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On the Characterization of Locally L^0-Convex Topologies Induced by a Family of L^0-Seminorms

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The purpose of this paper is to provide a characterization of locally L^0-convex modules induced by a family of L^0-seminorms using the gauge function for L^0-modules and discussing the notion of countable concatenation. Taking advantage of these ideas we will give a counterexample of a locally L^0-convex topology not induced by any family of L^0-seminorms.

Keywords: L^0-modules, locally L^0-convex modules, gauge function, countable concatenation, countable concatenation closure

Introduction

In [3], motivated by the financial applications, Filipovic, Kupper and Vogelpoth try to provide an appropriate theoretical framework in order to study the conditional risk measures and develop the classical convex analysis for topological L^0-modules.

To this end, they introduce the gauge function for L^0-modules and, in the same way as in the convex analysis, they claim that a topological L^0-module is locally L^0-convex if, and only if, its topology is induced by a family of L^0-seminorms. Namely, Theorem 2.4 of [3].

Nevertheless, in [5] T. Guo, S. Zhao and X. Zeng warn that there is a hole in the argument and introduce some theoretical considerations. They even conjecture that Theorem 2.4 of [3] does not hold but they do not delve into the subject and no counterexample is given, focusing the rest of their analysis only over locally L^0-convex topologies induced by a family of L^0-seminorms.

In this paper, we go further and provide a characterization of locally L^0-convex modules induced by a family of L^0-seminorms. Finally, taking advantage of these ideas, we will give a counterexample of a locally L^0-convex topology that cannot be induced by any family of L^0-seminorms.

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1. Some important concepts

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which will be fixed for the rest of this paper, we consider $L^0(\Omega, \mathcal{F}, \mathbb{P})$, the set of equivalence classes of real valued $\mathcal{F}$-measurable random variables, which will be denoted simply as L^0.

It is known that the triple $(L^0, +, \cdot)$ endowed with the partial order of the almost sure dominance is a lattice ordered ring.

For the convenience of the reader, precise some notation. We say "$X \geq Y$" if $\mathbb{P}(X \geq Y) = 1$. Likewise, we say "$X > Y$", if $\mathbb{P}(X > Y) = 1$. And, given $A \in \mathcal{F}$, we say that $X > Y$ (respectively, $X \geq Y$) on A, if $\mathbb{P}(X > Y \mid A) = 1$ (respectively , if $\mathbb{P}(X \geq Y \mid A) = 1$).

We also define

$$L^0_+ := \left\{Y \in L^0;\ Y \geq 0\right\},$$
$$L^0_{++} := \left\{Y \in L^0;\ Y > 0\right\}.$$

And denote by $\bar{L}^0$, the set of equivalence classes of $\mathcal{F}$-measurable random variables taking values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, and extend the partial order of the almost sure dominance to $\bar{L}^0$.

Let us see below, some notions and results that will be used in the development of this paper.

In A.5 of [4] is proved the proposition below:

Proposition 1.1. *Let ϕ be a subset of L^0. The following hold:*

1. *There exists $Y^* \in \bar{L}^0$ such that $Y^* \geq Y$ for all $Y \in \phi$, and such that any other Y' satisfying the same inequality, verifies $Y' \geq Y^*$.*
2. *Suppose that ϕ is directed upwards. Then there exists an increasing sequence $Y_1 \leq Y_2 \leq ...$ in ϕ, such that Y_n converges to Y^* almost surely.*

Definition 1.2. Under the conditions of the previous proposition, Y^* is the called essential supremum of ϕ, and we write

$$\operatorname{ess.sup} \phi = \operatorname*{ess.sup}_{Y \in \phi} Y := Y^*.$$

The essential infimum of ϕ is defined as

$$\operatorname{ess.inf} \phi = \operatorname*{ess.inf}_{Y \in \phi} Y := -\operatorname*{ess.sup}_{Y \in \phi} (-Y).$$

The order of the almost sure dominance also allows us to define a topology on L^0. Let us define

$$B_\varepsilon := \left\{Y \in L^0;\ |Y| \leq \varepsilon\right\}$$

the ball of radius $\varepsilon \in L^0_{++}$ centered at $0 \in L^0$. Then, for all $Y \in L^0$, $\mathcal{U}_Y := \left\{Y + B_\varepsilon;\ \varepsilon \in L^0_{++}\right\}$ is a neighborhood base of Y. Thus, it can be defined a topology on L^0, which is known as the topology induced by $|\cdot|$ and L^0 endowed with this topology will be denoted by $L^0[|\cdot|]$.

