Journal of Convex Analysis
Volume 24 (2017), No. 2, 501–503

A Note on an Alternative Iterative Method for Nonexpansive Mappings

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Received: April 4, 2014
Accepted: September 28, 2015

In this note we point out that results on the asymptotic behaviour of an alternative iterative method are corollaries of corresponding results on the well-known Halpern iteration.

1991 Mathematics Subject Classification: 47J25, 47H09

Let X be a geodesic space, $C \subseteq X$ a convex subset and $T : C \to C$ a nonexpansive mapping. We consider the following iteration:

$$x_0 \in C, \quad x_{n+1} = T(\lambda_{n+1}u + (1-\lambda_{n+1})x_n), \tag{1}$$

where $u \in C$ and (λ_n) is a sequence in $[0,1]$. This alternative iterative method was introduced for Banach spaces in [13] as a discretization of an approximating curve considered in [2]. A more general version of this iteration is studied in [1], while a cyclic version of (x_n) associated to a finite family of nonexpansive mappings is considered in [7].

Define for $n \geq 0$,

$$y_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})x_n. \tag{2}$$

*Laurenţiu Leuştean was supported by a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PN-II-ID-PCE-2011-3-0383.
†Adriana Nicolae was supported by a grant of the Romanian Ministry of Education, CNCS - UEFISCDI, project number PN-II-RU-PD-2012-3-0152.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Then for all $n \geq 1$, $x_n = Ty_n$ and the following holds:

Proposition 1. *For all* $n \geq 1$, $y_{n+1} = \lambda_{n+1}u + (1 - \lambda_{n+1})Ty_n$.

Thus, (y_n) is the well-known Halpern iteration [3, 11]. Since T is nonexpansive, we have that for all $m, n \geq 1$ and for any fixed point p of T,

$$d(x_m, x_n) \leq d(y_m, y_n) \quad \text{and} \quad d(x_n, p) \leq d(y_n, p).$$

Hence, one can obtain results concerning this iteration as immediate corollaries of corresponding results on the Halpern iteration.

A first example is the following: if (y_n) converges strongly to a fixed point p of T, then (x_n) also converges strongly to p. It follows that [13, Theorem 4.1], the main result on the iteration (x_n), is a corollary of [12, Theorem 3.1] and [10, 9]. Moreover, strong convergence results on the Halpern iteration, as well as their quantitative versions (see, e.g., [4, 5, 6, 8]), hold for the alternative iteration (x_n) too.

Let us give a second example. We say that a sequence (z_n) is asymptotically regular if $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$. A mapping $\Phi : (0, \infty) \to \mathbb{N}$ is a rate of asymptotic regularity for (z_n) if for all $\varepsilon > 0$ and for all $n \geq \Phi(\varepsilon)$, $d(z_n, z_{n+1}) < \varepsilon$. One has the following: if (y_n) is asymptotically regular with rate Φ, then (x_n) is asymptotically regular with the same rate. As a consequence, (quantitative) asymptotic regularity results for the Halpern iteration hold without changes for the alternative iteration (x_n).

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