space and Y a subspace of X. Then there exists a function $\varphi : B_{Y^*} \to X^*$ such that:*

- $\varphi(0) = 0$ *and* φ *is* w^*-w^*-*continuous at* 0*;*
- $q \circ \varphi = Id_{B_{Y^*}}$*;*
- $\|\varphi(y^*)\| \leq 2\|y^*\|$, *for each* $y^* \in B_{Y^*}$.

Proof. By [5, Theorem 5.6], the couple (X, Y) has the 2-EP. Since Y is separable, assertion (ii') in Proposition 1.7 holds with $\lambda = 2$. Since (B_{Y^*}, w^*) is metrizable, w^*-w^*-continuity and w^*-w^*-sequential continuity are equivalent and we are done. □

The following proposition is due to L. Veselý (private communication). It shows that the implication $(iii) \Rightarrow (iv)$ in Proposition 1.7 can be "reversed" in a sense.

Proposition 1.9. *If the couple (X, Y) has the* CE-*property then (X, Y) has the* λ-EP *for some $\lambda \geq 1$.*

Proof. Suppose on the contrary that the couple (X, Y) has the CE-property and that, for each $n \in \mathbb{N}$, there exist a nonempty set Γ_n and a linear norm-one operator $t_{n,0} : Y \to c_0(\ell_\infty(\Gamma_n))$ such that, if $t_n : X \to c_0(\ell_\infty(\Gamma_n))$ extends $t_{n,0}$, then $\|t_n\| \geq 4^n$. Put $\Gamma = \bigcup_n \Gamma_n$, then, without any loss of generality, we can suppose that $\Gamma = \Gamma_n$ for each $n \in \mathbb{N}$. Let us define $T_{n,0} : Y \to c_0(c_0(\ell_\infty(\Gamma)))$ as follows

$$[T_{n,0}(y)](m) = \begin{cases} t_{n,0}(y) & \text{if } m = n; \\ 0 & \text{otherwise.} \end{cases}$$

Now, if we put $T_0 = \sum_n \frac{1}{2^n} T_{n,0}$, it is clear that $T_0 : Y \to c_0(c_0(\ell_\infty(\Gamma)))$ is a bounded linear operator. Moreover, since $c_0(c_0(\ell_\infty(\Gamma)))$ is isometric isomorphic with $(c_0(\ell_\infty(\Gamma))$ and since the couple (X, Y) has the CE-property, T_0 admits a bounded linear extension $T : X \to c_0(c_0(\ell_\infty(\Gamma)))$. For $n \in \mathbb{N}$, let us denote by $P_n : c_0(c_0(\ell_\infty(\Gamma))) \to c_0(\ell_\infty(\Gamma))$ the canonical norm-one projection defined by $P_n\left(\{\{x_{h,k}\}_k\}_h\right) = \{x_{n,k}\}_k$. Then, for each $n \in \mathbb{N}$, $2^n P_n \circ T$ extends $2^n P_n \circ T_0 = t_{n,0}$ and hence, by our hypothesis, $\|2^n P_n \circ T\| \geq 4^n$. But this implies $\|T\| \geq 2^n$, for each $n \in \mathbb{N}$, a contradiction. □

The following lemma is an easy modification of [1, Proposition 3.2, (c)].

Lemma 1.10. *Let X_i $(i = 1, 2)$ and E be Banach spaces and $Y_i \subset X_i$ $(i = 1, 2)$ closed subspaces. Let $q_i : X_i \to X_i/Y_i$ $(i = 1, 2)$ be the canonical quotient maps* *and suppose that* $q_1(B_{X_1}) = B_{X_1/Y_1}$. *Suppose that* $X_1/Y_1 = X_2/Y_2$ *(isometrically) and that there exists a continuous linear operator* $Q : X_1 \to X_2$ *such that the following diagram commutes*

$$\begin{array}{ccc} X_1 & \xrightarrow{q_1} & X_1/Y_1 \\ {\scriptstyle Q}\Big\downarrow & & \Big\| \\ X_2 & \xrightarrow[q_2]{} & X_2/Y_2 \ . \end{array}$$

Let $\lambda \geq 1$ *and suppose that each linear continuous operator* $S_0 : Y_1 \to E$ *admits a linear extension* $S : X_1 \to E$ *such that* $\|S\| \leq \lambda\|S_0\|$.

Then each linear continuous operator $T_0 : Y_2 \to E$ *admits a linear extension* $T : X_2 \to E$ *such that* $\|T\| \leq (1 + \|Q\| + \lambda\|Q\|)\|T_0\|$.

Proof. Let $s : X_1/Y_1 \to X_1$ be a homogeneous lifting of q_1 such that $\|s(z)\| \leq \|z\|$, for each $z \in B_{X_1/Y_1}$ (this is possible since $q_1(B_{X_1}) = B_{X_1/Y_1}$). Let $S_0 = T_0 \circ (Q|_{Y_1})$ and let $S : X_1 \to E$ be a continuous linear extension of S_0 such that $\|S\| \leq \lambda\|S_0\| \leq \lambda\|T_0\|\|Q\|$. For each $x \in X_2$, put

$$T(x) = T_0(x - Qsq_2x) + S(sq_2x).$$

It is not difficult to verify that $T : X \to E$ is well defined, linear and extends T_0. An easy computation shows that $\|T\| \leq (1 + \|Q\| + \lambda\|Q\|)\|T_0\|$. $\square$

The remaining part of this section is an easy consequence of the results stated above and it is independent of the other sections of the present paper.

