Journal of Convex Analysis
Volume 24 (2017), No. 2, 519–524

On Gauge Functions for Convex Cones with Possibly Empty Interiors

Sorin-Mihai Grad*
Faculty of Mathematics,
Chemnitz University of Technology, 09107 Chemnitz, Germany
grad@mathematik.tu-chemnitz.de

Received: October 12, 2014
Revised manuscript received: December 14, 2015
Accepted: December 18, 2015

We extend for cones with possibly empty interior the analysis of the gauge functions for convex cones presented in a recent note due to Svaiter.

Keywords: Cones, gauge functions, quasi(-relative) interior, sublinear functions

1991 Mathematics Subject Classification: 90C29, 46N10, 42A50

1. Introduction and preliminaries

In the recent note [11] Svaiter has analyzed a class of sublinear functions which characterize the interior and the complement of a closed convex cone in a normed linear space. In some papers due to him and authors like Fliege, Graña-Drummond and Iusem (see for instance [8]), such functions play an important role in developing algorithms for solving vector optimization problems and are called *gauge functions* for the corresponding ordering cones. The scope of this note is to show that the main result in [11] can be extended for cones with empty interiors which have nonempty quasi-relative interiors. This may prove to be useful for constructing algorithms for solving vector optimization problems originating from mathematical finance, like the ones mentioned in [1, 12], where the ordering spaces have empty interiors, but nonempty quasi(-relative) interiors. Among the recent contributions to vector optimization dealing with such ordering cones we mention also [4, 5, 6, 7] and some references therein.

Consider the normed space $(X, \|\cdot\|)$ and its topological *dual space* $(X^*, \|\cdot\|_*)$. The dual space of the latter is $(X^{**}\|\cdot\|_{**})$, the *bidual space* of X. Denote by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. X can be identified with a subspace of X^{**}, and we denote by $\hat{x}$ the *canonical image* in X^{**} of the element $x \in X$ and $\widehat{U} = \{\hat{x} : x \in U\} \subseteq X^{**}$, where $U \subseteq X$.

*Research supported by DFG (German Research Foundation), project WA 922/8-1.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Let the nonempty convex cone $K \subseteq X$ and its *dual cone* $K^* = \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \; \forall x \in K\}$ and we denote $K^0 = \{x \in K : \langle x^*, x\rangle > 0 \; \forall x^* \in K^* \setminus \{0\}\}$. Given a subset U of X, by $\operatorname{int} U$, $\operatorname{core} U$, $\operatorname{cl} U$, $\operatorname{co} U$, $\operatorname{cone} U$, δ_U and σ_U we denote its *interior*, *algebraic interior*, *closure*, *convex hull*, *conical hull*, *indicator function* and *support function*, respectively. When U is convex, its *quasi-relative interior* is $\operatorname{qri} U = \{x \in U : \operatorname{cl}(\operatorname{cone}(U - x)) \text{ is a linear subspace of } X\}$, while its *quasi interior* is the set $\operatorname{qi} U = \big\{u \in U : \operatorname{cl}(\operatorname{cone}(U - u)) = X\big\}$. One has $\operatorname{int} U \subseteq \operatorname{core} U \subseteq \operatorname{qi} U \subseteq \operatorname{qri} U$ and when one of the sets in this chain of inclusions is nonempty it coincides with its mentioned supersets. When X is an Euclidean space one has $\operatorname{qi} U = \operatorname{int} U$ and $\operatorname{qri} U$ collapses to the relative interior of U. A typical example supporting the introduction (by Schaefer, and Borwein and Lewis, respectively) of the quasi(-relative) interior is a cone of nonnegative real sequences $\ell^p_+ \subseteq \ell^p$ with $1 \leq p < +\infty$ for which $\operatorname{int} \ell^p_+ = \operatorname{core} \ell^p_+ = \emptyset$, but $\operatorname{qi} \ell^p_+ = \operatorname{qri} \ell^p_+ = \big\{(x_n)_{n\in\mathbb{N}} \in \ell^p : x_n > 0 \; \forall n \in \mathbb{N}\big\}$. For more properties of the quasi(-relative) interior the reader can consult [2, 14] and some references therein. The largest lower semicontinuous function everywhere less than or equal to $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the *lower semicontinuous hull* of f, $\operatorname{cl} f : X \to \overline{\mathbb{R}}$, while the *conjugate function* of f is $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$ and its *subdifferential* at $x \in X$ where $f(x) \in \mathbb{R}$ is $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \; \forall y \in X\}$.

