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Lower Semicontinuity in SBV for Functionals with Nearly Linear Growth

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We present a lower semicontinuity result for free discontinuity energies with a quasiconvex volume term having non standard growth and a quite general surface term.

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1. Introduction

In this paper we study the lower semicontinuity with respect to the $L^1(\Omega, R^m)$ topology of integral functionals defined in $SBV(\Omega, \mathbb{R}^m)$, the space of special functions of bounded variation. A typical energy function on SBV is

$$\mathcal{F}(w) := \int_\Omega f(x, w, \nabla w)\, dx + \int_{J_w} \Phi(x, w^+, w^-, \nu_w)\, d\mathcal{H}^{N-1}, \tag{1}$$

where Ω is an open set of $\mathbb{R}^N$, $\mathcal{H}^{N-1}$ denotes the Hausdorff measure of dimension $(N-1)$, J_w denotes the set of approximate jump points of w, and the distributional derivative Dw is represented by $Dw = \nabla w \mathcal{L}^N + (w^+ - w^-) \otimes \nu_w \mathcal{H}^{N-1} \lfloor J_w$, with ν_w the normal to J_w and w^+, w^- the one-sided traces of w on both sides of J_w.

Actually, in the last years models involving bulk and interfacial energies have been used to describe phenomena in fracture mechanics, phase transitions, image segmentation and static theory of liquid crystals (see [5], [8], [9], [13], [25], [27], [29]). In [3] the author studied the lower semicontinuity of the functional (1) when the function f is quasi-convex and satisfies p-growth condition, while

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the function Φ fulfills suitable conditions (see also [20]). Moreover, in the recent papers [18] and [2], it has been also considered the lower semicontinuity of polyconvex functionals i.e. $f(A) = g(T(A))$ where g is convex and $T(A)$ denotes the set of all minors of the matrix $A \in M^{m\times N}$. Finally in [7] the authors addressed the case of a quasiconvex volume term having variable growth exponent.

In this paper we extend the l.s.c. theorem due to Ambrosio (see [3] or Theorem 5.29 in [4]) to the case when the absolute continuous part of the gradient is in an Orlicz space. We recall that Orlicz spaces generalize Lebesgue spaces: the role of the power t^p, $p \geq 1$, is played by more general convex functions.

Let us observe that in the Sobolev context this problem was considered in [14], [15], [23] and in [28]. In particular in [23] the authors considered very general growth assumptions.

Here we prove the following l.s.c. result concerning the bulk energy term.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let φ satisfy Assumption 2.3 (see Section 2). Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm} \to [0, +\infty)$ be a Carathéodory function such that*

$$-c + \varphi(|z|) \leq f(x, s, z) \leq a(x) + \psi(|s|)(1 + \varphi(|z|)) \tag{2}$$

for every $(x, s, z) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm}$, for some $c > 0$, $a \in L^1(\Omega)$ and some non-decreasing function $\psi : [0, +\infty) \to [0, +\infty)$. Let us assume that for every $(x, s) \in \Omega \times \mathbb{R}^m$ the function $z \mapsto f(x, s, z)$ is quasi-convex. Then

$$\liminf_{n\to\infty} \int_\Omega f(x, w^n, \nabla w^n)\, dx \geq \int_\Omega f(x, w, \nabla w)\, dx$$

for every sequence $\{w^n\} \subset SBV(\Omega, \mathbb{R}^m)$ converging in $L^1(\Omega, \mathbb{R}^m)$ to a function $w \in SBV(\Omega, \mathbb{R}^m)$ and satisfying $\sup_{n\in\mathbb{N}} \mathcal{H}^{N-1}(J_{w^n}) < +\infty$.

Let us point out here that we are able to recover the case of the so called *nearly linear growth* i.e. $\varphi(t) = t\log(1 + t)$, $\varphi(t) = t \log\log(t + e)$, $\varphi(t) = t \log\log\log(t + e^e)$, $t > 0$, etc..

Now let us explain the main ideas of the proof.

First of all we deal with a simpler situation in which Ω is the unit ball B_1 centered at 0, the limit function w is affine, the jump sets of the approximating functions w^n vanish as $n \to \infty$ and the density function vary. The general case will come by using a blow up argument.

The main tool in order to prove the result (in the simpler situation) is a Lipschitz approximation Lemma.

In the Sobolev setting this Lipschitz approximation Lemma is proved in [21] and [1], and then generalized by Ambrosio in [3] to the SBV context. More recently, Diening, Malék and Steinhauer in [10] extended the Lipschitz approximation to

Sobolev functions of variable exponent (see also [7]). Here, also in this general setting, using the maximal function of the total variation of w^n as in [3], we are able to construct a sequence $w^{n,j}$ of Lipschitz functions which approximate, in a suitable way, the sequence w^n belonging to $SBV^{\varphi}(\Omega, \mathbb{R}^m)$ (see Section 2 for the definition of the space $SBV^{\varphi}(\Omega, \mathbb{R}^m)$).

2. Notation and preliminary results

Throughout the paper $N, m > 1$ are fixed integers. Moreover Ω will be an open subset of $\mathbb{R}^N$. We denote the open ball in $\mathbb{R}^N$ with radius R and centered at x_0 and the open unit ball centered at 0 by $B_R(x_0)$ and B_1 respectively. To simplify the notation, the letter c will denote any positive constant, which may vary in the paper; relevant dependence upon parameters will be indicated in parentheses.

If $u \in L^1_{\rm loc}(\mathbb{R}^N, \mathbb{R}^m)$, for any $B_R(x_0)$ we set

$$u_{x_0,R} = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x)\, dx = \fint_{B_R(x_0)} u(x) dx.$$

The following definitions and results are standard in the context of $\mathcal{N}$–functions (see for instance [24]).

