Journal of Convex Analysis
Volume 24 (2017), No. 2, 505–517

$\mathbb{B}^{-1}$–Convex Functions

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Received: April 30, 2015
Revised manuscript received: July 30, 2015
Accepted: November 27, 2015

In this work, $\mathbb{B}^{-1}$-convex functions which are a new class of generalized convex functions are studied. Some theorems about important properties of $\mathbb{B}^{-1}$-convex functions are proved. Also, with the aim of regarding the relation between convex and $\mathbb{B}^{-1}$-convex functions, some examples are given.

Keywords: Abstract convexity, $\mathbb{B}^{-1}$-convexity, $\mathbb{B}^{-1}$-convex functions

2010 Mathematics Subject Classification: 52A01, 52A41, 26B25

1. Introduction

Nowadays, abstract convexity has attracted great attention, especially, generalized convex functions' classes have studied a lot. Some of these function classes are quasiconvex functions ([21]), multiplicatively convex functions ([20]), P-functions ([21]), Godnova-Levin functions ([12]), increasing convex-along-rays functions ([20]), s-convex functions ([16]), log-convex functions ([17]), r-convex functions ([5]), etc. Moreover, topical and anti-topical functions each of which is also of generalized convex function classes studied in [22, 23] in recent years.

Convexity concept has generalized as different abstract convexity types. Some of these types are max-plus convexity ([14, 15]), min-plus convexity ([15, 18]), tropical convexity ([11, 18]), $\mathbb{B}$-convexity ([7]), $\mathbb{B}^{-1}$-convexity ([2]) which are connected with each other. For instance, in [15], the equivalence of the theories of $\mathbb{B}-$convexity and max-plus convexity is indicated. In the same way, it can be shown that $\mathbb{B}^{-1}$-convexity and min-plus convexity are equal.

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ISSN 0944-6532 / $ 2.50

$\mathbb{B}$-convexity which is established in [7] is one of the generalizations of convexity. In this field, separation theorems were studied in [9], $\mathbb{B}$-convex sets and their properties are examined in [1], furthermore, halfspaces and Hahn-Banach like properties were considered in [8]. $\mathbb{B}$-convexity is also discussed in [4, 6, 13] and $\mathbb{B}$-convex functions are studied in [3].

$\mathbb{B}^{-1}$-convexity concept was given in [2, 10] and some important properties of $\mathbb{B}^{-1}$-convex sets were investigated in [2, 4]. The applications of $\mathbb{B}^{-1}$-convexity to economy were given in [10]. Separation theorems in $\mathbb{B}^{-1}$-convexity were analysed in [24]. In this paper, some theorems about important properties of $\mathbb{B}^{-1}$-convex function which is defined in [3] are proven.

The structure of this article is as follows. In second section, necessary definitions and theorems about $\mathbb{B}^{-1}$-convex sets are reminded. In Section 3, $\mathbb{B}^{-1}$-convex and $\mathbb{B}^{-1}$-concave functions are expressed and theorems which give sufficient and necessary conditions to these functions are proven. Afterwards, we show some examples that help us to understand the relation between convex and $\mathbb{B}^{-1}$-convex functions, in fourth section. Finally, in Section 5, some important properties of $\mathbb{B}^{-1}$-convex functions are studied.

We will use the following notation: $\mathbb{Z}_{-}$ is the set of negative integers; $K = \mathbb{R}\backslash\{0\}$; $\mathbb{R}^n$ is the n-dimensional vector space; $\mathbb{R}^n_{++} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i > 0, i = 1, ..., n\}$; for $x, y \in \mathbb{R}^n$, $x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, ..., \min\{x_n, y_n\})$; $\text{epi}(f) = \{(x,\mu)|x \in U, \mu \in \mathbb{R}, \mu \geq f(x)\}$; $\text{hyp}(f) = \{(x,\mu)|x \in U, \mu \in \mathbb{R}, \mu \leq f(x)\}$; $\text{epi}^0(f) = \{(x,\mu)|x \in U, \mu \in K, \mu \geq f(x)\}$; $\text{hyp}^{++}(f) = \{(x,\mu)|x \in U, 0 < \mu \leq f(x)\}$.

2. $\mathbb{B}^{-1}$-convex Sets

For $r \in \mathbb{Z}_{-}$, where $\mathbb{Z}_{-}$ is the set of negative integers, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $K = \mathbb{R}\backslash\{0\}$ to itself; $x = (x_1, x_2, ..., x_n) \to \Phi_r(x) = (\varphi_r(x_1), \varphi_r(x_2), ..., \varphi_r(x_n))$ is homeomorphism from K^n to itself. For $k_1, k_2 \in K$ the indexed sum is

$$k_1 \overset{r}{+} k_2 = (k_1^{2r+1} + k_2^{2r+1})^{\frac{1}{2r+1}}$$

and, with the help of $\varphi_r(k_1k_2) = \varphi_r(k_1)\varphi_r(k_2)$, the indexed product is

$$k_1 \overset{r}{\cdot} k_2 = k_1k_2\,.$$

For $k_i \in K$ and $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in K^n, i = 1, 2, ..., m$ we have

$$\sum_{i=1,...,m}^{r} k_i \overset{r}{\cdot} x_i = \Phi_r^{-1}\left(\sum_{i=1}^{m} \varphi\left(k_i\right)\Phi_r(x_i)\right)$$

$$= \left(\left[\sum_{i=1}^{m}(k_ix_{i1})^{2r+1}\right]^{\frac{1}{2r+1}}, \left[\sum_{i=1}^{m}(k_ix_{i2})^{2r+1}\right]^{\frac{1}{2r+1}}, ..., \left[\sum_{i=1}^{m}(k_ix_{in})^{2r+1}\right]^{\frac{1}{2r+1}}\right).$$