In the literature one can find different so-called *gauge* functions, see for instance the one from see also [10, Section 15] or the Minkowski functional attached to a set. In our presentation we use the one considered in [8, 11] and some of the references therein, namely we say that a function $\varphi : X \to \mathbb{R}$ which coincides at each $x \in X$ with $\sigma_C(\hat{x})$, where $C \subseteq K^* \setminus \{0\}$ is a weak*-compact set fulfilling $K^* = \operatorname{cl}(\operatorname{co}(\operatorname{cone} C))$, is a *gauge function for the convex cone* K. In [8] the set C is said to be a *subbase* of K^* and one can note that a base of a cone is a subbase of it, too. Moreover, the intersection of K^* with the boundary of the corresponding closed unit ball is always a subbase of K^*.

We begin with some statements which will be useful in our presentation. The first one can be derived via [3, Proposition 2.1.1] and [5, Proposition 1.4] (see also [6, Proposition 1]).

Lemma 1.1.

(a) *One always has* $\operatorname{qi} K \subseteq K^0 \subseteq \operatorname{qri} K$.
(b) *If* $K^0 \neq \emptyset$, *then* $\operatorname{qri} K \subseteq K^0$.
(c) *If* K *is closed, then* $\operatorname{qi} K = K^0$.

Lemma 1.2. *If* $U \subseteq X$ *is a convex set, then* $\widehat{\operatorname{qri} U} = \operatorname{qri} \widehat{U}$ *and, when* $\operatorname{qi} U \neq \emptyset$, *it holds* $\widehat{\operatorname{qi} U} = \operatorname{qri} \widehat{U}$.

Proof. When $x \in \operatorname{qri} U$, then $x \in U$ hence $\hat{x} \in \widehat{U}$, $\hat{x} \in \widehat{\operatorname{qri} U}$ and $\operatorname{cl}(\operatorname{cone}(U - x))$ is a subspace of X. Then $\operatorname{cl} \widehat{\operatorname{cone}(U - x)} = \operatorname{cl}(\operatorname{cone} \widehat{U - x}) = \operatorname{cl}(\operatorname{cone}(\widehat{U} - \hat{x}))$ is a subspace of X^{**}, consequently $\hat{x} \in \operatorname{qri} \widehat{U}$.

Viceversa, when $w \in \operatorname{qri} \widehat{U}$, it follows that $w \in \widehat{U}$, thus there exists an $x \in U$ such that $w = \hat{x}$. Moreover, $\operatorname{cl}(\operatorname{cone}(\widehat{U} - \hat{x}))$ is a subspace of X^{**} that coincides with $\operatorname{cl}(\widehat{\operatorname{cone}(U - x)})$ and is included in $\widehat{X}$. Therefore, $\operatorname{cl}(\operatorname{cone}(U - x))$ is a subspace of X and since $x \in U$ it follows that $x \in \operatorname{qri} U$, hence $\hat{x} \in \widehat{\operatorname{qri} U}$.

The last implication is a consequence of the fact that $\operatorname{qi} U \neq \emptyset$ yields $\operatorname{qi} U = \operatorname{qri} U$. $\square$

Remark 1.3. If $U \subseteq X$ is a convex set and $x \in \operatorname{qi} U$, one can show like in the proof of Lemma 1.2 that $\operatorname{cl}(\operatorname{cone}(\widehat{U} - \hat{x})) = \widehat{X}$. However, this yields $\widehat{\operatorname{qi} U} = \operatorname{qi} \widehat{U}$ only when $\widehat{X} = X^{**}$, i.e. X is reflexive.

2. Characterizations of gauge functions

In [11] the class of the gauge functions for K was equivalently characterized within the one of the sublinear functions. However, since some hypotheses are not necessary for both implications, we begin characterizing the gauge functions for K by analyzing their properties. Note also that some steps of our proofs are inspired from their correspondents from [11], but we do not skip them here for the sake of completeness.

Theorem 2.1. *If $K^0 \neq \emptyset$ and $\varphi : X \to \mathbb{R}$ is a gauge function for K, then φ is continuous, sublinear and takes negative values over $-\operatorname{qri} K$ and only positive values in the complement of $-\operatorname{cl} K$.*

Proof. The continuity and sublinearity of φ follow from the definition. From Lemma 1.1 and Lemma 1.2 one obtains $\langle x^*, w \rangle < 0$ for all $x^* \in K^* \setminus \{0\}$ and all $w \in -\operatorname{qri} \widehat{K}$. Because the set $C \subseteq K^* \setminus \{0\}$ that exists due to the definition of φ is weak*-compact, it follows that $\sigma_C(w) < 0$ for all $w \in \widehat{-\operatorname{qri} K}$, hence $\varphi(x) < 0$ whenever $x \in -\operatorname{qri} K$.