A *Young* function is a continuously increasing function

$$\varphi \,:\, [0, +\infty) \to [0, +\infty), \quad \varphi(0) = 0, \quad \lim_{t \to +\infty} \varphi(t) = \infty.$$

A convex function $\varphi : [0, +\infty) \to [0, +\infty)$ is said to be an $\mathcal{N}$-function if it satisfies the following conditions:

$$\lim_{t \to 0} \frac{\varphi(t)}{t} = 0 \tag{3}$$

and

$$\lim_{t \to +\infty} \frac{\varphi(t)}{t} = \infty. \tag{4}$$

It follows from the definition that there exists the derivative φ' of φ; this derivative is right continuous for $t > 0$, non-decreasing and satisfies $\varphi'(0) = 0$, $\lim_{t \to +\infty} \varphi(t) = \infty$, and $\varphi'(t) > 0$ for $t > 0$.

For example, the functions $\varphi_{p,\alpha}(t) = t^p \log^{\alpha}(1 + t)$, for $p > 1$ and $\alpha \geq 0$ or $p = 1$ and $\alpha > 0$, are $\mathcal{N}$-functions. See also Table 2.1 for other examples.

In the sequel, an essential role will be played by the growth at infinity of an $\mathcal{N}$-function. So, it is convenient to introduce the following notation. We shall write

$$\varphi_1(t) \preceq \varphi_2(t)$$

if there exist positive constants t_0 and k such that

$$\varphi_1(t) \leq \varphi_2(kt) \quad \text{if } t \geq t_0.$$

We shall say that the $\mathcal{N}$-functions φ_1 and φ_2 are *equivalent* and write $\varphi_1(t) \sim \varphi_2(t)$ if $\varphi_1(t) \preceq \varphi_2(t)$ and $\varphi_2(t) \preceq \varphi_1(t)$. It is clear that the $\mathcal{N}$-functions φ_1 and φ_2 satisfying the condition

$$\lim_{t \to +\infty} \frac{\varphi_1(t)}{\varphi_2(t)} = a > 0,$$

are equivalent.

Let us observe that only the growth at infinity really matters in the definition of $\mathcal{N}$-functions. Indeed given a continuous and convex function $Q : [0, +\infty) \to [0, +\infty)$ satisfying (4), it can be proved that Q is the *principal part* of an $\mathcal{N}$-function φ, that is $Q(t) = \varphi(t)$, for large values of t. Thus, we will refer as $\mathcal{N}$-functions to the functions $\varphi_0(t) = t^{\log(t+e)}$, $\varphi_\beta(t) = e^{t^\beta} - 1$, $\beta > 0$, which have not superlinear growth at 0.

We say that the $\mathcal{N}$-function φ satisfies the Δ_2-condition for large values of t if there exist $c_1 > 0$ and $t_0 \geq 0$ such that

$$\varphi(2t) \leq c_1 \varphi(t) \quad \text{if } t \geq t_0.$$

Since $\varphi(t) \leq \varphi(2t)$ the Δ_2-condition implies $\varphi(2t) \sim \varphi(t)$.

Take for instance $\varphi_{p,\alpha}(t) = t^p \log^\alpha(1+t)$ for $p > 1$ and $\alpha \geq 0$ or $p = 1$ and $\alpha > 0$, then $\varphi_{p,\alpha}$ satisfies the Δ_2-condition, while $\varphi_0(t) = (t+1)^{\log(t+1)}$ and $\varphi_\beta(t) = e^{t^\beta} - 1$ for any $\beta > 0$ do not verify the Δ_2-condition.

Let us also note that, if φ satisfies the Δ_2-condition then any $\mathcal{N}$-function which is equivalent to φ satisfies this condition too.

If φ is an $\mathcal{N}$-function satisfying the Δ_2-condition, by $L^\varphi(\Omega)$ and $W^{1,\varphi}(\Omega)$ (with Ω a bounded open set) we denote the classical Orlicz and Orlicz-Sobolev spaces, i.e. $u \in L^\varphi(\Omega)$ iff $\int_\Omega \varphi(|u|)\, dx < \infty$ and $u \in W^{1,\varphi}(\Omega)$ iff $u, \nabla u \in L^\varphi(\Omega)$. The Luxembourg norm is defined as follows:

$$\|u\|_{L^\varphi(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \varphi\left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

With this norm $L^\varphi(\Omega)$ is a Banach space.

Actually, by the very definition of the norm and the convexity of φ, it follows:

$$\|u\|_{L^\varphi(\Omega)} \leq 1 \Longrightarrow \int_\Omega \varphi(|u|) dx \leq \|u\|_{L^\varphi(\Omega)}, \tag{5}$$

as well as

$$\|u\|_{L^{\varphi}(\Omega)} \leq 1 + \int_{\Omega} \varphi(|u|)\, dx.$$

If there is no misunderstanding, we will write $\|u\|_{\varphi}$. Moreover, we denote by $W_0^{1,\varphi}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ functions with respect to the norm

$$\|u\|_{1,\varphi} = \|u\|_{\varphi} + \|\nabla u\|_{\varphi}$$

and by $W^{-1,\varphi}(\Omega)$ its dual.

Let us recall the following useful property (see [14]).

Proposition 2.1. *Let φ be an $\mathcal{N}$-function satisfying the Δ_2 condition. Then, for every family $\{u_\alpha\}_{\alpha\in I} \subseteq L^{\varphi}(\Omega)$ we have*

$$\sup_I \|u_\alpha\|_{L^{\varphi}(\Omega)} < +\infty$$

if and only if

$$\sup_I \int_{\Omega} \varphi(|u_\alpha|)dx < +\infty.$$

2.1. Maximal function.

For $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we define the non-centered maximal function of u by

$$Mu(x) := \sup_{B\ni x} \fint_B |u(y)|\, dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^N$ which contain x.

The following Proposition can be found in [19] (see Proposition 3.1).