For a finite nonempty set $A = \{x_1, x_2, ..., x_m\} \subset K^n$ the Φ_r-convex hull (shortly r-convex hull) of A, which we denote $\mathrm{Co}^r(A)$, is given by

$$\mathrm{Co}^r(A) = \left\{ \Phi_r^{-1}\left(\sum_{i=1}^{m} t_i \Phi_r(x_i)\right) : t_i \geq 0, \sum_{i=1}^{m} t_i = 1 \right\}.$$

Thus $\mathbb{B}^{-1}$-polytopes can be defined as follows:

Definition 2.1 ([2]). The Kuratowski-Painleve upper limit of the sequence of sets $\{\mathrm{Co}^r(A)\}_{r\in\mathbb{Z}^-}$, where A is finite, denoted by $\mathrm{Co}^{-\infty}(A)$ is called $\mathbb{B}^{-1}$-polytope of A; that is the set of point $x \in K^n$ for which there exist a decreasing sequence $\{r_k\}_{k\in\mathbb{N}}$ and points $x_{r_k} \in \mathrm{Co}^{r_k}(A)$ such that $x = \lim_{k\to\infty} x_{r_k}$.

In the following theorem, a definition which is equivalent to the Definition 2.1 in $\mathbb{R}^n_{++}$ is obtained.

Theorem 2.2 ([2]). *For all nonempty finite subsets $A = \{x_1, ..., x_m\} \subset \mathbb{R}^n_{++}$ we have*

$$\mathrm{Co}^{-\infty}(A) = \lim_{r\to-\infty} \mathrm{Co}^r(A) = \left\{ \mathop{\wedge}_{i=1}^{m} t_i x_i : t_i \geq 1, \min_{1\leq i\leq m} t_i = 1 \right\}.$$

Here, we denote by $\wedge_{i=1}^{m} x_i$ the greatest lower bound of $x_1, x_2, ..., x_m \in \mathbb{R}^n$, that is:

$$\mathop{\wedge}_{i=1}^{m} x_i = (\min\{x_{1,1}, ..., x_{m,1}\}, ..., \min\{x_{1,n}, ..., x_{m,n}\})$$

where $x_{i,j}$ denotes jth coordinate of the ith point.

Now, we can give the definition of $\mathbb{B}^{-1}$-convex sets.

Definition 2.3 ([2]). A subset U of K^n is called a $\mathbb{B}^{-1}$-convex set if for all finite subsets $A \subset U$ the $\mathbb{B}^{-1}$-polytope $\mathrm{Co}^{-\infty}(A)$ is contained in U.

By Theorem 2.2, for subsets of $\mathbb{R}^n_{++}$, the above definition can be expressed in the following form:

A subset U of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}$-convex if and only if, for all $x_1, ..., x_m \in U$ and all $t_1, ..., t_m \in [1, \infty)$ such that $\min\{t_1, ..., t_m\} = 1$ one has $\wedge_{i=1}^{m} t_i x_i \in U$.

The following theorem presents more simple definition of $\mathbb{B}^{-1}$-convex sets.

Theorem 2.4 ([2]). *A subset U of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}$-convex if and only if, for all $x_1, x_2 \in U$*

$$\mathrm{Co}^{-\infty}(\{x_1, x_2\}) \subset U$$

i.e., the subset U of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}$-convex if and only if, for all $x_1, x_2 \in U$ and all $t \in [1, \infty)$ one has $tx_1 \wedge x_2 \in U$.

3. $\mathbb{B}^{-1}$–convex Functions

Firstly, we recall the following definitions which will be required to define $\mathbb{B}^{-1}$-convex and $\mathbb{B}^{-1}$-concave functions.

Definition 3.1 ([19]). Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R} \bigcup \{\pm\infty\}$. The set

$$\{(x, \mu) \,|x \in U, \mu \in \mathbb{R}, \mu \geq f(x)\}$$

is called the epigraph of f and is denoted by epi(f).

Definition 3.2 ([20]). Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R} \bigcup \{\pm\infty\}$. The set

$$\{(x, \mu) \,|x \in U, \mu \in \mathbb{R}, \mu \leq f(x)\}$$

is called the hypograph of f and is denoted by hyp(f).

Now, we can give the definition of $\mathbb{B}^{-1}$-convex function.