Given $\lambda \geq 1$, a Banach space Z is said to be λ-*separably injective* iff for each separable Banach space X and each subspace $Y \subset X$, every bounded linear operator $T_0 \colon Y \to Z$ can be extended to a bounded linear operator $T \colon X \to Z$ such that $\|T\| \leq \lambda\|T_0\|$. Several authors gave different proofs of the following result, each with its own estimate for the constant C (see [1] and the references therein).

Theorem 1.11. *Let* Z *be a* λ*-separably injective Banach space. Then there exists a constant* C *such that, whenever* Y *is a closed subspace of a Banach space* X *such that* X/Y *is separable, every continuous linear operator* $T_0 \colon Y \to c_0(Z)$ *can be extended to a linear operator* $T \colon X \to c_0(Z)$ *such that* $\|T\| \leq C\|T_0\|$.

In [5], the author and L. Veselý, generalized the theorem above to topological vector spaces. Following the proof of [5, Theorem 5.5] and using Lemma 1.10 instead of [1, Proposition 3.2, (c)], we obtain the following result.

Fact 1.12. *The thesis of Theorem* 1.11 *holds for any constant* $C > 2\lambda + 4$.

For the sake of completeness we give a proof of this fact.

Proof. Fix a set Γ such that $Z \subset \ell_\infty(\Gamma)$. Since $c_0(Z) \subset c_0(\ell_\infty(\Gamma))$, [5, Theorem 5.6] implies that T_0 can be extended to a continuous linear operator $S\colon X \to c_0(\ell_\infty(\Gamma))$ such that $\|S\| \le 2\|T_0\|$. We can write $S = (S_n)_{n\in\mathbb{N}}$ where $S_n\colon X \to \ell_\infty(\Gamma)$ for each n. Then $E_n := \overline{Z + S_n(X)}$ is a closed subspace of $\ell_\infty(\Gamma)$.

Fix $n \in \mathbb{N}$. It is easy to see that the quotient E_n/Z is separable, let $p_n : E_n \to E_n/Z$ be the canonical projection. Then, by [6, Proposition 5.1], there exists W_n a closed subspace of ℓ_1 such that $\ell_1/W_n = E_n/Z$ (isometrically) and such that $q_n(B_{\ell_1}) = B_{\ell_1/W_n}$, where q_n is the canonical projection. By the *lifting property* of ℓ_1 ([6, Proposition 5.2]), there exist Q_n a linear operator such that the following diagram commutes

$$\begin{array}{ccc} \ell_1 & \xrightarrow{q_n} & \ell_1/W_n \\ {\scriptstyle Q_n}\big\downarrow & & \big\| \\ E_n & \xrightarrow[p_n]{} & E_n/Z\ . \end{array}$$

Moreover, since p_n is the canonical quotient map, we can (and do) assume that $1 + \|Q_n\| + \lambda\|Q_n\| \le C/2$. By Lemma 1.10, we get a linear projection $P_n : E_n \to Z$ such that $\|P_n\| \le 1 + \|Q_n\| + \lambda\|Q_n\|$.

Now, we can consider the linear operator $T = (T_n)_n : X \to c_0(Z)$ defined by $T_n = P_n \circ S_n$ ($n \in \mathbb{N}$). It easy to see that T extends T_0 and $\|T\| \le C\|T_0\|$. $\square$

2. Sums of normed spaces

In the present section, we introduce certain sums of normed spaces which include, as particular cases, the usual c_0-sums and ℓ_p-sums ($1 \le p < \infty$) of normed spaces. This allows us to give a unified treatment of the main results of the present paper, contained in the next section.

In the sequel, we denote by E a Banach space with a symmetric Schauder basis $\{u_n\}_{n\in\mathbb{N}}$, equipped with a symmetric norm $\|\cdot\|_E$ (i.e., $\|\sum_{n=1}^\infty a_n u_n\|_E = \|\sum_{n=1}^\infty \theta_n a_n u_{\pi(n)}\|_E$, for each permutation π of the integers and each choice of signs $\theta_n = \pm 1$). Let us observe that in particular $\{u_n\}_{n\in\mathbb{N}}$ is an unconditional basis with unconditional basis constant equal to 1 (see e.g. [6, Proposition 4.36]).

Let us introduce the definition of *decomposition invariant* norm. Likely, this notion (maybe under another terminology) has already been considered in the literature, but we have not found any reference.

Definition 2.1. The norm $\|\cdot\|_E$ of E will be called *decomposition invariant* iff, whenever $N_1, \ldots, N_n \subset \mathbb{N}$ are pairwise disjoint finite sets and $N = N_1 \cup \ldots \cup N_n$, the equality

$$\Big\|\sum_{i\in N} a_i u_i\Big\|_E = \Big\|\sum_{i=1}^{n}\Big\|\sum_{j\in N_i} a_j u_j\Big\|_E u_i\Big\|_E$$

holds, for each sequence $\{a_n\}_n$ of reals.

We have the following fact.