Proposition 2.2. *Given an $\mathcal{N}$-function φ, we define a Young function $\gamma : [0,+\infty) \to [0,+\infty)$ by*

$$\gamma(t) = \varphi(t) - \int_0^t \frac{\varphi(s)}{s} ds. \tag{6}$$

Then for each positive $h \in L^1(\Omega)$ we have

$$\int_{\Omega} \gamma(M(h))\, dx \leq 2\cdot 3^n \int_{\Omega} \varphi(2h)\, dx. \tag{7}$$

Conversely,

$$\fint_{\Omega} \varphi(h)\, dx \leq \varphi\left(\fint_{\Omega} h\, dx\right) + 2^N \fint_{\Omega} \gamma(Mh)\, dx.$$

Moreover we do not gain any higher integrability if the function $t \to t^{-p}\varphi(t)$ is increasing for some $p > 1$. In this case

$$\gamma(t) \leq \varphi(t) \leq \frac{p}{p-1}\gamma(t)$$

and we have the following asymptotically sharp bound for p near 1*:*

$$\int_\Omega \gamma(M(h))\,dx \le \frac{4\cdot 3^n p}{p-1}\int_\Omega \gamma(h)\,dx.$$

Throughout the paper we will consider the following assumption on the function φ.

Assumption 2.3. Let φ be an $\mathcal{N}$-function satisfying the Δ_2-condition; we will assume that there exists an integer $l \ge 0$ such that the function

$$\frac{\gamma(t)}{t}\log t\log(\log t)\cdots\underbrace{\log(\log(\ldots(\log t)))}_{l-\text{logharitms}}$$

is non-decreasing for $t \ge t_0 > \underbrace{\exp(\exp(\ldots(\exp(1))))}_{l+1-\text{times}}$.

The following table exhibits some triples of functions equivalent to φ, γ, and the quotient $L(t) = \frac{\varphi(t)}{\gamma(t)}$ (see also Table 2 in [19]). In the last column we specified the corresponding index l for which Assumption 2.3 holds.

$\varphi(t)$	$\gamma(t)$	$L(t)$	l
$t^p(\log t)^\alpha$, $p>1$, $\alpha\ge 0$	$t^p(\log t)^\alpha$	1	0
$t\log t$	t	$\log t$	0
$t\log\log t$	$\frac{t}{\log t}$	$(\log t)\log\log t,$	1
$t(\log\log t)^\lambda$, $\lambda\ge 1$	$\frac{\lambda t(\log\log t)^{\lambda-1}}{\log t}$	$\frac{1}{\lambda}(\log t)\log\log t$	1
$t(\log t)^\alpha$, $\alpha>0$	$\alpha t(\log t)^{\alpha-1}$	$\frac{1}{\alpha}\log t$	1
$t\log\log\log t$	$\frac{t}{(\log t)\log\log t}$	$(\log t)(\log\log t)\log\log\log t$	2

Table 2.1:

Actually, other illustrative examples for the function φ are $\varphi(t) = tA(t)$ with

$$A(t) \sim \log\log\log\log t, \qquad A(t) \sim \log\log\log\log\log t, \qquad \text{etc.};$$

roughly speaking, the function A may approach infinity with increasing t as slowly as one wants. The corresponding γ will be equivalent to

$$\frac{t}{(\log t)(\log\log t)\log\log\log t}, \qquad \text{etc.}$$

For the sake of simplicity we confine our analysis to functions satisfying Assumption 2.3 for $l = 1$, since already this case incorporate all the technicality needed to obtain our result; without serious additional efforts we could consider all the other cases. Hence from now on we consider only functions which satisfy Assumption 2.3 with $l = 1$.

Remark 2.4. Let us notice that Assumption 2.3 reveals that there exists a positive constant C such that

$$L(t) \leq C(\log t) \log\log t \quad \text{if } t \geq t_0 = e^e. \tag{8}$$

In fact, let γ be defined by Proposition 2.2 or, equivalently,

$$\varphi(t) = \gamma(t) + t \int_0^t s^{-2} \gamma(s)\, ds.$$

Then Assumption 2.3 yields, when $t \geq t_0$,

$$\begin{aligned} \varphi(t) &= \gamma(t) + ct + t \int_{t_0}^t s^{-2}\gamma(s)\, ds \\ &\leq \gamma(t) + ct + \gamma(t)(\log t) \log\left(\frac{\log t}{\log t_0}\right) \\ &\leq C\gamma(t)(\log t)\log\log t, \end{aligned} \tag{9}$$

where the constant C depends on t_0. Under Assumption 2.3 we may also notice that

$$\gamma(2^{n+1}) - \gamma(2^n) \geq \frac{1}{3}\gamma(2^n) \quad \forall n \geq n_0, \tag{10}$$

where $n_0 \in \mathbb{N}$ is such that $2^{n_0} \geq t_0$.

An easy computation in fact shows that, for $n \geq n_0$,

$$\frac{\gamma(2^{n+1})}{2^{n+1}} \log 2^{n+1} \geq \frac{\gamma(2^n)}{2^n} \log 2^n,$$

that is

$$\frac{n+1}{2}\gamma(2^{n+1}) \geq \gamma(2^n) n,$$

which, in turn, reveals that $\gamma(2^{n+1}) \geq \frac{4}{3}\gamma(2^n)$, since $n \geq 2$.

2.2. BV functions.

If $w \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ and $x \in \Omega$, the *approximate limit of* w *at* x is defined as the unique value $\widetilde{w}(x) \in \mathbb{R}^m$ such that

$$\lim_{\rho \to 0^+} \frac{1}{\rho^N} \int_{B_\rho(x)} |w(y) - \widetilde{w}(x)|\, dy = 0\,.$$

The set of points in Ω where the approximate limit is not defined is called the *approximate singular set* of w and denoted by S_w, while $\Omega \setminus S_w$ consists of approximate continuity points.

Let $w \in L^1_{\rm loc}(\Omega, \mathbb{R}^m)$ and $x \in \Omega$. We say that x is an approximate jump point of w if there exist $a, b \in \mathbb{R}^m$ and $\nu \in \mathbb{S}^{N-1}$, such that $a \neq b$ and

$$\lim_{\rho\to 0^+} \fint_{B^+_\rho(x,\nu)} |w(y) - a|\, dy = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \fint_{B^-_\rho(x,\nu)} |w(y) - b|\, dy = 0$$

where $B^\pm_\rho(x,\nu) := \{y \in B_\rho(x) \ : \ \langle y - x, \nu\rangle \gtrless 0\}$. The triplet (a, b, ν) is uniquely determined by the previous formulas, up to a permutation of a, b and a change of sign of ν, and it is denoted by $(w^+(x), w^-(x), \nu_w(x))$. The Borel functions w^+ and w^- are called the *upper* and *lower approximate limit* of w at the point $x \in \Omega$. The set of approximate jump points of w is denoted by J_w. The set J_w is a Borel subset of S_w.