Definition 3.3 ([13]). For $U \subset K^n$, a function $f : U \to K$ is called a $\mathbb{B}^{-1}$-convex function if $\mathrm{epi}^0(f) := \{(x, \mu) \,|x \in U, \mu \in K, \mu \geq f(x)\}$ is a $\mathbb{B}^{-1}$-convex set.

In $\mathbb{R}^n_{++}$, we can give the following fundamental theorem which provides a sufficient and necessary condition for $\mathbb{B}^{-1}$-convex functions [13].

Theorem 3.4. *Let $U \subset \mathbb{R}^n_{++}$ and $f : U \to \mathbb{R}_{++}$. The function f is $\mathbb{B}^{-1}$-convex if and only if the set U is $\mathbb{B}^{-1}$-convex and one has the inequality:*

$$f(\lambda x \wedge y) \leq \lambda f(x) \wedge f(y) \tag{1}$$

for all $x, y \in U$ and all $\lambda \in [1, +\infty)$.

Proof. Firstly, we assume that f is a $\mathbb{B}^{-1}$-convex function. In this case, from Definition 3.3 which gives the $\mathbb{B}^{-1}$-convexity of $\mathrm{epi}^0(f)$, for all $(x, \mu), (y, \nu) \in \mathrm{epi}^0(f)$ and all $\lambda \in [1, +\infty)$ we have $\lambda (x, \mu) \wedge (y, \nu) \in \mathrm{epi}^0(f)$. Thus, it is obvious that

$$(x, \mu) \in \mathrm{epi}^0(f) \Rightarrow \begin{cases} x \in U, & (2) \\ f(x) \leq \mu, & (3) \end{cases}$$

$$(y, \nu) \in \mathrm{epi}^0(f) \Rightarrow \begin{cases} y \in U, & (4) \\ f(y) \leq \nu, & (5) \end{cases}$$

$$(\lambda x \wedge y, \lambda\mu \wedge \nu) \in \mathrm{epi}^0(f) \Rightarrow \begin{cases} \lambda x \wedge y \in U, & (6) \\ f(\lambda x \wedge y) \leq \lambda\mu \wedge \nu\,. & (7) \end{cases}$$

We deduce that U is a $\mathbb{B}^{-1}$-convex set from (2), (4) and (6). Also, from (3), (5), (7), we obtain that

$$f(\lambda x \wedge y) \leq \lambda f(x) \wedge f(y)$$

for all $x, y \in U$ and all $\lambda \in [1, +\infty)$.

For the proof of sufficiency, let us assume that $U \subset \mathbb{R}^n_{++}$ is a $\mathbb{B}^{-1}$-convex set and the inequality (1) holds for the function $f : U \to \mathbb{R}_{++}$ and all $x, y \in U$, $\lambda \in [1, +\infty)$. We have to show that f is a $\mathbb{B}^{-1}$-convex function, namely, $\mathrm{epi}^0(f)$ is a $\mathbb{B}^{-1}$-convex set. For $(x, \mu), (y, \nu) \in \mathrm{epi}^0(f)$ it is clear that

$$(x, \mu) \in \mathrm{epi}^0(f) \Rightarrow \begin{cases} x \in U, & (8) \\ f(x) \le \mu, & (9) \end{cases}$$

$$(y, \nu) \in \mathrm{epi}^0(f) \Rightarrow \begin{cases} y \in U, & (10) \\ f(y) \le \nu\,. & (11) \end{cases}$$

If we consider (8), (10) and $\mathbb{B}^{-1}$-convexity of U, then we get

$$\lambda x \wedge y \in U \tag{12}$$

for all $\lambda \in [1, +\infty)$. Afterwards, using (9), (11), (12) and the inequality (1), we obtain $\lambda(x, \mu) \wedge (y, \nu) = (\lambda x \wedge y, \lambda\mu \wedge \nu) \in \mathrm{epi}^0(f)$ and this gives us that $\mathrm{epi}^0(f)$ is a $\mathbb{B}^{-1}$-convex set. □

Remark 3.5. For $c, d \in \mathbb{R}$, since $\min\{c, d\} = \frac{1}{2}[c + d - |c - d|]$, we have

$$\begin{aligned} \lambda x \wedge y = &\ (\min\{\lambda x_1, y_1\}, \min\{\lambda x_2, y_2\}, ..., \min\{\lambda x_n, y_n\}) \\ = &\left(\frac{1}{2}[\lambda x_1 + y_1 - |\lambda x_1 - y_1|], \frac{1}{2}[\lambda x_2 + y_2 - |\lambda x_2 - y_2|], \right. \\ &\left. ..., \frac{1}{2}[\lambda x_n + y_n - |\lambda x_n - y_n|]\right) \\ = &\sum_{i=1}^{n} \frac{1}{2}[\lambda x_i + y_i - |\lambda x_i - y_i|]\, \boldsymbol{e_i} \end{aligned}$$

where $\boldsymbol{e_i} = (0, ..., 0, \underbrace{1}_{i.}, 0, ..., 0) \in \mathbb{R}^n,\ i = 1, ..., n.$ In this case, we can express the inequality (1) in a following algebraic form:

$$f\left(\sum_{i=1}^{n} \frac{1}{2}[\lambda x_i + y_i - |\lambda x_i - y_i|]\, \boldsymbol{e_i}\right) \le \frac{1}{2}[\lambda f(x) + f(y) - |\lambda f(x) - f(y)|]\ .$$

Jensen Inequality for $\mathbb{B}^{-1}$-convex functions, provides another sufficient and necessary condition, can be proven as follows.