Fact 2.2. *Suppose that the norm $\|\cdot\|_E$ is decomposition invariant. Then, whenever $N_1, N_2, \ldots \subset \mathbb{N}$ are pairwise disjoint sets such that $\mathbb{N} = \bigcup_n N_n$, we get that the series $\sum_{i\in\mathbb{N}} a_i u_i$ converges iff the series $\sum_{j\in N_i} a_j u_j$ $(i = 1, 2, \ldots)$ and $\sum_{i\in\mathbb{N}}\big\|\sum_{j\in N_i} a_j u_j\big\|_E u_i$ converge. Moreover, in the case all the series converge, the following equality holds*

$$\Big\|\sum_{i\in\mathbb{N}} a_i u_i\Big\|_E = \Big\|\sum_{i\in\mathbb{N}}\Big\|\sum_{j\in N_i} a_j u_j\Big\|_E u_i\Big\|_E. \tag{1}$$

Proof. Suppose that the series $\sum_{i\in\mathbb{N}} a_i u_i$ converges. Clearly, since $\{u_n\}_{n\in\mathbb{N}}$ is a symmetric (and hence an unconditional) Schauder basis, the series $\sum_{j\in N_i} a_j u_j$ $(i = 1, 2, \ldots)$ converge. Let us prove that the series $\sum_{i\in\mathbb{N}}\big\|\sum_{j\in N_i} a_j u_j\big\|_E u_i$ converges. Fix $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ be such that if $n \geq m \geq n_0$ then $\|\sum_{i=m}^{n} a_i u_i\|_E \leq \varepsilon$. Let $m_0 \in \mathbb{N}$ be such that if $i \geq m_0$ then $N_i \cap \{1, \ldots, n_0\} = \emptyset$. Fix $h \geq k \geq m_0$, by the continuity of the norm, we can find finite sets $N_i' \subset N_i$ $(i = k, \ldots, h)$ such that

$$\Big\|\sum_{i=k}^{h}\Big\|\sum_{j\in N_i} a_j u_j\Big\|_E u_i\Big\|_E \leq \Big\|\sum_{i=k}^{h}\Big\|\sum_{j\in N_i'} a_j u_j\Big\|_E u_i\Big\|_E + \varepsilon.$$

Since $N = \bigcup_{i=k}^{h} N_i' \subset \{n \in \mathbb{N};\ n > n_0\}$ and

$$\Big\|\sum_{i=k}^{h}\Big\|\sum_{j\in N_i'} a_j u_j\Big\|_E u_i\Big\|_E = \Big\|\sum_{i\in N} a_i u_i\Big\|_E,$$

we have that $\big\|\sum_{i=k}^{h}\big\|\sum_{j\in N_i} a_j u_j\big\|_E u_i\big\|_E \leq 2\varepsilon$.

Let us prove the other implication. Fix $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ be such that

$$\Big\|\sum_{i=n_0+1}^{\infty}\Big\|\sum_{j\in N_i} a_j u_j\Big\|_E u_i\Big\|_E \leq \varepsilon.$$

Let $F_i \subset N_i$ $(i = 1, \ldots, n_0)$ be finite sets such that $\big\|\sum_{j\in N_i\setminus F_i} a_j u_j\big\|_E \leq \frac{\varepsilon}{n_0\|u_i\|_E}$, for each $i = 1, \ldots, n_0$. Let $F = \bigcup_{i=1}^{n_0} F_i$ and let $m_0 = \sup F + 1$. For $n \geq m \geq m_0$, let us denote $\sigma := \{i \in \mathbb{N};\ N_i \cap \{m, \ldots, n\} \neq \emptyset\}$ and $\sigma_0 := \sigma \cap \{1, \ldots, n_0\}$.

If $i \in \sigma_0$, put $N_i' := N_i \cap \{m, \dots, n\}$ and observe that $N_i' \cap F_i = \emptyset$; if $i \in \sigma \setminus \sigma_0$, let us consider a finite set N_i' such that $N_i \cap \{m, \dots, n\} \subset N_i' \subset N_i$ and such that

$$\Big\| \sum_{i \in \sigma \setminus \sigma_0} \Big\| \sum_{j \in N_i'} a_j u_j \Big\|_E u_i \Big\|_E \leq 2\varepsilon.$$

Since $\{u_n\}_{n \in \mathbb{N}}$ is an unconditional basis with unconditional basis constant equal to 1 and the norm $\|\cdot\|_E$ is decomposition invariant, we have

$$\begin{aligned} \|\textstyle\sum_{i=m}^n a_i u_i\|_E &\leq \Big\| \sum_{i \in \sigma_0} \Big\| \sum_{j \in N_i'} a_j u_j \Big\|_E u_i \Big\|_E + \Big\| \sum_{i \in \sigma \setminus \sigma_0} \Big\| \sum_{j \in N_i'} a_j u_j \Big\|_E u_i \Big\|_E \\ &\leq n_0 \|u_i\|_E \cdot \Big\| \sum_{j \in N_i \setminus F_i} a_j u_j \Big\|_E + 2\varepsilon \leq 3\varepsilon. \end{aligned}$$

Now, suppose that all the series converge and let us prove the equality in (1). For $n \in \mathbb{N}$ let σ_n be the set of all $i \in \mathbb{N}$ such that

$$N_i^n := N_i \cap \{1, \dots, n\} \neq \emptyset.$$

Since $\{u_n\}_{n \in \mathbb{N}}$ is a Schauder basis and the norm $\|\cdot\|_E$ is decomposition invariant, we can observe that

$$\Big\| \sum_{i \in \mathbb{N}} a_i u_i \Big\|_E = \lim_n \Big\| \sum_{i \in \sigma_n} \Big\| \sum_{j \in N_i^n} a_j u_j \Big\|_E u_i \Big\|_E = \Big\| \sum_{i \in \mathbb{N}} \Big\| \sum_{j \in N_i} a_j u_j \Big\|_E u_i \Big\|_E. \qquad \square$$

Let Γ be a nonempty set and X_γ ($\gamma \in \Gamma$) normed spaces. Suppose that $x \in \bigotimes_{\gamma \in \Gamma} X_\gamma := \{f : \Gamma \to \big(\bigcup_{\gamma \in \Gamma} X_\gamma\big);\ f(\gamma) \in X_\gamma\}$ then we denote by $\mathrm{supp}(x)$ the support of x, i.e. the set $\{\gamma \in \Gamma;\ x(\gamma) \neq 0\}$.