We recall that the space $BV(\Omega, \mathbb{R}^m)$ of *functions of bounded variation* is defined as the set of all $w \in L^1(\Omega, \mathbb{R}^m)$ whose distributional gradient Dw is a bounded Radon measure on Ω with values in the space $\mathbb{M}^{m\times N}$ of $m \times N$ matrices.

We recall the usual decomposition

$$Dw = \nabla w\, \mathcal{L}^N + D^c w + (w^+ - w^-) \otimes \nu_w\, \mathcal{H}^{N-1} \lfloor J_w\,,$$

where ∇w is the Radon-Nikodým derivative of Dw with respect to the Lebesgue measure and $D^c w$ is the *Cantor part* of Dw. For the sake of simplicity, we denote by $D^s w = D^c w + (w^+ - w^-) \otimes \nu_w\, \mathcal{H}^{N-1} \lfloor J_w$.

If $w \in BV(\Omega, \mathbb{R}^m)$ and x is such that $\tilde{w}(x)$ exists, then $\nabla w(x)$ is the *approximate differential* of w at $x \in \Omega$, i.e. the matrix $\nabla w(x)$ satisfies

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)} \frac{|w(y) - w(x) - \nabla w(x)(y - x)|}{|y - x|}\, dy = 0. \tag{11}$$

An important consequence of (11) is the fact that $\nabla w(x) = 0$ for almost every x in the set $\{x \in \Omega : w(x) = 0\}$. In particular we have

$$w, v \in BV(\Omega) \Longrightarrow \nabla w(x) = \nabla v(x) \ \text{ for a.e. } x \in \Omega \text{ s.t. } w(x) = v(x). \tag{12}$$

We recall that the space $SBV(\Omega, \mathbb{R}^m)$ of *special functions of bounded variation* is defined as the set of all $w \in BV(\Omega, \mathbb{R}^m)$ such that the Cantor part $D^c w$ of the derivative is 0.

The space $SBV^\varphi(\Omega, \mathbb{R}^m)$ is defined as the set of all functions $w \in SBV(\Omega, \mathbb{R}^m)$ with $\nabla w \in L^\varphi(\Omega, \mathbb{M}^{m\times N})$ and $\mathcal{H}^{N-1}(S_w) < \infty$.

For a general survey on the spaces of BV, SBV and SBV^p (i.e. when $\varphi(t) = t^p$) we refer for instance to [4].

2.3. Quasi-convex functions

Let $f : \mathbb{R}^{Nm} \to \mathbb{R}$ be a continuous function. We say that f is quasiconvex if

$$\fint_{B_1} f(z + \nabla\eta(x))\, dx \geq f(z) \ \ \forall z \in \mathbb{R}^{Nm}, \ \forall \eta \in C^1_0(B_1, \mathbb{R}^m).$$

This property, introduced by Morrey (see [26]), is necessary for the sequential lower semicontinuity of the functional

$$F(w) = \int_{B_1} f(\nabla w)\,dx$$

with respect to the weak* topology of $W^{1,\infty}(B_1, \mathbb{R}^m)$. The following result was proven in [1] (see also [22]).

Theorem 2.5. *Let $f(x, s, z) : B_1 \times \mathbb{R}^m \times \mathbb{R}^{Nm} \to \mathbb{R}$ be a Carathéodory function, quasi-convex in z for any $x \in B_1$ and any $s \in \mathbb{R}^m$, such that*

$$\sup\left\{ f(x, s, z) : x \in B_1, s \in \mathbb{R}^m, z \in \mathbb{R}^{Nm}, |s| + |z| \le \lambda \right\} < +\infty$$

for any $\lambda > 0$. Then,

$$\liminf_{n\to+\infty} \int_{B_1} f(x, w^n, \nabla w^n)\,dx \ge \int_{B_1} f(x, w, \nabla w)\,dx$$

for any bounded sequence $\{w^n\} \in W^{1,\infty}(B_1, \mathbb{R}^m)$ uniformly converging in B_1 to $w \in W^{1,\infty}(B_1, \mathbb{R}^m)$.

2.4. Lusin approximation in BV

For every positive, finite Radon measure μ in $\mathbb{R}^N$, using the Besicovitch covering theorem (see e.g. [12]), it can be proved the existence of a constant ξ depending on N such that

$$\mathcal{L}^N\left(\left\{x \in \mathbb{R}^N : M(\mu)(x) > \lambda\right\}\right) \le \frac{\xi}{\lambda}\mu(\mathbb{R}^N). \tag{13}$$

Next result can be found in [7] (see Theorem 2.2).

Lemma 2.6. *There exists a constant $c = c(N, m)$ such that for every $w \in BV(\mathbb{R}^N, \mathbb{R}^m)$ and $\vartheta, \lambda > 0$ there exists a Lipschitz function $v : \mathbb{R}^N \to \mathbb{R}^m$ such that $\|v\|_\infty \le \vartheta$, $\|\nabla v\|_\infty \le c\lambda$ and up to a null set*

$$\{v \ne w\} \subseteq \{M(|w|) > \vartheta\} \cup \{M(|Dw|) > \lambda\}\,. \tag{14}$$

Moreover if $\Omega \subset \mathbb{R}^N$ is a bounded open set with the property that there exists a constant $A \ge 1$ such that for all $x \in \Omega$

$$\mathcal{L}^N(B_{2\mathrm{dist}(x,\Omega^c)}(x)) \le A\mathcal{L}^N(B_{2\mathrm{dist}(x,\Omega^c)}(x) \cap \Omega^c), \tag{15}$$

and if w has compact support in Ω then $v = 0$ in $\mathbb{R}^N \setminus \Omega$, $\|\nabla v\|_\infty \le c\lambda$ with $c = c(N, m, A)$, and up to a null set

$$\{v \ne w\} \subseteq \Omega \cap (\{M(|w|) > \vartheta\} \cup \{M(|Dw|) > \lambda\})\,. \tag{16}$$

Remark 2.7. If $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary, then Ω satisfies (15).