Take an $x \in X \setminus (-\operatorname{cl} K)$. Then there exists an $x^* \in K^*$ such that $\langle x^*, x \rangle \geq 0$, thus $K^* = \operatorname{cl}(\operatorname{co}(\operatorname{cone} C))$ yields $\varphi(x) \geq 0$. For an $\bar{x} \in -\operatorname{qri} K$ one can find some $t \in]0, 1[$ such that $tx + (1-t)\bar{x} \notin -\operatorname{cl} K$. Then $0 \leq \varphi(tx + (1-t)\bar{x}) \leq t\varphi(x) + (1-t)\varphi(\bar{x})$, which if $\varphi(x) = 0$ yields the contradiction $0 < 0$. Consequently $\varphi(x) > 0$ for all $x \in X \setminus (-\operatorname{cl} K)$. $\square$

A consequence of this result for the case when $\operatorname{qi} K \neq \emptyset$ follows.

Corollary 2.2. *If K is also closed with $\operatorname{qi} K \neq \emptyset$ and $\varphi : X \to \mathbb{R}$ is a gauge function for it, then φ is continuous, sublinear and takes negative values over $-\operatorname{qi} K$ and only positive values in the complement of $-\operatorname{cl} K$.*

Now let us see how can one obtain a gauge for K from an extended real valued function with suitable properties.

Theorem 2.3. *When $K^0 \neq \emptyset$ and $\varphi : X \to \overline{\mathbb{R}}$ is a proper sublinear function that is continuous at* 0*, then, if* $\operatorname{cl}\varphi(x) < 0$ *if and only if* $x \in -\operatorname{qri} K$, $\operatorname{cl}\varphi$ *is a gauge function for* K.

Proof. Because φ is sublinear, via [13, Theorem 2.4.14] it follows that $\varphi(0) = 0$, $\varphi^* = \delta_C$, where $C = \partial\varphi(0)$, and $\operatorname{cl}\varphi(x) = \sigma_C(\hat{x})$ for all $x \in X$. On the other hand, [13, Theorem 2.4.9] yields that C is nonempty and weak*-compact.

Since $\operatorname{cl}\varphi$ is convex, via [13, Exercise 2.5] one obtains that $\{x \in X : \operatorname{cl}\varphi(x) \leq 0\} = \operatorname{cl}\{x \in X : \operatorname{cl}\varphi(x) < 0\} = \operatorname{cl}(-\operatorname{qri} K) = -\operatorname{cl} K$, where the last equality follows via [2, Proposition 20.3]. Thus $\operatorname{cl}\varphi(x) \leq 0$ for all $x \in -K$, hence $\sigma_C(\hat{x}) \leq 0$ whenever $x \in -K$, consequently $C \subseteq K^*$. Moreover, for any $x \in -\operatorname{qri} K$ it holds $\operatorname{cl}\varphi(x) = \sigma_C(\hat{x}) < 0$, that yields $0 \notin C$.

Denoting $P = \operatorname{cone} C$, it follows that $P \subseteq K^*$. If $P \neq K^*$ there exists a $w^* \in K^* \setminus P$, hence $C \cap \operatorname{cone}\{w^*\} = \emptyset$ and Tuckey's separation statement yields the existence of an $\bar{x} \in X \setminus \{0\}$ such that $\sup_{z^* \in C}\langle z^*, \bar{x}\rangle < \inf_{t \geq 0} t\langle w^*, \bar{x}\rangle$. Consequently, $\langle w^*, \bar{x}\rangle \geq 0$ and $\sigma_C(\hat{\bar{x}}) < 0$. Then the hypothesis yields $\bar{x} \in -\operatorname{qri} K = -K^0$, followed by $\langle w^*, \bar{x}\rangle < 0$, which is a contradiction, hence $P = K^*$, therefore $K^* = \operatorname{cl}(\operatorname{co}(\operatorname{cone} C))$, i.e. $\operatorname{cl}\varphi$ is a gauge for K. □

Some consequences of this result follow, but before listing them we include a small observation.