Theorem 3.6 (Jensen Inequality). *Let $f : U \subset \mathbb{R}^n_{++} \to \mathbb{R}_{++}$. f is a $\mathbb{B}^{-1}$-convex function if and only if $U \subset \mathbb{R}^n_{++}$ is a $\mathbb{B}^{-1}$-convex set and for all $x_1, ..., x_m \in U$, all $\lambda_1, ..., \lambda_m \in [1, +\infty)$ such that $\min_{i=1,...,m}\{\lambda_i\} = 1$ we have the inequality*

$$f\left(\bigwedge_{i=1}^{m} \lambda_i x_i\right) \le \bigwedge_{i=1}^{m} \lambda_i f(x_i)\ . \tag{13}$$

Proof. Let f be a $\mathbb{B}^{-1}$-convex function. With the help of Theorem 3.4, for all $x_1, ..., x_m \in U$ and all $\lambda_1, ..., \lambda_m \in [1, +\infty)$ such that $\min_{i=1,...,m}\{\lambda_i\} = 1$ we have $\lambda_1 x_1 \wedge \lambda_2 x_2 \wedge ... \wedge \lambda_m x_m \in U$. Without losing generality, we can take $\lambda_m = 1$. Hence, if we denote $y = \lambda_2 x_2 \wedge \lambda_3 x_3 \wedge ... \wedge \lambda_m x_m$, then we get $y \in U$. Again, from Theorem 3.4 the inequality

$$f(\lambda_1 x_1 \wedge y) \leq \lambda_1 f(x_1) \wedge f(y)$$

holds. Similarly, suppose that $z = \lambda_3 x_3 \wedge ... \wedge \lambda_m x_m$, then we obtain $z \in U$. If we consider $\mathbb{B}^{-1}$-convexity of f, then we conclude that

$$\begin{aligned} f\left(\bigwedge_{i=1}^{m} \lambda_i x_i\right) &\leq \lambda_1 f(x_1) \wedge f(y) = \lambda_1 f(x_1) \wedge f(\lambda_2 x_2 \wedge z) \\ &\leq \lambda_1 f(x_1) \wedge \lambda_2 f(x_2) \wedge f(z)\ . \end{aligned}$$

If we continue applying this process, then we deduce the inequality (13).

To prove the sufficiency, we suppose that the inequality (13) is provided and $m = 2$. In this case, because of $\lambda_1, \lambda_2 \in [1, +\infty)$ and $\min\{\lambda_1, \lambda_2\} = 1$, we can take $\lambda_2 = 1$. Thus, we derive the inequality

$$f(\lambda_1 x_1 \wedge x_2) \leq \lambda_1 f(x_1) \wedge f(x_2)$$

which means that f is a $\mathbb{B}^{-1}$-convex function. □

The following gives the definition of $\mathbb{B}^{-1}$-concave function in $\mathbb{R}^n_{++}$.

Definition 3.7. A function $f : U \subset \mathbb{R}^n_{++} \to \mathbb{R}_{++}$is called a $\mathbb{B}^{-1}$-concave function if the set $\text{hyp}^{++}(f) := \{(x, \mu) \,|x \in U, 0 < \mu \leq f(x)\}$ is $\mathbb{B}^{-1}$-convex.

In the next theorem, a sufficient and necessary condition is expressed for $\mathbb{B}^{-1}$-concave function.

Theorem 3.8. *Let $U \subset \mathbb{R}^n_{++}$ and $f : U \to \mathbb{R}_{++}$. The function f is $\mathbb{B}^{-1}$-concave if and only if the set U is $\mathbb{B}^{-1}$-convex and for all $x, y \in U$ and all $\lambda \in [1, +\infty)$ the following inequality holds:*

$$f(\lambda x \wedge y) \geq \lambda f(x) \wedge f(y)\ . \tag{14}$$

Proof. For the necessary part of the proof, we assume that f is a $\mathbb{B}^{-1}$-concave function. From the $\mathbb{B}^{-1}$-concavity of f, $\text{hyp}^{++}(f)$ is a $\mathbb{B}^{-1}$-convex set. In the other words; for all $(x, \mu), (y, \nu) \in \text{hyp}^{++}(f)$ and all $\lambda \in [1, +\infty)$, we have $\lambda(x, \mu) \wedge (y, \nu) \in \text{hyp}^{++}(f)$. Hence, it is clear that

$$(x, \mu) \in \text{hyp}^{++}(f) \Rightarrow \begin{cases} x \in U, & (15) \\ 0 < \mu \leq f(x), & (16) \end{cases}$$

$$(y, \nu) \in \text{hyp}^{++}(f) \Rightarrow \begin{cases} y \in U, & (17) \\ 0 < \nu \leq f(y), & (18) \end{cases}$$

$$(\lambda x \wedge y, \lambda\mu \wedge \nu) \in \mathrm{hyp}^{++}(f) \Rightarrow \begin{cases} \lambda x \wedge y \in U, & (19) \\ 0 < \lambda\mu \wedge \nu \leq f(\lambda x \wedge y) . & (20) \end{cases}$$