Definition 2.3. Let Γ be a nonempty set and X_γ ($\gamma \in \Gamma$) normed spaces. We denote by $Z(\Gamma) = \big(\bigoplus_{\gamma \in \Gamma} X_\gamma\big)_E$ the linear space of all elements $x \in \bigotimes_{\gamma \in \Gamma} X_\gamma$ such that $\mathrm{supp}(x)$ is an at most countable set $\{\gamma_n\}_{n \in N}$ ($N \subset \mathbb{N}$) and the series $\sum_{n \in N} \|x(\gamma_n)\| u_n$ converges. We endow $Z(\Gamma)$ with the norm

$$\|x\|_{Z(\Gamma)} = \Big\| \sum_{n \in N} \|x(\gamma_n)\| u_n \Big\|_E.$$

Moreover, in the case $X_\gamma = \mathbb{R}$, for each $\gamma \in \Gamma$, we denote $E(\Gamma) := Z(\Gamma)$ and we denote by e_γ ($\gamma \in \Gamma$) the characteristic function of the set $\{\gamma\}$, i.e. the function defined by $e_\gamma(\gamma') = \delta_{\gamma\gamma'}$.

Remark 2.4.

(a) It is clear that, in the case $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ is an infinite countable set, $E(\Gamma)$ is isometric with E and an isometry between $E(\Gamma)$ and E is given by $e_{\gamma_n} \mapsto u_n$.

(b) The Banach spaces c_0 and ℓ_p ($p \in [1,\infty)$), endowed with the canonical norms and the canonical Schauder bases, are examples of spaces with symmetric decomposition invariant norms.

(c) Suppose that Γ is a nonempty set, X_γ ($\gamma \in \Gamma$) are normed spaces, and $E = c_0$ (respectively, $E = \ell_p$, $p \in [1,\infty)$). In this case $Z(\Gamma)$ coincides with the $c_0(\Gamma)$-sum (respectively, the $\ell_p(\Gamma)$-sum) of the spaces X_γ. Moreover, $E(\Gamma) = c_0(\Gamma)$ (respectively, $E(\Gamma) = \ell_p(\Gamma)$).

The following proposition is an easy consequence of Fact 2.2.

Proposition 2.5. *Let Γ be a nonempty set and suppose that $\|\cdot\|_E$ is decomposition invariant. Let X_γ ($\gamma \in \Gamma$) be normed spaces and let $\Gamma = \bigcup_{\alpha\in A}\Gamma_\alpha$ be a decomposition of Γ in pairwise disjoint nonempty sets. Let $x \in \bigotimes_{\gamma\in\Gamma} X_\gamma$ be such that $x|_{\Gamma_\alpha} \in Z(\Gamma_\alpha)$ for each $\alpha \in A$. Then $x \in Z(\Gamma)$ iff $\{\|x|_{\Gamma_\alpha}\|\}_{\alpha\in A} \in E(A)$. Moreover, in the case $x \in Z(\Gamma)$, the following equality holds*

$$\|x\|_{Z(\Gamma)} = \left\|\{\|x|_{\Gamma_\alpha}\|_{Z(\Gamma_\alpha)}\}_{\alpha\in A}\right\|_{E(A)}. \tag{2}$$

Proof. Let $\{\gamma_n\}_{n\in N} = \operatorname{supp}(x)$ ($N \subset \mathbb{N}$) and let $\{\alpha_n\}_{n\in M} = \{\alpha \in A;\ \operatorname{supp}(x)\cap \Gamma_\alpha \neq \emptyset\}$ ($M \subset \mathbb{N}$). For $i \in M$ define $N_i := \{n \in N;\ \gamma_n \in \Gamma_{\alpha_i}\}$, and for $n \in N$ define $a_n := \|x(\gamma_n)\|$. Observe that the sets N_i ($i \in M$) are pairwise disjoint and such that $\bigcup_{i\in M} N_i = N$. Moreover,

(i) $x \in Z(\Gamma)$ iff $\sum_{n\in N} a_n u_n$ converges,

(ii) $\{\|x|_{\Gamma_\alpha}\|\}_{\alpha\in A} \in E(A)$ iff $\sum_{i\in M}\left\|\sum_{j\in N_i} a_j u_j\right\|_E u_i$ converges.

Hence, the first part of the proposition follows by the first part of Fact 2.2. The equality in (2) follows by the equality in (1). □

Fact 2.6. *Let Γ be a nonempty set and X_γ ($\gamma \in \Gamma$) normed spaces. Then the topological dual of $Z(\Gamma)$ is isometric with the linear space*

$$Z^*(\Gamma) := \{x^* \in \bigotimes_{\gamma\in\Gamma} X^*_\gamma;\ \|x^*\|_* := \sup_{x\in B_{Z(\Gamma)}} \sum_{\gamma\in\Gamma} |x^*(\gamma)x(\gamma)| < \infty\},$$

equipped with the norm $\|\cdot\|_*$. Moreover the isometric correspondence between $v^* \in [Z(\Gamma)]^*$ and $w \in Z^*(\Gamma)$ is given by $w(\gamma) = v^*|_{X_\gamma}$ ($\gamma \in \Gamma$).