3. Lusin approximation in SBV^φ

In this section we introduce the method of Lipschitz approximations of $SBV^\varphi(\Omega, \mathbb{R}^m)$ functions. Let us recall that, in the Sobolev setting, the basic idea is that Sobolev functions in $W_0^{1,1}$ can be approximated by λ-Lipschitz functions that coincide with the originals up to sets of small Lebesgue measure. The Lebesgue measure of these non-coincidence sets is bounded by the Lebesgue measure of the sets where the Hardy-Littlewood maximal function of the gradients are above λ.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set satisfying* (15) *and let φ satisfy Assumption* 2.3. *Let $v^n \in \mathrm{SBV}^\varphi(\mathbb{R}^N, \mathbb{R}^m)$ be a sequence of functions with compact support in Ω; we assume that*

$$\|v^n\|_1 \to 0, \qquad \sup_n \|v^n\|_\infty < +\infty, \tag{17}$$

$$\sup_n \int_\Omega \varphi(|\nabla v^n|)dx < +\infty \tag{18}$$

so that in particular

$$\|v^n\|_{L^\varphi(\Omega)} \to 0 \quad \textit{and} \quad \int_\Omega \varphi(|v^n|)dx \to 0 \ (\textit{as } n \to +\infty). \tag{19}$$

Let $\vartheta_n > 0$ be such that

$$\vartheta_n \to 0 \quad \textit{and} \quad \gamma_n := \frac{\int_\Omega \varphi(|v^n|)dx}{\gamma(\vartheta_n)} \to 0 \quad (\textit{as } n \to +\infty).$$

Then there exist sequences $\mu_j, \lambda_{n,j} > 1$ satisfying, for every $n, j \in \mathbb{N}$,

$$\mu_j \le \lambda_{n,j} \le \mu_{j+1}, \tag{20}$$

and a sequence $v^{n,j} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^m)$, $v^{n,j} = 0$ in $\mathbb{R}^N \setminus \Omega$, such that, for every $n, j \in \mathbb{N}$,

$$\|v^{n,j}\|_\infty \le \vartheta_n\,; \tag{21}$$

$$\|\nabla v^{n,j}\|_\infty \le C\lambda_{n,j}\,; \tag{22}$$

moreover, up to a null set,

$$\left\{v^{n,j} \ne v^n\right\} \subseteq \Omega \cap \left(\{M(|v^n|) > \vartheta^n\} \cup \left\{M(|Dv^n|) > 3\lambda^{n,j}\right\}\right). \tag{23}$$

Then, in particular, for every $j \in \mathbb{N}$ and for n tending to infinity,

$$\nabla v^{n,j} \rightharpoonup 0 \quad \textit{weakly}^* \textit{ in } L^\infty(\Omega, \mathbb{R}^{Nm}). \tag{24}$$

Finally, there exists a sequence $\varepsilon_j > 0$ with $\liminf_{j\to+\infty} \varepsilon_j = 0$ such that for every $j \in \mathbb{N}$

$$\begin{aligned} &\limsup_{n\to+\infty} \int_\Omega \varphi(|\nabla v^{n,j}|)\chi_{\{M(|v^n|)>\vartheta_n\}\cup\{M(|\nabla v^n|)>2\lambda_{n,j}\}}dx \\ &\le C \limsup_{n\to+\infty} \int_\Omega \varphi(\lambda_{n,j})\chi_{\{M(|v^n|)>\vartheta_n\}\cup\{M(|\nabla v^n|)>2\lambda_{n,j}\}}dx \le \varepsilon_j. \end{aligned} \tag{25}$$

Proof. The proof follows the lines of the proof of Theorem 2.13 in [6] (see also [10], [7], [11]) but we can not apply exactly the same technique.

By (18), (19), using Proposition 2.2 together with the Δ_2-condition, we get the existence of a constant K such that

$$K := \sup_n \int_{\mathbb{R}^N} \gamma(M(|v^n|))dx + \int_{\mathbb{R}^N} \gamma(M(|\nabla v^n|))dx < \infty.$$

Next, we note that for a measurable function g with $\gamma(|g|) \in L^1(\mathbb{R}^N)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \gamma(|g(x)|)\,dx &= \int_{\mathbb{R}^N} \int_0^\infty \gamma'(t)\chi_{\{|g|>t\}}\,dt\,dx \\
&\geq \int_{\mathbb{R}^N} \sum_{m\geq n_0} \int_{2^m}^{2^{m+1}} \gamma'(t)\chi_{\{|g|>t\}}\,dt\,dx \\
&\geq \frac{1}{3}\int_{\mathbb{R}^N} \sum_{m\geq n_0} \gamma(2^m)\chi_{\{|g|>2^{m+1}\}}\,dx \qquad (26)\\
&\geq \frac{1}{3}\sum_{j\geq j_0}\sum_{k=2^j}^{2^{j+1}-1}\int_{\mathbb{R}^N} \gamma(2^k)\chi_{\{|g|>2^{k+1}\}}\,dx,
\end{aligned}$$

where we used (10) in the third line and where j_0 is such that $2^{j_0} \geq n_0$. The choice $g = \chi_\Omega M(|\nabla v^n|)$ yields

$$\sum_{j\geq j_0}\sum_{k=2^j}^{2^{j+1}-1}\int_\Omega \gamma(2^k)\chi_{\{|M(|\nabla v^n|)|>2^{k+1}\}}\,dx \leq 3K.$$

We can rewrite the last inequality as

$$\sum_{j\geq j_0} b_j^n \leq 3K,$$

with an obvious definition for b_j^n. Since the sum (which defines b_j^n) contains 2^j addenda, there is at least one index $k_{n,j}$ such that

$$\int_\Omega \gamma(2^{k_{n,j}})\chi_{\{|M(|\nabla v^n|)|>2\cdot 2^{k_{n,j}}\}}\,dx \leq 2^{-j}b_j^n. \qquad (27)$$