Definition 1.3. A topological L^0-module $E[\tau]$ is a L^0-module E endowed with a topology τ such that

1. $E[\tau] \times E[\tau] \longrightarrow E[\tau], (X, X') \mapsto X + X'$ and
2. $L^0[|\cdot|] \times E[\tau] \longrightarrow E[\tau], (Y, X) \mapsto YX$

are continuous with the corresponding product topologies.

Definition 1.4. A topology τ on a L^0-module E is locally L^0-convex if there is a neighborhood base $\mathcal{U}$ of $0 \in E$ such that each $U \in \mathcal{U}$ is

1. L^0-convex, i.e. $YX_1 + (1-Y)X_2 \in U$ for all $X_1, X_2 \in U$ and $Y \in L^0$ with $0 \leq Y \leq 1$,
2. L^0-absorbent, i.e. for all $X \in E$ there is a $Y \in L^0_{++}$ such that $X \in YU$,
3. L^0-balanced, i.e. $YX \in U$ for all $X \in U$ and $Y \in L^0$ with $|Y| \leq 1$.
   In this case, $E[\tau]$ is a locally L^0-convex module.

Definition 1.5. A function $\|\cdot\| : E \to L^0_+$ is a L^0-seminorm on E if:

1. $\|YX\| = |Y| \, \|X\|$ for all $Y \in L^0$ and $X \in E$.
2. $\|X_1 + X_2\| \leq \|X_1\| + \|X_2\|$, for all $X_1, X_2 \in E$.

If, moreover,

3. $\|X\| = 0$ implies $X = 0$,

then $\|\cdot\|$ is a L^0-norm on E.

Definition 1.6. Let $\mathcal{P}$ be a family of L^0-seminorms on a L^0-module E. Given a finite subset $Q \subset \mathcal{P}$ and $\varepsilon \in L^0_{++}$, we define

$$U_{Q,\varepsilon} := \left\{X \in E;\ \sup_{\|\cdot\| \in Q} \|X\| \leq \varepsilon\right\}.$$

Then for all $X \in E$, $\mathcal{U}_X := \left\{X + U_{Q,\varepsilon};\ \varepsilon \in L^0_{++},\ Q \subset \mathcal{P} \text{ finite}\right\}$ is a neighborhood base of X. Thereby, we define a topology on E, which is known as the topology induced by $\mathcal{P}$ and E endowed with this topology will be denoted by $E[\mathcal{P}]$.

Furthermore, it is proved in Lemma 2.16 of [3] that $E[\mathcal{P}]$ is a locally L^0-convex module.

2. The gauge function and the countable concatenation closure.

Let us recall the notion of gauge function given in [3]:

Definition 2.1. Let E be a L^0-module. The gauge function $p_K : E \to \bar{L}^0_+$ of a set $K \subset E$ is defined by

$$p_K(X) := \operatorname{ess.inf}\left\{Y \in L^0_+;\ X \in YK\right\}.$$

In addition, in [3] the properties below are proved:

Proposition 2.2. *The gauge function p_K of a L^0-convex and L^0-absorbent set $K \subset E$ satisfies:*

1. $1_A p_K(1_A X) = 1_A p(X)$, *for all $A \in \mathcal{F}$ and $X \in E$.*
2. $p_K(X) = \text{ess.}\inf\left\{Y \in L^0_{++};\ X \in YK\right\}$ *for all $X \in E$.*
3. $Y p_K(X) = p_K(YX)$ *for all $X \in E$ and $Y \in L^0_+$*
4. $p_K(X+Y) \leq p_K(X) + p_K(Y)$ *for X, $Y \in E$.*
5. *For all $X \in E$ there exists a sequence $\{Z_n\}$ in L^0_{++} such that $Z_n \searrow p_K(X)$ almost surely and such that $X \in Z_n K$ for all n.*
6. *If, in addition, K is L^0-balanced then $p_K(YX) = |Y|\, p_K(X)$ for all $Y \in L^0$ and $X \in E$.*

In particular, p_K is an L^0-seminorm.

We also have the next result (see Proposition 2.5 of [3] and Proposition 2.22 of [5]):

Proposition 2.3. *The gauge function p_U of a L^0-convex and L^0-absorbent set $U \subset E$ satisfies:*

1. $p_U(X) \geq 1$ *on B for all $X \in E$ with $1_A X \notin 1_A U$ for all $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, $A \subset B$.*
2. *If, in addition, E is a locally L^0-convex module, then*

$$\overset{0}{U} \subset \{X \in E;\ p_U(X) < 1\}.$$

Proceeding in the same way as the classical convex analysis, given a L^0-convex, L^0-absorbent and L^0-balanced set $U \subset E$, one can expect that $\{X \in E;\ p_U(X) < 1\} \subset U$ holds. If this held, we could prove that any topological L^0-module is locally L^0-convex module if, and only if, its topology is induced by a family of L^0-seminorms.