As we take (15), (17) and (19) into consideration, we reach the $\mathbb{B}^{-1}$-convexity of U. Then from (16), (18) and (20), for all $x, y \in U$ and all $\lambda \in [1, +\infty)$, we deduce that

$$f(\lambda x \wedge y) \geq \lambda f(x) \wedge f(y) .$$

For the sufficiency, let $U \subset \mathbb{R}^n_{++}$ be a $\mathbb{B}^{-1}$-convex set and the inequality (14) holds. To show $\mathbb{B}^{-1}$-concavity of f, we have to prove that $\mathrm{hyp}^{++}(f)$ is a $\mathbb{B}^{-1}$-convex set. Let $(x, \mu), (y, \nu) \in \mathrm{hyp}^{++}(f)$, namely

$$(x, \mu) \in \mathrm{hyp}^{++}(f) \Rightarrow \begin{cases} x \in U, & (21) \\ 0 < \mu \leq f(x) , & (22) \end{cases}$$

$$(y, \nu) \in \mathrm{hyp}^{++}(f) \Rightarrow \begin{cases} y \in U, & (23) \\ 0 < \nu \leq f(y) . & (24) \end{cases}$$

From (21), (23) and $\mathbb{B}^{-1}$-convexity of U, we obtain that

$$\lambda x \wedge y \in U, \quad \forall \lambda \in [1, +\infty) . \tag{25}$$

If we take (22), (24), (25) and the inequality (14) into consideration, then we have $\lambda(x, \mu) \wedge (y, \nu) = (\lambda x \wedge y, \lambda\mu \wedge \nu) \in \mathrm{hyp}^{++}(f)$. □

A function is called a $\mathbb{B}^{-1}$-affine function if the function is both $\mathbb{B}^{-1}$-convex and $\mathbb{B}^{-1}$-concave function.

From Theorem 3.4 and Theorem 3.8, the following equality can be easily shown for the $\mathbb{B}^{-1}$-affine function:

The function $f : U \subset \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is $\mathbb{B}^{-1}$-affine if and only if U is a $\mathbb{B}^{-1}$-convex set and we have the equality

$$f(\lambda x \wedge y) = \lambda f(x) \wedge f(y) \tag{26}$$

for all $x, y \in U$ and all $\lambda \in [1, +\infty)$.

The following theorem includes Jensen Inequality for $\mathbb{B}^{-1}$-concave functions and can be proven similarly to Theorem 3.6.

Theorem 3.9. *The function $f : U \subset \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is $\mathbb{B}^{-1}$-concave if and only if the set U is $\mathbb{B}^{-1}$-convex and the inequality*

$$f\left(\bigwedge_{i=1}^{m} \lambda_i x_i\right) \geq \bigwedge_{i=1}^{m} \lambda_i f(x_i) \tag{27}$$

holds for all $x_1, ..., x_m \in U$ and all $\lambda_1, ..., \lambda_m \in [1, +\infty)$ such that $\min_{i=1,...,m}\{\lambda_i\} = 1$.

4. Examples

In this section, we give some examples of $\mathbb{B}^{-1}$-convex functions.

Example 4.1. Let $f : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ be Cobb-Douglas function, namely $f(z_1, ..., z_n) = \prod_{i=1}^n z_i^{\alpha_i}$. For $\alpha_i \geq 0 (\forall i = 1, ..., n)$ such that $\sum_{i=1}^n \alpha_i \leq 1$, the function f is $\mathbb{B}^{-1}$-convex.

To show the $\mathbb{B}^{-1}$-convexity of f, we have to prove the inequality

$$f(\lambda x \wedge y) = \prod_{i=1}^n (\lambda x_i \wedge y_i)^{\alpha_i} \leq \lambda \left(\prod_{i=1}^n x_i^{\alpha_i} \right) \wedge \left(\prod_{i=1}^n y_i^{\alpha_i} \right) = \lambda f(x) \wedge f(y) \quad (28)$$

for all $x, y \in \mathbb{R}^n_{++}$ and all $\lambda \in [1, +\infty)$.