Therefore multiplying both sides by $L(2^{k_{n,j}})$, we gain

$$\int_\Omega \varphi(2^{k_{n,j}})\chi_{\{|M(|\nabla v^n|)|>2\cdot 2^{k_{n,j}}\}}\,dx \leq cL(2^{k_{n,j}})2^{-j}b_j^n.$$

Thanks to (8) we have that

$$\int_\Omega \varphi(2^{k_{n,j}})\chi_{\{|M(|\nabla v^n|)|>2\cdot 2^{k_{n,j}}\}}\,dx \leq c\log(2^{k_{n,j}})\log\log(2^{k_{n,j}})2^{-j}b_j^n \leq cj\,b_j^n.$$

Since the sequence b_j^n is uniformly bounded, i.e. $b_j^n \le 3K$ for all j, n, by a diagonal argument we can construct a subsequence $\{b_j^{n_i}\}_{i=1}^{+\infty}$ so that, for each fixed j , the sequence $b_j^{n_i} \to \delta_j$ as $i \to +\infty$. For the sake of simplicity we denote again the sequence $\{b_j^{n_i}\}$ with $\{b_j^n\}$.

This proves that, for all $j \ge j_0$

$$\limsup_{n\to+\infty} \int_\Omega \varphi(2^{k_{n,j}}) \chi_{\{|M(|\nabla v^n|)|>2\cdot 2^{k_{n,j}}\}}\, dx \le cj\delta_j. \tag{28}$$

Moreover

$$3K \ge \liminf_{n\to+\infty} \sum_j b_j^n \ge \sum_j \liminf_{n\to+\infty} b_j^n = \sum_j \delta_j$$

and as a consequence $\liminf_{j\to+\infty} j\delta_j = 0$ (if $\liminf_{j\to+\infty} j\delta_j = a > 0$, then $j\delta_j \ge \frac{a}{2}$, for $j \ge \tilde{j}$, which in turn gives a contradiction with the fact that δ_j defines a convergent series).

Let us define $\epsilon_j = cj\delta_j$, $\lambda_{n,j} := 2^{k_{n,j}}$ and $\mu_j := 2^{2^j}$, then

$$\mu_j = 2^{2^j} \le \lambda_{n,j} \le 2^{2^{j+1}} = \mu_{j+1}. \tag{29}$$

Therefore by using Proposition 2.2, since $\gamma(t)$ is an increasing function, we get

$$\begin{aligned} &\int_\Omega \varphi(\lambda_{n,j}) \chi_{\{M(|v^n|)>\vartheta_n\}} dx \\ &\le \varphi(\lambda_{n,j}) \int_\Omega \chi_{\{\gamma(M(|v^n|))>\gamma(\vartheta_n)\}} dx \\ &\le c\varphi(\lambda_{n,j}) \int_\Omega \frac{\gamma(M(|v^n|))}{\gamma(\vartheta_n)} dx \le c\varphi(\lambda_{n,j}) \int_\Omega \frac{\varphi(|v^n|)}{\gamma(\vartheta_n)} dx \le c\varphi(\mu_{j+1})\gamma_n. \end{aligned} \tag{30}$$

Now we apply Lemma 2.6 (with ϑ_n and $3\lambda_{n,j}$) and we find, for each $n, j \in \mathbb{N}$, a sequence $v^{n,j} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^m)$, $v^{n,j} = 0$ in $\mathbb{R}^N \setminus \Omega$, such that

$$\|v^{n,j}\|_\infty \le \vartheta_n, \qquad \|\nabla v^{n,j}\|_\infty \le C\lambda_{n,j}, \tag{31}$$

and, up to a null set,

$$\left\{v^{n,j} \ne v^n\right\} \subseteq \Omega \cap (\{M(|v^n|) > \vartheta_n\} \cup \{M(|Dv^n|) > 3\lambda_{n,j}\}) \tag{32}$$

holds. Finally (24) follows by (31) while, using (30) together with (28), and (31) we obtain (25). $\square$

4. Lower semicontinuity

In this section, we perform the proof of Theorem 1.1 and we give a lower semicontinuity result for the surface term.

As in [3] (see also [4]), in order to prove Theorem 1.1, first of all we need to prove the result in the particular case when Ω is the unit ball B_1 centered at 0, the limit function u is linear and $\mathcal{H}^{N-1}(J_{u_n})$ is infinitesimal and the integrand functions vary.

Theorem 4.1. *Let φ satisfy Assumption 2.3 and $g_n : B_1 \times \mathbb{R}^m \times \mathbb{R}^{Nm} \to [0, +\infty)$ be a sequence of Carathéodory functions such that*

$$-c_1 + c_2\varphi(|z|) \le g_n(y, s, z) \le a_n(y) + \psi(|s|)(1 + \varphi(|z|)) \tag{33}$$

for every $(y, s, z) \in B_1 \times \mathbb{R}^m \times \mathbb{R}^{Nm}$; here c_1, c_2 are positive constants, $\psi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function, and finally $a_n \in L^1(B_1)$ is such that $a_n \to \bar{a}$ in L^1. Let us assume that there exists a $\mathcal{L}^N$-negligible set $E \subset B_1$ and a quasi-convex function $g : \mathbb{R}^{Nm} \to [0, +\infty)$, such that $\lim_{n\to\infty} g_n(y, s, z) = g(z)$ locally uniformly in $\mathbb{R}^{m+Nm}$ for every $y \in B_1 \setminus E$. Then

$$\liminf_{n\to\infty} \int_{B_1} g_n(y, w^n, \nabla w^n)\, dy \ge \int_{B_1} g(\nabla w)\, dy \tag{34}$$

for every sequence $\{w^n\} \subset SBV(B_1, \mathbb{R}^m)$ converging in $L^1(B_1, \mathbb{R}^m)$ to a linear function w and satisfying $\mathcal{H}^{N-1}(J_{w^n}) \to 0$ as $n \to \infty$.