Not in vain, this statement is set as valid in Theorem 2.4 of [3].

However, in [4] the authors point out that the argument that leads to Theorem 2.4 given in [3] has a hole and conjecture that, according to their observations, in general the topology of a locally L^0-convex module is not necessary induced by a family of L^0-seminorms, but no counterexample is given, and they focus the rest of their analysis only over locally L^0-convex topologies induced by a family of L^0-seminorms. Presumably, this is so because the spaces usually used for financial applications are always endowed with a family of L^0-seminorms.

We go a little bit further in this subject and provide an example (see Example 2.7) of a locally L^0-convex topology not induced by any family of L^0-seminorms. Therefore, Theorem 2.4 of [3] does not hold.

In addition, this example shows that there exists a L^0-convex, L^0-absorbent and L^0-balanced set $U \subset E$ such that $\{X \in E;\ p_U(X) < 1\} \nsubseteq U$.

The fact that the order of almost sure dominance is not a total order leads to difficulties when one wishes to move from real convex analysis to locally L^0-convex modules. With the aim to overcome this difficulty, we use the following notions and the subsequent lemma.

Let E be a L^0-module and $K \subset E$ a subset, and denote by $\Pi(\Omega, \mathcal{F})$ the set of countable partitions of Ω to $\mathcal{F}$.

- Given a sequence $\{X_n\}_{n\in\mathbb{N}}$ in E and a partition $\{A_n\}_{n\in\mathbb{N}} \in \Pi(\Omega, \mathcal{F})$, we define the set of countable concatenations of $\{X_n\}_n$ and $\{A_n\}_n$

$$\mathrm{cc}\left(\{A_n\}_n, \{X_n\}_n\right) := \{X \in E;\ 1_{A_n} X_n = 1_{A_n} X \text{ for each } n \in \mathbb{N}\}.$$

- We say that K is *closed under countable concatenations*, if for each sequence $\{X_n\}_{n\in\mathbb{N}}$ in K and each partition $\{A_n\}_{n\in\mathbb{N}} \in \Pi(\Omega, \mathcal{F})$ it holds

$$\mathrm{cc}\left(\{A_n\}_n, \{X_n\}_n\right) \subset K.$$

- We say that K has *the countable concatenation property*, if it is closed under countable concatenations and for each sequence $\{X_n\}_{n\in\mathbb{N}}$ in K and each partition $\{A_n\}_{n\in\mathbb{N}} \in \Pi(\Omega, \mathcal{F})$ it holds

$$\mathrm{cc}\left(\{A_n\}_n, \{X_n\}_n\right) \text{ is non empty.}$$

- We call the *countable concatenation closure of* K the set defined by

$$\overline{K}^{\mathrm{cc}} := \bigcup \mathrm{cc}\left(\{A_n\}_n, \{X_n\}_n\right)$$

  where $\{A_n\}_{n\in\mathbb{N}}$ runs through $\Pi(\Omega, \mathcal{F})$ and $\{X_n\}_{n\in\mathbb{N}}$ runs through all sequences in K. Or, written in another way,

$$\overline{K}^{\mathrm{cc}} = \{X \in E;\ \exists \{A_n\}_{n\in\mathbb{N}} \in \Pi(\Omega, \mathcal{F}) \text{ with } 1_{A_n} X \in 1_{A_n} K \text{ for all } n\}.$$

It is clear that K is closed under countable concatenations if, and only if, $\overline{K}^{\mathrm{cc}} = K$.

In [4], the countable concatenation property is broadly used as an effective tool. But we shall use the property of "being closed under countable concatenations" rather than "'having the concatenation property", because the earlier is a slightly weaker property which will fit better for our aim.

Lemma 2.4. *If C is a bounded below (resp. above) and closed under countable concatenations subset of L^0, then for each $\varepsilon \in L^0_{++}$ there exists $Y_\varepsilon \in C$ such that*

$$\mathrm{ess.\,inf}\, C \le Y_\varepsilon < \mathrm{ess.\,inf}\, C + \varepsilon$$

(resp. $\mathrm{ess.\,sup}\, C \ge Y_\varepsilon > \mathrm{ess.\,sup}\, C - \varepsilon$*).*

In particular, let be given a L^0-module E and $K \subset E$ a L^0-convex, L^0-absorbent and closed under countable concatenations subset. We have that, for $\varepsilon \in L^0_{++}$, there exists $Y_\varepsilon \in L^0_{++}$ with $X \in Y_\varepsilon K$ such that

$$p_K(X) \le Y_\varepsilon < p_K(X) + \varepsilon.$$

Proof. Firstly, let us see that C is downwards directed. Indeed, given $Y, Y' \in C$, define $A := (Y < Y')$. Then, since C is closed under countable concatenations, $1_A Y + 1_{A^c} Y' = Y \wedge Y' \in C$.