Let $I = \{1, ..., n\}$ and let us denote K as a subset of I which consist of the element $i \in I$ such that $\lambda x_i \wedge y_i = \lambda x_i$. Namely, we have

$$(\lambda x_i \wedge y_i)^{\alpha_i} = \begin{cases} (\lambda x_i)^{\alpha_i} & \text{if } i \in K, \\ (y_i)^{\alpha_i} & \text{if } i \in I \setminus K. \end{cases}$$

Hence, we obtain that the following two possibilities.

$$\begin{aligned} f(\lambda x \wedge y) = & \prod_{i=1}^n (\lambda x_i \wedge y_i)^{\alpha_i} = \prod_{i \in K} (\lambda x_i)^{\alpha_i} \prod_{i \in I \setminus K} (y_i)^{\alpha_i} \\ \leq & \prod_{i \in I} (\lambda x_i)^{\alpha_i} = \lambda^{\sum_{i \in I} \alpha_i} \prod_{i \in I} x_i^{\alpha_i} \leq \lambda \prod_{i \in I} x_i^{\alpha_i} = \lambda f(x) \end{aligned}$$

and

$$f(\lambda x \wedge y) = \prod_{i=1}^n (\lambda x_i \wedge y_i)^{\alpha_i} = \prod_{i \in K} (\lambda x_i)^{\alpha_i} \prod_{i \in I \setminus K} (y_i)^{\alpha_i} \leq \prod_{i \in I} (y_i)^{\alpha_i} = f(y) \ .$$

Finally, the inequality (28) holds, namely the function f is $\mathbb{B}^{-1}$-convex.

Example 4.2. The $\mathbb{B}^{-1}$-convexity of the following function can be easily proven.

For $0 < \rho \leq 1$, $0 < \alpha_i \leq 1, i = 1, ..., n$ and $a_i \geq 0, i = 1, ..., n$ (all a_i-s are not zero at the same time), $g : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$, $g(z_1, ..., z_n) = (\sum_{i=1}^n a_i z_i^{\alpha_i})^\rho$.

The $\mathbb{B}^{-1}$-convex functions f and g are examples for non-convex functions except the case when $\rho = 1$ and $\alpha_i = 1, i = 1, ..., n$ for g.

But, in the case of $\rho = 1$ and $\alpha_i = 1, i = 1, ..., n$ for the function g, it becomes a linear function, namely it is a convex function. Thus, it is example of function which is both $\mathbb{B}^{-1}$-convex and convex.

On the contrary, the following example shows that a convex function does not have to be $\mathbb{B}^{-1}$-convex.

Example 4.3. For $\rho \geq 1$, $\alpha_i \geq 1, i = 1, ..., n$ and $a_i \geq 0, i = 1, ..., n$ (all a_i-s are not zero at the same time), $g : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$, $g(z_1, ..., z_n) = \left(\sum_{i=1}^n a_i z_i^{\alpha_i}\right)^\rho$ is a convex function. But g is not a $\mathbb{B}^{-1}$-convex function when we take at least one $\alpha_i > 1$.

Indeed, we assume that any $\alpha_i > 1$. For $x = (1, 1, ..., 1), y = (2, 2, ..., 2) \in \mathbb{R}^n_{++}$, we get

$$\lambda x \wedge y = \begin{cases} \lambda x = \lambda(1, 1, ..., 1) & \text{if } \lambda \in [1, 2), \\ y = (2, 2, ..., 2) & \text{if } \lambda \in [2, +\infty). \end{cases}$$

If we examine the case of $\lambda \in (1, 2)$, then we have

$$\begin{aligned} g(\lambda x \wedge y) = g(\lambda x) &= (a_1\lambda^{\alpha_1} + a_2\lambda^{\alpha_2} + ... + a_n\lambda^{\alpha_n})^\rho \\ &> \left(\lambda(a_1 + a_2 + ... + a_n)\right)^\rho \\ &\geq \lambda(a_1 + a_2 + ... + a_n)^\rho = \lambda g(x) \end{aligned}$$

which means that the inequality (1) is not valid for g. Namely, the function g is not $\mathbb{B}^{-1}$-convex.

5. Some Properties of $\mathbb{B}^{-1}$-convex Functions

In this part of the paper, we give some important properties of the $\mathbb{B}^{-1}$-convex and $\mathbb{B}^{-1}$-concave functions.

We can express that supremum of the family of $\mathbb{B}^{-1}$-convex functions is also $\mathbb{B}^{-1}$-convex in the following theorem.

Theorem 5.1. *Let* $\left\{f_\alpha \,|\, f_\alpha : U \subset \mathbb{R}^n_{++} \to \mathbb{R}_{++}\right\}$ *be a family of* $\mathbb{B}^{-1}$*-convex functions. Then the function* f *defined by* $f(x) = \sup_\alpha \{f_\alpha(x)\}$ *is a* $\mathbb{B}^{-1}$*-convex function.*

Proof. Let $\{f_\alpha\}$ be a family of $\mathbb{B}^{-1}$-convex functions and $f(x) = \sup_\alpha \{f_\alpha(x)\}$. From the definition of the function f, we have $\mathrm{epi}^0(f) = \bigcap_\alpha \mathrm{epi}^0(f_\alpha)$. Since, the intersection of $\mathbb{B}^{-1}$-convex sets is $\mathbb{B}^{-1}$-convex from [4] then $\mathrm{epi}^0(f)$ is a $\mathbb{B}^{-1}$-convex set. Consequently, the function f is $\mathbb{B}^{-1}$-convex. □

The following theorem is about the level sets of a $\mathbb{B}^{-1}$-convex function.