Proof. The result can be achieved following the lines of Theorem 4.1 in [7]. Let $L = \nabla w$. Possibly replacing w^n by $w^n - w$ and $g_n(y, u, z)$ by $g_n(y, u+Ly, z+L)$ and taking into account the Δ_2-condition, we can assume that $w = 0$, that is w^n converges to 0 in L^1. Without loss of generality we can suppose that the liminf in (34) is a finite limit, so that, by (33), we easily get that

$$\limsup_{n\to\infty} \int_{B_1} \varphi(|\nabla w^n|)\, dy < \infty. \tag{35}$$

Thus the proof of Step 1 in Proposition 5.37 of [4] can be carried out exactly in the same way since the only assumptions needed are the equiintegrability of $\{|\nabla w^n|\}$ and the fact that the size of the jumps of w^n is infinitesimal: from now on we assume that $\|w^n\|_\infty \le 3$. Letting $\rho \in (0, 1)$, the Theorem is proved if we show that

$$\liminf_{n\to\infty} \int_{B_\rho} g_n(y, w^n, \nabla w^n)\, dy \ge \mathcal{L}^N(B_\rho) g(0) \tag{36}$$

and letting ρ tend to 1. Moreover, possibly multiplying w^n by a cut off function $\chi \in C_c^\infty(B_1)$ identically equal to 1 in B_ρ, we can assume, without loss of generality, that $w^n \in SBV^\varphi(\mathbb{R}^N, \mathbb{R}^m)$ have compact support in B_1. Notice that, by the L^∞ bound on w^n and the Δ_2-condition for φ, the sequence $\{\nabla(w^n\chi)\}$ still satisfies (35).

We are now in position to apply Theorem 3.1 to w^n, obtaining a sequence $w^{n,j} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^m)$, $w^{n,j} = 0$ in $\mathbb{R}^N \setminus B_1$, such that (21), (22), (23) and (24) hold, and

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_\Omega \varphi(|\nabla w^{n,j}|)\chi_{\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2\lambda_{n,j}\}}dx \\
&\le C \limsup_{n\to+\infty} \varphi(\lambda_{n,j}) \int_\Omega \chi_{\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2\lambda_{n,j}\}}dx \le \varepsilon_j,
\end{aligned} \tag{37}$$

with $\mu_j, \lambda_{n,j} > 1$ such that $\mu_j \le \lambda_{n,j} \le \mu_{j+1}$. We have:

$$\begin{aligned}
&\int_{B_\rho} g_n(y, w^n, \nabla w^n)\, dy \ge \int_{B_\rho\cap\{w^n=w^{n,j}\}} g_n(y, w^{n,j}, \nabla w^{n,j})\, dy \\
&\ge \int_{B_\rho\setminus E} [g_n(y, w^{n,j}, \nabla w^{n,j}) - g(\nabla w^{n,j})]\, dy + \int_{B_\rho} g(\nabla w^{n,j})\, dy \\
&\quad - \int_{B_\rho\cap\{w^n\neq w^{n,j}\}} g_n(y, w^{n,j}, \nabla w^{n,j})\, dy,
\end{aligned} \tag{38}$$

obtaining, when n tends to infinity,

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{B_\rho} g_n(y, w^n, \nabla w^n)\, dy \\
&\ge \liminf_{n\to\infty} \int_{B_\rho} g(\nabla w^{n,j})\, dy - \limsup_{n\to\infty} \int_{B_\rho\cap\{w^n\neq w^{n,j}\}} g_n(y, w^{n,j}, \nabla w^{n,j})\, dy \\
&\ge \mathcal{L}^N(B_\rho)g(0) - C \limsup_{n\to\infty} \int_{B_\rho\cap\{w^n\neq w^{n,j}\}} [a_n(y) + \psi(|w^{n,j}|)(1 + \varphi(|\lambda_{n,j}|))]\, dy,
\end{aligned} \tag{39}$$

where we used the Lebesgue dominated convergence theorem, Theorem 2.5, and (33). Let us now focus our attention on the last term in (39). Thanks to (23) and taking into account that $\psi(|w^{n,j}|)$ is bounded by a constant c, we can estimate the last term in (39) as follows:

$$\begin{aligned}
&\int_{B_\rho\cap\{w^n\neq w^{n,j}\}} [a_n(y) + c(1 + \varphi(\lambda_{n,j}))]\, dy \\
&\le \int_{B_1\cap(\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2\lambda_{n,j}\})} [a_n(y) + c(1 + \varphi(\lambda_{n,j}))]\, dy \\
&\quad + \int_{B_1\cap(\{M(|D^j w^n|)>\lambda_{n,j}\})} [a_n(y) + c(1 + \varphi(\lambda_{n,j}))]\, dy \\
&\le \omega(n) + I_1^{n,j} + I_2^{n,j} + I_3^{n,j},
\end{aligned} \tag{40}$$

where $\omega(n)$ is infinitesimal and

$$I_1^{n,j} = \int_{B_1\cap(\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2\lambda_{n,j}\})} (\bar a(y)+c)dy + \int_{B_1\cap\{M(|D^jw^n|)>\lambda_{n,j}\})} (\bar a(y)+c)dy,$$

with $\bar a$ the L^1-limit of a_n,

$$I_2^{n,j} = \int_{B_1\cap(\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2\lambda_{n,j}\})} c\,\varphi(\lambda_{n,j})\,dy,$$

and

$$I_3^{n,j} = \int_{B_1\cap(\{M(|D^jw^n|)>\lambda_{n,j}\})} c\,\varphi(\lambda_{n,j})\,dy.$$