In the case $X_\gamma = \mathbb{R}$, for each $\gamma \in \Gamma$, we denote $E^*(\Gamma) := Z^*(\Gamma)$

Definition 2.7. Let Γ, X_γ ($\gamma \in \Gamma$), $Z(\Gamma)$ and $Z^*(\Gamma)$ be as above. We denote by $Z^*_0(\Gamma)$ the linear space $\{x^* \in Z^*(\Gamma);\ \operatorname{supp}(x^*) \text{ is finite}\}$ and by $\overline{Z^*_0(\Gamma)}$ the norm-closure of $Z^*_0(\Gamma)$ in $Z^*(\Gamma)$.

Let Γ, Γ_0 be nonempty sets such that $\Gamma_0 \subset \Gamma$, and X_γ ($\gamma \in \Gamma$) normed spaces. We denote by $R_{\Gamma_0} : Z(\Gamma) \to Z(\Gamma)$ the canonical projection defined by

$$[R_{\Gamma_0}(x)](\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in \Gamma_0; \\ 0 & \text{otherwise.} \end{cases}$$

Let us observe that $R_{\Gamma_0}(Z(\Gamma))$ is isometrically isomorphic with $Z(\Gamma_0)$.

Lemma 2.8. *Let Γ be a nonempty set. Suppose that $\{u_n\}_n \subset E$ is shrinking, then $\overline{Z_0^*(\Gamma)} = Z^*(\Gamma)$.*

Proof. Suppose on the contrary that $\overline{Z_0^*(\Gamma)} \neq Z^*(\Gamma)$, then there exist $f \in Z^*(\Gamma)$ and $\varepsilon > 0$ such that $\mathrm{dist}(f, \overline{Z_0^*(\Gamma)}) > \varepsilon$, and hence such that

$$\sup\{f(z); z \in B_{Z(\Gamma \setminus F)}\} > \varepsilon \quad \forall F \subset \Gamma \text{ finite.} \tag{3}$$

The following induction argument shows that there exists an infinite countable subset Γ_0 of Γ such that (3) holds if Γ is replaced by Γ_0. Fix a countable subset Γ_1 of Γ. By induction, we can construct countable sets $\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma$ such that, for each $n \in \mathbb{N}$ and for each finite set $F \subset \Gamma_n$, there exists $z \in B_{Z(\Gamma_{n+1} \setminus F)}$ such that $f(z) > \varepsilon$. Put $\Gamma_0 = \bigcup_{n \in \mathbb{N}} \Gamma_n$.

Now, for each $\gamma \in \Gamma_0$, define $F(\gamma) = \|f(\gamma)\|_{X_\gamma^*}$. Then $F \in E^*(\Gamma_0)$ and

$$\sup\{F(x); x \in B_{E(\Gamma_0 \setminus F)}\} > \varepsilon \quad \forall F \subset \Gamma_0 \text{ finite.}$$

Hence $\{e_\gamma\}_{\gamma \in \Gamma_0} \subset E(\Gamma_0) = E$ is not shrinking (see e.g., [6, Fact 4.13]), a contradiction. $\square$

3. Main results

The following lemma is an easy variation of [8, Lemma 2].

Lemma 3.1. *Let Γ be an uncountable set and X_γ ($\gamma \in \Gamma$) separable normed spaces. Let Y be a subspace of $Z(\Gamma)$ closed in the $\sigma\big(Z(\Gamma), \overline{Z_0^*(\Gamma)}\big)$-topology. Then Γ can be decomposed into a family $\{\Gamma_\alpha\}_{\alpha \in A}$ of pairwise disjoint at most countable sets such that if $\alpha \in A$ then $R_{\Gamma_\alpha}(Y) \subset Y$.*

Proof. We proceed as in the proof of [8, Lemma 2]. First of all, observe that if $\{\Delta_\beta\}_{\beta \in B}$ are pairwise disjoint subsets of Γ then, for every $x \in Z(\Gamma)$, the series $\sum_{\beta \in B} R_{\Delta_\beta}(x)$ converges in norm to $R_\Delta(x)$, where $\Delta = \bigcup_{\beta \in B} \Delta_\beta$. Moreover, observe that if $\Delta \subset \Gamma$ and $R_\Delta(Y) \subset Y$ then $R_{\Gamma \setminus \Delta}(Y) \subset Y$. We shall need the following claim.

Claim. *For each countable subset Δ_0 of Γ, there is a countable subset Ω of Γ such that $\Delta_0 \subset \Omega$ and $R_\Omega(Y) \subset Y$.*

Proof of the claim. Let us inductively construct countable sets $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \ldots$ and separable subspaces $Y_0 \subset Y_1 \subset Y_2 \subset \ldots$ of Y such that, for each $n = 0, 1, 2, \ldots$,

(a) $R_{\Delta_n}(B_{Y_n})$ is dense in $R_{\Delta_n}(B_Y)$;

(b) $Y_n \subset R_{\Delta_{n+1}}(Z(\Gamma))$.

Since $R_{\Delta_0}(B_Y) \subset B_{Z(\Delta_0)}$ and the normed spaces X_γ ($\gamma \in \Delta_0$) are separable, we have that $R_{\Delta_0}(B_Y)$ is separable. Let $D_0 \subset B_Y$ be a countable set such that $R_{\Delta_0}(D_0)$ is dense in $R_{\Delta_0}(B_Y)$. Define $Y_0 = \overline{\operatorname{span}}(D_0)$ and $\Delta_1 = \left(\bigcup_{d\in D_0} \operatorname{supp}(d)\right) \cup \Delta_0$. Observe that Y_0 is separable, Δ_1 is countable and $Y_0 \subset R_{\Delta_1}(Z(\Gamma))$.