Therefore, by Proposition 1.1, for $\varepsilon \in L^0_{++}$, there exists a decreasing sequence $\{Y_k\}_{k\in\mathbb{N}}$ in C converging to $\text{ess.}\inf C$ almost surely.

Thereby, we consider the sequence of sets

$$A_0 := \phi \quad \text{and} \quad A_k := (Y_k < \text{ess.}\inf C + \varepsilon) - A_{k-1} \text{ for } k > 0.$$

Then $\{A_k\}_{k\in\mathbb{N}}$ is a partition of Ω.

We define

$$Y_\varepsilon := \sum_{k\in\mathbb{N}} 1_{A_k} Y_k.$$

Since C is closed under countable concatenations, it follows that $Y_\varepsilon \in C$.

For the second part, it suffices to see that if K is closed under countable concatenations, then the set below

$$\left\{Y \in L^0_{++};\ X \in YK\right\}$$

is closed under countable concatenations as well.

Indeed, if $\{A_k\}_{k\in\mathbb{N}} \in \Pi(\Omega, \mathcal{F})$ and $\{Y_k\}_{k\in\mathbb{N}}$ is a sequence in L^0_{++} such that $X \in Y_k K$ for each $k \in \mathbb{N}$, we take $Y := \sum_{n\in\mathbb{N}} Y_n 1_{A_n} \in L^0_{++}$. Then we have

$$X/Y \in \text{cc}\left(\{A_k\}_k, \{X/Y_k\}_k\right)$$

and

$$X/Y_k \in K.$$

Therefore $X/Y \in K$, as K is closed under countable concatenations. □

Now, we have the following proposition:

Proposition 2.5. *Let $E[\tau]$ be a locally L^0-convex module and $U \subset E$ a L^0-convex, L^0-absorbent, L^0-balanced and closed under countable concatenations subset. Then*

$$\overset{0}{U} \subset \{X \in E;\ p_U(X) < 1\} \subset U \subset \{X \in E;\ p_U(X) \leq 1\}.$$

Proof. It suffices to show that $\{X \in E;\ p_U(X) < 1\} \subset U$. Indeed, let $X \in E$ be such that $p_U(X) < 1$.

By Lemma 2.4 there exists $Y \in L^0_{++}$ such that $X \in YU$ and $p_U(X) \leq Y < 1$. Then, since U is L^0-convex, $X = Y \cdot \frac{X}{Y} + (1 - Y) \cdot 0 \in U$. And the proof is complete. □

Finally, in the theorem below we provide a characterization of the locally L^0-convex topologies induced by a family of L^0-seminorms. This statement differs from Theorem 2.4 of [3] in requiring an extra condition over the elements of the neighborhood base of $0 \in E$, namely, being closed under countable concatenations.

Theorem 2.6. *Let $E[\tau]$ be a topological L^0-module. Then τ is induced by a family of L^0-seminorms if, and only if, there is a neighborhood base of $0 \in E$ for which each $U \in \mathcal{U}$ is*

1. *L^0-convex,*
2. *L^0-absorbent,*
3. *L^0-balanced and*
4. *closed under countable concatenations, i.e, $U = \overline{U}^{\mathrm{cc}}$.*

Proof. Suppose that τ is induced by a family of L^0-seminorms. If $Q \subset \mathcal{P}$ is finite and $\varepsilon \in L^0_{++}$, it can be easily checked that $U_{Q,\varepsilon}$ is L^0-convex, L^0-absorbent and L^0-balanced. Besides, $U_{Q,\varepsilon}$ is closed under countable concatenations. Indeed, if $X = \sum_n 1_{A_n} X_n$ with $X_n \in U_{Q,\varepsilon}$ for all $n \in \mathbb{N}$ and $\{A_n\}_{n\in\mathbb{N}}$ is a partition of Ω with $A_n \in \mathcal{F}$, it holds

$$\begin{aligned}\|X\| &= \left(\sum_n 1_{A_n}\right)\|X\| = \sum_n 1_{A_n}\|X\| \\ &= \sum_n \|1_{A_n}X\| = \sum_n \|1_{A_n}X_n\| = \sum_n 1_{A_n}\|X_n\| \leq \varepsilon.\end{aligned}$$