Theorem 5.2. *For any* $\mathbb{B}^{-1}$*-convex function* $f : U \subset \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ *and any* $\alpha \in (0, +\infty]$*, the level sets* $\{x \,|\, f(x) < \alpha\}$ *and* $\{x \,|\, f(x) \leq \alpha\}$ *are* $\mathbb{B}^{-1}$*-convex.*

Proof. Let $f : U \subset \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ be a $\mathbb{B}^{-1}$-convex function. From Theorem 3.4, for all $x, y \in U$ and all $\lambda \in [1, +\infty)$ the following inequality holds:

$$f(\lambda x \wedge y) \leq \lambda f(x) \wedge f(y)\ .$$

If we get $x', y' \in \{x \,|\, f(x) < \alpha\}$, from the above inequality we have

$$f(\lambda x' \wedge y') \leq \lambda f(x') \wedge f(y')\ .$$

Additionally, since $f(x') < \alpha$ and $f(y') < \alpha$, we deduce that

$$f(\lambda x' \wedge y') < \lambda\alpha \wedge \alpha = \alpha\,. \tag{29}$$

The inequality (29) gives that $\lambda x' \wedge y' \in \{x \,|f(x) < \alpha\}$, namely the level set $\{x \,|f(x) < \alpha\}$ is $\mathbb{B}^{-1}$-convex.

It can be shown similarly that the level set $\{x \,|f(x) \leq \alpha\}$ is also a $\mathbb{B}^{-1}$-convex set. □

We now prove a theorem for positively homogeneous $\mathbb{B}^{-1}$-convex functions. Therefore, let us recall the definition of positively homogeneous function.

Definition 5.3. A function f on $\mathbb{R}^n$ is said to be positively homogeneous (of degree 1) if for every $x \in \mathbb{R}^n$ one has

$$f(\lambda x) = \lambda f(x)\,, \quad 0 < \lambda < +\infty\,.$$

Theorem 5.4. *A positively homogeneous function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is $\mathbb{B}^{-1}$-convex if and only if for every $x, y \in \mathbb{R}^n_{++}$, the following inequality holds:*

$$f(x \wedge y) \leq f(x) \wedge f(y)\ . \tag{30}$$

Proof. It is obvious that if f is a positively homogeneous $\mathbb{B}^{-1}$-convex function, then (30) is valid.

On the other side, when we take the inequality (30) and positive homogeneity of f into consideration, we have

$$f(\lambda x \wedge y) \leq f(\lambda x) \wedge f(y) = \lambda f(x) \wedge f(y)\ .$$

Namely, f is $\mathbb{B}^{-1}$-convex. □

Theorem 5.4 has a following consequence.

Corollary 5.5. *If $f : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is a positively homogeneous $\mathbb{B}^{-1}$-convex function, then*

$$f(\lambda_1 x_1 \wedge \lambda_2 x_2 \wedge \ldots \wedge \lambda_m x_m) \leq \lambda_1 f(x_1) \wedge \lambda_2 f(x_2) \wedge \ldots \wedge \lambda_m f(x_m)\ . \tag{31}$$

whenever $\lambda_1 > 0, \ldots, \lambda_m > 0$.

Proof. If $f : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is positively homogeneous $\mathbb{B}^{-1}$-convex function, then the inequality (30) is valid. For $\lambda_i > 0$ and $x_i \in \mathbb{R}^n_{++}$, $i = 1, \ldots, m$, we have both $\lambda_i x_i \in \mathbb{R}^n_{++} (\forall i = 1, \ldots, m)$ and $\lambda_2 x_2 \wedge \ldots \wedge \lambda_m x_m \in \mathbb{R}^n_{++}$. With the above assumptions, we obtain that

$$\begin{aligned}
f(\lambda_1 x_1 \wedge \lambda_2 x_2 \wedge \ldots \wedge \lambda_m x_m) &\leq f(\lambda_1 x_1) \wedge f(\lambda_2 x_2 \wedge \ldots \wedge \lambda_m x_m) \\
&= \lambda_1 f(x_1) \wedge f(\lambda_2 x_2 \wedge \lambda_3 x_3 \wedge \ldots \wedge \lambda_m x_m) \\
&\leq \lambda_1 f(x_1) \wedge f(\lambda_2 x_2) \wedge f(\lambda_3 x_3 \wedge \ldots \wedge \lambda_m x_m) \\
&\ \ \vdots \\
&\leq \lambda_1 f(x_1) \wedge \lambda_2 f(x_2) \wedge \ldots \wedge \lambda_m f(x_m)\ .
\end{aligned}$$

□

Theorem 5.6 (The operations with $\mathbb{B}^{-1}$-convex functions).