Let $n \in \mathbb{N}$ and suppose that Δ_n and Y_{n-1} have been already defined. Since $R_{\Delta_n}(B_Y) \subset B_{Z(\Delta_n)}$ and the normed spaces X_γ ($\gamma \in \Delta_n$) are separable, we have that $R_{\Delta_n}(B_Y)$ is separable. Let $D_n \subset B_Y$ be a countable set such that $R_{\Delta_n}(D_n)$ is dense in $R_{\Delta_n}(B_Y)$. Define $Y_n = \overline{\operatorname{span}}(D_n \cup Y_{n-1})$ and $\Delta_{n+1} = \left(\bigcup_{d\in D_n} \operatorname{supp}(d)\right) \cup \Delta_n$. Observe that Y_n is separable, Δ_{n+1} is countable and $Y_n \subset R_{\Delta_{n+1}}(Z(\Gamma))$.

Put $\Omega = \bigcup_{n=1}^{\infty} \Delta_n$. Let $y \in B_Y$ and let $y_n \in B_{Y_n}$ so that $\|R_{\Delta_n}(y - y_n)\|_{Z(\Gamma)} \to 0$ as $n \to \infty$. Then also $\|R_\Omega(y) - R_{\Delta_n}(y_n)\|_{Z(\Gamma)} \to 0$ as $n \to \infty$. Now, since $\operatorname{supp}(y_n - R_{\Delta_n} y_n) \subset \Delta_{n+1} \setminus \Delta_n$ ($n \in \mathbb{N}$), the terms of the sequence $\{y_n - R_{\Delta_n}(y_n)\}_n$ have disjoint support. Hence, for each $F \in Z_0^*(\Gamma)$, $F\big(y_n - R_{\Delta_n}(y_n)\big) \to 0$ as $n \to \infty$. Moreover the sequence $\{y_n - R_{\Delta_n}(y_n)\}_n$ is bounded and hence converges to 0 in the $\sigma\big(Z(\Gamma), \overline{Z_0^*(\Gamma)}\big)$-topology. This proves that the sequence $\{y_n\}_n$ tends to $R_\Omega(y)$ in the $\sigma\big(Z(\Gamma), \overline{Z_0^*(\Gamma)}\big)$-topology, and hence that $R_\Omega(y)$ is in Y. This concludes the proof of the claim. □

Now, let us consider the family $\mathcal{F}$ of all collections $F = \{\Gamma_\alpha\}_{\alpha\in A}$ of pairwise disjoint nonempty at most countable subsets of Γ such that if $\alpha \in A$ then $R_{\Gamma_\alpha}(Y) \subset Y$. By our claim $\mathcal{F}$ is nonempty. Moreover, if $\mathcal{F}$ is ordered by the inclusion relation, it is easy to see that every chain in $\mathcal{F}$ has an upper-bound, namely the union of all elements of the chain. By Zorn's lemma there exists a maximal element $\overline{F} = \{\overline{\Gamma}_\alpha\}_{\alpha\in\overline{A}}$ in $\mathcal{F}$.

We just have to prove that $\overline{\Gamma} := \bigcup_{\alpha\in\overline{A}} \overline{\Gamma}_\alpha = \Gamma$. Suppose that this is not the case and suppose that $\Gamma \setminus \overline{\Gamma}$ is nonempty. Let us observe that $R_{\overline{\Gamma}}(Y) \subset Y$ and hence $R_{\Gamma\setminus\overline{\Gamma}}(Y) \subset Y$, too. Moreover, since $\overline{F}$ is maximal, $\Gamma \setminus \overline{\Gamma}$ is uncountable. By our claim, we can find a nonempty countable set Ω in $\Gamma \setminus \overline{\Gamma}$ such that

$$R_\Omega(Y) = R_\Omega\left(R_{\Gamma\setminus\overline{\Gamma}}(Y)\right) \subset R_{\Gamma\setminus\overline{\Gamma}}(Y) \subset Y.$$

Hence $\overline{F} \subsetneq (\overline{F} \cup \{\Omega\}) \in \mathcal{F}$, this contradiction concludes the proof. □

Proposition 3.2. *Let Γ be an uncountable set and X_γ ($\gamma \in \Gamma$) separable normed spaces. Suppose that $\|\cdot\|_E$ is decomposition invariant and let Y be a subspace of $Z(\Gamma)$. Then the couple $(Z(\Gamma), Y)$ has the* 2-EP *provided Y is closed in the $\sigma\big(Z(\Gamma), \overline{Z_0^*(\Gamma)}\big)$-topology.*

Proof. Let $\{\Gamma_\alpha\}_{\alpha\in A}$ be the decomposition guaranteed by Lemma 3.1. For every $\alpha \in A$, let $Z_\alpha := R_{\Gamma_\alpha} Z(\Gamma)$ and $Y_\alpha := R_{\Gamma_\alpha} Y = Y \cap Z_\alpha$ (the last equality holds by Lemma 3.1). Let us observe that, for each $\alpha \in A$, since Γ_α is countable,

Z_α is a separable normed space; then, by Corollary 1.8, there exists $\varphi_\alpha : B_{Y^*_\alpha} \to Z^*_\alpha$ such that $\varphi_\alpha(0) = 0$, φ_α is w^*-w^*-sequentially continuous at 0, $\|\varphi_\alpha(y^*)\| \leq 2\|y^*\|$ ($y^* \in B_{Y^*_\alpha}$) and $q_\alpha \circ \varphi_\alpha = Id_{B_{Y^*_\alpha}}$, where $q_\alpha : Z^*_\alpha \to Y^*_\alpha$ is the canonical restriction map.