Reciprocally, let $\mathcal{U}$ be a neighborhood base of $0 \in E$ for which each $U \in \mathcal{U}$ is L^0-convex, L^0-absorbent, L^0-balanced and closed under countable concatenations. Let us consider the family of L^0-seminorms $\{p_U\}_{U\in\mathcal{U}}$ and let us show that it induces the topology τ. Given $U \in \mathcal{U}$, it is clear that $U \subset U_{p_U,1}$. Therefore, for $\varepsilon \in L^0_{++}$ there exists $U' \in \mathcal{U}$ such that $\frac{1}{\varepsilon}U' \subset U \subset U_{p_U,1}$. Thus, $U' \subset U_{p_U,\varepsilon}$. On the other hand, for $U \in \mathcal{U}$, it holds that $U_{p_U,\frac{1}{2}} \subset \{X \in E;\, p_U(X) < 1\} \subset U$. $\square$

Taking advantage of the ideas of the last theorem, we provide an example of a locally L^0-convex topology not induced by any family of L^0-seminorms.

Example 2.7. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an infinite partition $\{A_n\}_{n\in\mathbb{N}}$ of Ω with $A_n \in \mathcal{F}$ and $\mathbb{P}(A_n) > 0$ (for example, $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ the σ-algebra of Borel, $A_n = [\frac{1}{2^n}, \frac{1}{2^{n-1}})$ with $n \in \mathbb{N}$ and $\mathbb{P}$ the Lebesgue measure).

Let $\varepsilon \in L^0_{++}$. We define the set

$$U_\varepsilon := \left\{Y \in L^0;\, \exists\, I \subset \mathbb{N} \text{ finite},\ |Y 1_{A_i}| \leq \varepsilon\, \forall\, i \in \mathbb{N} - I\right\}.$$

Then, it is easily shown that U_ε is L^0-convex, L^0-absorbent and L^0-balanced, and $\mathcal{U} := \left\{U_\varepsilon;\ \varepsilon \in L^0_{++}\right\}$ is a neighborhood base of $0 \in E$ which generates a topology for which L^0 is a locally L^0-convex module.

Furthermore, it holds that U_ε is not closed under countable concatenations.

Indeed, it satisfies that $\varepsilon+1 \notin U_\varepsilon$, but $\varepsilon+1 = \sum_n (\varepsilon+1) 1_{A_n}$ with $(\varepsilon+1) 1_{A_n} \in U_\varepsilon$.

Easily, it can be shown that any neighborhood base of $0 \in E$ generating the same topology verifies that its proper elements are not closed under countable concatenations on L^0.

Therefore, due to Theorem 2.6, L^0 endowed with the topology generated by $\mathcal{U}$, is a locally L^0-convex topology, which is not induced by any family of L^0-seminorms.

Besides, it has to be met that $\{X \in L^0;\ p_{U_\varepsilon}(X) < 1\} \nsubseteq U_\varepsilon$ for some $\varepsilon \in L^0_{++}$. Otherwise, the family of L^0-seminorms $\{p_{U_\varepsilon}\}_{\varepsilon \in L^0_{++}}$ would induce the topology.

In fact, we claim that $p_U(X) = 0$ for all $X \in L^0$ and $U \in \mathcal{U}$. It suffices to show that $p_{U_1}(1) = 0$, since p_{U_1} is a L^0-seminorm. By way of contradiction, assume $p_{U_1}(1) > 0$. Then, there exists $m \in \mathbb{N}$ such that $\mathbb{P}((p_{U_1}(1) > 0) \cap A_m) > 0$. Define $A := (p_{U_1}(1) > 0) \cap A_m$, $\nu := \frac{p_{U_1(1)} + 1_{A^c}}{2}$, $Y := 1_{A^c} + \nu 1_A$ and $X := 1_{A^c} + \frac{1}{\nu} 1_A$. Thus, we have $1 = YX \in YU_1$ and $\mathbb{P}(p_{U_1}(1) > Y) > 0$. We have a contradiction.

Theorem 2.6 gives rise to give a new more restrictive definition of locally L^0-convex module. Namely, we propose the following definition:

Definition 2.8. A locally L^0-convex module is a L^0-module such that there is a neighborhood base of $0 \in E$ for which each $U \in \mathcal{U}$ is

1. L^0-convex,
2. L^0-absorbent,
3. L^0-balanced and
4. closed under countable concatenations.

Thus, under this new definition, we can rewrite Theorem 2.6 as follow:

Theorem 2.9. *A L^0-module $E[\tau]$ is locally L^0-convex if, and only if, τ is induced by a family of L^0-seminorms.*

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