(i) *Let $U \subset \mathbb{R}^n_{++}$, $f : U \to \mathbb{R}_{++}$ be a $\mathbb{B}^{-1}$-convex function. Multiplying f by a positive scalar, we obtain also a $\mathbb{B}^{-1}$-convex function.*

(ii) *Let $f : U \subset \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ be a $\mathbb{B}^{-1}$-convex function. The restriction of f to a $\mathbb{B}^{-1}$-convex subset of U is also a $\mathbb{B}^{-1}$-convex function.*

(iii) *If $f : U \subset \mathbb{R}^n_{++} \to V \subset \mathbb{R}_{++}$ is a $\mathbb{B}^{-1}$-convex function and $g : V \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a non-decreasing $\mathbb{B}^{-1}$-convex function, then $g \circ f$ is $\mathbb{B}^{-1}$-convex.*

(iv) *Suppose that $f : U \subset \mathbb{R}_{++} \to V \subset \mathbb{R}_{++}$ is a bijection. If f is increasing, then f is $\mathbb{B}^{-1}$-convex if and only if f^{-1} is $\mathbb{B}^{-1}$-concave. If f is a decreasing bijection, then f and f^{-1} are of the same type of $\mathbb{B}^{-1}$-convexity.*

Proof. (i) We assume that f is a $\mathbb{B}^{-1}$-convex function. From Theorem 3.4, the inequality (1) holds. For any $c > 0$,

$$cf(\lambda x \wedge y) = c[f(\lambda x \wedge y)] \le c[\lambda f(x) \wedge f(y)] = \lambda (cf)(x) \wedge (cf)(y)$$

which means that multiplying f by a positive scalar is a $\mathbb{B}^{-1}$-convex function.

(ii) For a $\mathbb{B}^{-1}$-convex function f, the inequality (1) is valid for all $x, y \in U$ and all $\lambda \in [1, +\infty)$. Since, (1) holds on every $\mathbb{B}^{-1}$-convex subset A of U, the restriction of f to A is a $\mathbb{B}^{-1}$-convex function.

(iii) Let $f : U \subset \mathbb{R}^n_{++} \to V \subset \mathbb{R}_{++}$ be $\mathbb{B}^{-1}$-convex. In this case, for all $x, y \in U$ and all $\lambda \in [1, +\infty)$ we have

$$f(\lambda x \wedge y) \le \lambda f(x) \wedge f(y) \ . \tag{32}$$

Let $g : V \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ be non-decreasing $\mathbb{B}^{-1}$-convex function. Namely, for all $s, t \in V$ and all $\mu \in [1, +\infty)$ one has the inequality

$$g(\mu s \wedge t) \le \mu g(s) \wedge g(t) \ . \tag{33}$$

Thus, from (32), (33) and non-decreasing of g, we obtain that

$$(g \circ f)(\lambda x \wedge y) \le g(\lambda f(x) \wedge f(y)) \le \lambda (g \circ f)(x) \wedge (g \circ f)(y) \ .$$

(iv) We assume that bijection f is increasing and $\mathbb{B}^{-1}$-convex. From $\mathbb{B}^{-1}$-convexity of f, for all $x, y \in U$ and all $\lambda \in [1, +\infty)$, the inequality (1) holds. Let $I_V\ (I_U)$ be the identity function on $V\ (U)$. Since $f \circ f^{-1} = I_V$ and $f^{-1} \circ f = I_U$, for all $s, t \in V$ and all $\mu \in [1, +\infty)$, we derive

$$\begin{aligned} f\left(f^{-1}(\mu s \wedge t)\right) &= \mu s \wedge t = \mu f\left(f^{-1}(s)\right) \wedge f\left(f^{-1}(t)\right) \\ &\ge f\left(\mu f^{-1}(s) \wedge f^{-1}(t)\right) \ . \end{aligned}$$

When we use the above inequality and increasing of f^{-1} which is obtained from increasing of f, we have

$$f^{-1}(\mu s \wedge t) \ge \mu f^{-1}(s) \wedge f^{-1}(t) \ .$$

Consequently, f^{-1} is $\mathbb{B}^{-1}$-concave.

Now, let f^{-1} be $\mathbb{B}^{-1}$-concave. Namely, for all $s, t \in V$ and all $\mu \in [1, +\infty)$, the inequality (14) is provided. If we take $f^{-1}(s) = x$, $f^{-1}(t) = y$, then it is obvious that $f(x) = s$, $f(y) = t$. From (14) and increasing of f, we obtain that

$$\mu f(x) \wedge f(y) \geq f(\mu x \wedge y) .$$

Finally, we have proven that f is a $\mathbb{B}^{-1}$-convex function.

Similarly, we can prove the other hypothesis in (iv). □

Remark 5.7. As in classic convexity, product of two increasing (decreasing) $\mathbb{B}$-convex functions is also a $\mathbb{B}$-convex function([3]). However, it is not valid for $\mathbb{B}^{-1}$-convexity. For instance, let $f : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$, $f(z_1, ..., z_n) = \left(\sum_{i=1}^n z_i\right)^{\frac{1}{2}}$ and $g : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$, $g(z_1, ..., z_n) = \sum_{i=1}^n z_i$. It is obvious that both of the functions are increasing (with respect to the coordinate-wise order relation) and from Example 4.2, it can be seen that the functions f and g are $\mathbb{B}^{-1}$-convex functions. But, by using the same way in Example 4.3, we can show that product of these two functions is not $\mathbb{B}^{-1}$-convex.

Acknowledgements. The authors are very thankful to editor and the anonymous referees for their constructive comments on this article. This work was supported by Akdeniz University, Mersin University and TUBITAK (The Scientific and Technological Research Council of Turkey).

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