Concerning $I_3^{n,j}$ we derive from (13), the convexity of φ, and (20) that

$$I_3^{n,j} \le c\frac{\xi}{\lambda_{n,j}}\varphi(\lambda_{n,j})|D^jw^n|(\mathbb{R}^N) \le c\,\xi\,\varphi'(\mu_{j+1})\,\mathcal{H}^{N-1}(J_{w^n}), \tag{41}$$

so that

$$\lim_{n\to\infty} I_3^{n,j} = 0. \tag{42}$$

Let us consider now $I_2^{n,j}$; thanks to (37) we have

$$\liminf_{j\to\infty}\limsup_{n\to\infty} I_2^{n,j} = 0. \tag{43}$$

Finally, let us deal with $I_1^{n,j}$. We already estimated the Lebesgue measure of $\{M(|D^jw^n|)>\lambda_{n,j}\}$ in (41) obtaining that it goes to zero as n tends to infinity. On the other hand, by Chebyshev inequality, Proposition 2.2, the monotonicity of γ, and (35)

$$\begin{aligned}\mathcal{L}^N(\{B_1\cap M(|\nabla w^n|)>2\lambda_{n,j}\}) &\le c\int_{\{\gamma(M(|\nabla w^n|))>\gamma(\mu_j)\}} \frac{\gamma(M(|\nabla w^n|))}{\gamma(\mu_j)}\,dy \\ &\le \frac{c}{\gamma(\mu_j)}\int_{B_1}\varphi(|\nabla w^n|)\,dy \le \frac{c}{\gamma(\mu_j)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathcal{L}^N(\{M(|w^n|)>\vartheta_n\}) &\le \frac{c}{\gamma(\vartheta_n)}\int_{B_1}\gamma(M(|w^n|))\,dy \\ &\le \frac{c}{\gamma(\vartheta_n)}\int_{B_1}\varphi(|w^n|)dy \to 0,\end{aligned}$$

by the choice of ϑ_n. Thus we can infer that

$$\lim_{j\to\infty}\limsup_{n\to+\infty} I_1^{n,j} = 0. \tag{44}$$

Finally, combining (42), (43), and (44) we complete the proof. $\square$

Now a blow up argument allows us to reduce the general problem to the special case of Theorem 4.1.

Proof of Theorem 1.1. The proof follows that of Theorem 5.29 in [4] (see also Theorem 4.3 in [3]). We will consider only the main points of the proof reminding for the rest to [4] or [3].

The aim is to show that the Radon measure μ, that is the weak* limit of (a subsequence of) $f(x, w^n, \nabla w^n)\mathcal{L}^N$, will be greater than $f(x, w, \nabla w)\mathcal{L}^N$. This is obtained by proving that

$$\limsup_{\rho \to 0} \frac{\mu(B_\rho(x_0))}{\mathcal{L}^N(B_\rho(x_0))} \geq f(x_0, w(x_0), \nabla w(x_0)),$$

for a.e. $x_0 \in \Omega$. Choosing suitable points x_0, Theorem 4.1 is applied to the sequences $g_i(y, u, z) := f(x_0 + \rho_i y, w(x_0) + \rho_i u, z)$, $u^i(y) := [w^{n_i}(x_0 + \rho_i y) - w(x_0)]/\rho_i$, where $\rho_i \to 0$ as $i \to \infty$ and $u^i \to \nabla w(x_0)\, y$ in $L^1(B_1, \mathbb{R}^m)$, while $\mathcal{H}^{N-1}(J_{u^i}) \to 0$. On the other hand the functions g_i satisfy all the conditions of Theorem 4.1, besides the fact that

$$\int_{B_1} g_i(y, u^i, \nabla u^i)\, dy \leq \frac{\mu(B_{\rho_i}(x_0))}{\rho_i^N} + \rho_i. \tag{45}$$

Thus

$$\begin{aligned} \liminf_{i \to \infty} \int_{B_1} g_i(y, u^i, \nabla u^i)\, dy &\geq \int_{B_1} f(x_0, w(x_0), \nabla w(x_0))\, dy \\ &= \mathcal{L}^N(B_1) f(x_0, w(x_0), \nabla w(x_0)), \end{aligned}$$

and this inequality, together with (45) gives the result. □

Finally, for the reader's convenience, we recall a lower semicontinuity result concerning the surface term in (1). We assume that $\Phi : \mathbb{R}^N \times \mathbb{R}^N \times S^{n-1} \to [0, \infty)$ is jointly convex, i.e. there exists a sequence of continuous functions $g^n : \mathbb{R}^N \to \mathbb{R}^n$ such that

$$\Phi(u, v, \nu) = \sup_n \langle g^n(u) - g^n(v), \nu \rangle \quad \text{for all } (u, v, \nu) \in \mathbb{R}^N \times \mathbb{R}^N \times S^{n-1}.$$

Examples of jointly convex functions are easily constructed (see [[4], Example 5.23]) by taking functions of the form $\Phi(u, v, \nu) = \vartheta(|u - v|)\psi(\nu)$, where $\vartheta : \mathbb{R} \to [0, \infty)$ is a lower semicontinuous, increasing and subadditive function and $\psi : \mathbb{R}^n \to [0, \infty)$ is an even, positively 1-homogeneous convex function.

Theorem 4.2. *Let $\Phi : \mathbb{R}^N \times \mathbb{R}^N \times S^{n-1} \to [0, \infty)$ be a jointly convex function with the property that there exist $M > 0$ and $\mu > 0$ such that*

$$\Phi(u, v, \nu) \geq \mu \quad \text{for all } (u, v, \nu) \in \mathbb{R}^N \times \mathbb{R}^N \times S^{n-1}.$$

Let w^n be a a sequence in $SBV(\Omega,\mathbb{R}^N)$ converging in $L^1(\Omega,\mathbb{R}^N)$ to $w \in SBV(\Omega,\mathbb{R}^N)$, such that $|\nabla w^n|$ is equi-integrable and, for any $n \in \mathbb{N}$, $|w^n(x)| < M$ for a.e. $x \in \Omega$. Then

$$\int_{J_w} \Phi(w^+, w^-, \nu_w) d\mathcal{H}^{n-1} \leq \liminf_n \int_{J_{w^n}} \Phi((w^n)^+, (w^n)^-, \nu_{w^n}) d\mathcal{H}^{n-1}.$$

For the proof of Theorem 4.2 see [4, Theorem 5.22].

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