Define $\varphi : B_{Y^*} \to Z^*(\Gamma)$ as follows: let $\gamma \in \Gamma$ and let $\alpha \in A$ be such that $\gamma \in \Gamma_\alpha$, define $\varphi(y^*)(\gamma) = \varphi_\alpha(y^*|_{Y_\alpha})(\gamma)$. First of all, observe that if $x \in Z(\Gamma)$ then

$$\begin{aligned}\sum_{\gamma\in\Gamma} |\varphi(y^*)(\gamma)x(\gamma)| &= \sum_{\alpha\in A}\sum_{\gamma\in\Gamma_\alpha} |\varphi_\alpha(y^*|_{Y_\alpha})(\gamma)(R_{\Gamma_\alpha}x)(\gamma)| \\ &\leq 2\sum_{\alpha\in A} \|y^*|_{Y_\alpha}\|\|x|_{\Gamma_\alpha}\|_{Z(\Gamma_\alpha)} \\ &\leq 2\|y^*\|\,\|x\|_{Z(\Gamma)},\end{aligned}$$

where the last inequality follows by the claim below.

Claim. *Let $y^* \in Y^*$, then $\{\|y^*|_{Y_\alpha}\|\}_{\alpha\in A} \in E^*(A)$ and*

$$\big\|\{\|y^*|_{Y_\alpha}\|\}_{\alpha\in A}\big\|_{E^*(A)} \leq \|y^*\|.$$

Proof of the claim. Let $(\beta_\alpha)_\alpha \in E(A)$, fix $\varepsilon > 0$ and, for each $\alpha \in A$, let $y_\alpha \in Y_\alpha$ be such that $\|y_\alpha\| = |\beta_\alpha|$ and $(1+\varepsilon)y^*|_{Y_\alpha}(y_\alpha) = \|y^*|_{Y_\alpha}\||\beta_\alpha|$. Put $y(\gamma) = y_\alpha(\gamma)$ if $\gamma \in \Gamma_\alpha$. Since Y is closed in the $\sigma\big(Z(\Gamma), Z^*_0(\Gamma)\big)$-topology, Proposition 2.5 implies that $y \in Y$ and $\|y\|_{Z(\Gamma)} = \|(\beta_\alpha)_\alpha\|_{E(A)}$. Then

$$\begin{aligned}\sum_{\alpha\in A} \|y^*|_{Y_\alpha}\|\,|\beta_\alpha| &= (1+\varepsilon)\sum_{\alpha\in A} y^*|_{Y_\alpha}(y_\alpha) \\ &= (1+\varepsilon)y^*y \\ &\leq (1+\varepsilon)\,\|y^*\|\,\|(\beta_\alpha)_\alpha\|_{E(A)}.\end{aligned}$$

The arbitrariness of $\varepsilon > 0$ concludes the proof of the claim. □

So, φ is well-defined and $\|\varphi(y^*)\| \leq 2\|y^*\|$ ($y^* \in B_{Y^*}$), moreover, it is easily checked that φ extends functionals. Now, let $\{y^*_n\}_n \subset B_{Y^*}$ be a w^*-null sequence and put $x^*_n = \varphi(y^*_n)$ ($n \in \mathbb{N}$). Fix $\gamma \in \Gamma$ and let $\alpha \in A$ be such that $\gamma \in \Gamma_\alpha$, since $\{y^*_n|_{Y_\alpha}\}_n \subset Y^*_\alpha$ is a w^*-null sequence and φ_α is w^*-w^*-sequentially continuous at the origin, $x^*_n(\gamma) = \varphi_\alpha(y^*_n|_{Y_\alpha})(\gamma) \to 0$ in the w^*-topology of X^*_γ. Since $\{x^*_n\}_n$ is bounded, this prove that φ is w^*-w^*-sequentially continuous at 0. Apply Proposition 1.7. □

Corollary 3.3. *Let Γ be an uncountable set and X_γ ($\gamma \in \Gamma$) separable normed spaces. Suppose that $\|\cdot\|_E$ is decomposition invariant and let Y be a subspace of $Z(\Gamma)$. Then the couple $(Z(\Gamma), Y)$ has the* 2-EP *provided the Schauder basis $\{u_n\}_n \subset E$ is shrinking.*

Proof. Without any loss of generality, we can suppose that Y is closed in the norm (and hence in the weak) topology. By Lemma 2.8, since $\{u_n\}_n \subset E$ is shrinking, Y is closed in the $\sigma\big(Z(\Gamma), \overline{Z_0^*(\Gamma)}\big)$-topology. The thesis follows by Proposition 3.2. $\square$

Corollary 3.4. *Let Y be a subspace of a normed space Z, Γ be a nonempty set and X_γ $(\gamma \in \Gamma)$ separable normed spaces. Then the couple (Z, Y) has the* 2-EP *(and hence the* CE*-property) provided* at least one *of the following conditions is satisfied.*

(*a*) $\quad Z = (\bigoplus_{\gamma\in\Gamma} X_\gamma)_{c_0}$ *or* $Z = (\bigoplus_{\gamma\in\Gamma} X_\gamma)_{\ell_p}$ *$(1 < p < \infty)$.*

(*b*) $\quad Z = (\bigoplus_{\gamma\in\Gamma} X_\gamma)_{\ell_1}$ *and Y is closed in the $\sigma\big(Z, (\bigoplus_{\gamma\in\Gamma} X_\gamma^*)_{c_0}\big)$-topology.*

Moreover, if Z/Y is isomorphic with $\ell_1(\Gamma)/W$, where W is a w^*-closed subspace of $\ell_1(\Gamma)$ (with respect to $c_0(\Gamma)$), then the couple (Z, Y) has the CE*-property.*

Proof. In the case (a) or (b) are satisfied, the proof follows by Corollary 3.3 and Proposition 3.2, respectively.

Now, to prove the second part, we can suppose without any loss of generality that X and Y are Banach spaces and that Z/Y is isomorphic with $\ell_1(\Gamma)/W$, where W is a w^*-closed subspace of $\ell_1(\Gamma)$. Since W is w^*-closed then the couple $(\ell_1(\Gamma), W)$ has the 2-EP. Let $|\cdot|$ be an equivalent norm of $\ell_1(\Gamma)$ such that, if $X_1 = (\ell_1(\Gamma), |\cdot|)$ and $q_1 : X_1 \to X_1/W$ is the canonical quotient map, then $q_1(B_{X_1}) = B_{X_1/W}$ and $X_1/W = Z/Y$ isometrically (see e.g. [2, Lemma 2.5]). Since the couple $(\ell_1(\Gamma), W)$ has the 2-EP then the couple (X_1, W) has the λ_0-EP for some $\lambda_0 \geq 1$. By the *lifting-property* of $\ell_1(\Gamma)$ ([6, Proposition 5.2]) we can consider a continuous linear operator $Q : X_1 \to Z$ such that the following diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{q_1} & X_1/W \\ {\scriptstyle Q}\big\downarrow & & \big\| \\ Z & \xrightarrow[q_2]{} & Z/Y\ . \end{array}$$

Put $\lambda = 1 + \|Q\| + \lambda_0\|Q\|$. Lemma 1.10, applied with $E = c_0(\ell_\infty(\Delta))$ (Δ a nonempty set), shows that the couple (Z, Y) has the λ-EP and hence the CE-property. $\square$

Remark 3.5.

(a) The condition about the w^*-closedness of W in the corollary above is necessary. To see this, let W be a subspace of $\ell_1(\mathfrak{c})$ such that $\ell_1(\mathfrak{c})/W = \ell_\infty/c_0$. Then the couple $(\ell_1(\mathfrak{c}), W)$ does not have the CE-property. In fact, if we suppose that the couple $(\ell_1(\mathfrak{c}), W)$ has the CE-property then, proceeding as in the proof of the corollary above, we get that the the couple (ℓ_∞, c_0) has the CE-property, too. A contradiction by [4, Proposition 4.4].

(b) In the case $Z = c_0(\Gamma)$ or $Z = \ell_p(\Gamma)$ $(1 < p < \infty)$, something more can be said, using known results about extension of operators with values in $C(K)$ spaces.
It follows by the proof of [8, Theorem 4] and Proposition 1.7 that, for every $\varepsilon > 0$ and for every subspace $Y \subset c_0(\Gamma)$, $(c_0(\Gamma), Y)$ has the $(1+\varepsilon)$-EP.
If Y is a subspace of $\ell_p(\Gamma)$ and $y^* \in B_{Y^*}$, the uniform convexity of the unit ball of $\ell_p^*(\Gamma)$ yields the existence of a unique functional x^* which extends y^* and such that $\|y^*\| = \|x^*\|$, it is not difficult to see that the map $y^* \mapsto x^*$ is w^*-w^* continuous on B_{Y^*} (see e.g. [11]). Proposition 1.7 proves that $(\ell_p(\Gamma), Y)$ has the 1-EP.

Acknowledgements. The author would like to thank L. Veselý for numerous comments and suggestions that helped him in preparing this paper. The author also would like to thank the anonymous referee for several comments that helped him in preparing the final version of this paper.

References

[1] A. Avilés, F. Cábello Sanchez, J. M. F. Castillo, M. Gonzáles, Y. Moreno: On separably injective Banach spaces, Adv. Math. 234 (2013) 192–216.

[2] P. Bandyopadhyay, G. Godefroy: Linear structures in the set of norm-attaining functionals on a Banach space, J. Convex Analysis 13 (2006) 489–497.

[3] J. Borwein, V. Montesinos, J. Vanderwerff: Boundedness, differentiability and extensions of convex functions, J. Convex Analysis 13 (2006) 587–602.

[4] C. A. De Bernardi, L. Veselý: Extension of continuous convex functions from subspaces I, J. Convex Analysis 21 (2014) 1065–1084.

[5] C. A. De Bernardi, L. Veselý: Extension of continuous convex functions from subspaces II, J. Convex Analysis 22 (2015) 101–116.

[6] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler: Banach Space Theory. The Basis for Linear and Nonlinear Analysis, CMS Books in Mathematics, Springer, New York (2011).

[7] K. Floret: Weakly Compact Sets, Lecture Notes in Mathematics 801, Springer, Berlin (1980).

[8] W. B. Johnson, M. Zippin: Extension of operators from subspaces of $c_0(\Gamma)$ into $C(K)$ spaces, Proc. Amer. Math. Soc. 107 (1989) 751–754.

[9] H. P. Rosenthal: The complete separable extension property, J. Oper. Theory 43 (2000) 329–374.

[10] L. Veselý, L. Zajíček: On extensions of d.c. functions and convex functions, J. Convex Analysis 17 (2010) 427–440.

[11] M. Zippin: Extension of bounded linear operators, in: Handbook of the Geometry of Banach Spaces, Vol. II, W. B. Johnson et al. (ed.), North-Holland, Amsterdam (2003) 1703–1741.