Remark 2.4. A byproduct of the proof of Theorem 2.3 is that one can see that $\operatorname{cl}\varphi$ takes only positive values in the complement of $-\operatorname{cl} K$ even before knowing that it is a gauge for K, since $\{x \in X : \operatorname{cl}\varphi(x) \leq 0\} = -\operatorname{cl} K$.

Corollary 2.5. *If $K^0 \neq \emptyset$ and $\varphi : X \to \overline{\mathbb{R}}$ is a proper sublinear function that is lower semicontinuous at* 0 *and with $\partial f(0)$ weak*-compact, and one also has* $\operatorname{cl}\varphi(x) < 0$ *if and only if* $x \in -\operatorname{qri} K$*, then* $\operatorname{cl}\varphi$ *is a gauge function for* K.

Corollary 2.6. *If $K^0 \neq \emptyset$ and $\varphi : X \to \overline{\mathbb{R}}$ is a proper sublinear lower semicontinuous function that is continuous at* 0 *and satisfies* $\varphi(x) < 0$ *if and only if* $x \in -\operatorname{qri} K$*, then* φ *is a gauge function for* K.

Corollary 2.7. *If* $\operatorname{qi} K \neq \emptyset$ *and $\varphi : X \to \overline{\mathbb{R}}$ is a proper sublinear function that is continuous at* 0 *and with* $\operatorname{cl}\varphi(x) < 0$ *if and only if* $x \in -\operatorname{qi} K$*, then* $\operatorname{cl}\varphi$ *is a gauge function for* K.

Remark 2.8. In Theorem 2.1 and Theorem 2.3 and their consequences we generalize the statements from [11, Theorem 3], which are recovered as special cases when K is closed and pointed and $\operatorname{int} K \neq \emptyset$, using the fact that for a continuous convex function $f : X \to \mathbb{R}$ whose restriction to $-\operatorname{int} K$ takes only negative values one obtains via [13, Exercise 2.4] that $f(x) < 0$ if and only if $x \in -\operatorname{int} K$.

In the following we present a concrete situation where [11, Theorem 3] cannot be applied, unlike our results, justifying thus our generalizations.

Example 2.9. Take $X = c_0(\mathbb{N})$, the space of null sequences, and $K = c_0^+ = \{x \in c_0 : x_n \geq 0\ \forall n \geq 0\}$. Then $\operatorname{qi} K = \{x \in c_0 : x_n > 0\ \forall n \in \mathbb{N}\}$, but $\operatorname{int} K = \emptyset$, $X^* = \ell^1$ and $K^* = \ell^1_+ = \{x \in \ell^1 : x_n \geq 0\ \forall n \in \mathbb{N}\}$. Using the element $u \in \ell^\infty$ given by $u_n = 1$ for all $n \in \mathbb{N}$ one can construct (cf. [9, Example 1]) a closed and bounded base $B = \{x \in \ell^1_+ : \langle u, x\rangle = 1\}$ for ℓ^1_+ that does not contain the origin and lies within the corresponding weak*-compact closed unit ball, so it is weak*-compact, too. Then the restriction to $\widehat{c_0}$ of σ_B delivers a gauge function for the cone c_0^+.

Remark 2.10. In [11] it is noted that in some cases the set C which generates K^* and defines the gauge function for K needs not necessarily be weak*-compact, even if $\operatorname{int} K \neq \emptyset$. Taking a look at the proofs of the main statements in this paper, one can note that in the one of Theorem 2.1 the weak*-compactness of C plays a key role. However, in the demonstration of Theorem 2.3 it is actually needed only for applying Tuckey's statement. A natural question that can be raised is whether it is possible to replace this result with another separation theorem, maybe one using the fact that the quasi(-relative) interior of K is nonempty, like [5, Theorem 3.2] or the ones listed in [2, Section 20] or in [14]. In such a case it would suffice in Theorem 2.3 to ask φ to be only lower semicontinuous at 0 and not continuous.

Remark 2.11. Our investigations can be extended to more general frameworks, for instance by taking X to be a Hausdorff locally convex space and, in order to avoid unnecessary complications, its dual endowed with the corresponding weak* topology.

References

[1] C. D. Aliprantis, M. Florenzano, V. F. Martins-da-Rocha, R. Tourky: Equilibrium analysis in financial markets with countably many securities, J. Math. Econ. 40(6) (2004) 683–699.

[2] R. I. Boţ: Conjugate Duality in Convex Optimization, Lecture Notes in Economics and Mathematical Systems 637, Springer, Berlin (2010).

[3] R.I. Boţ, S.-M. Grad, G. Wanka: Duality in Vector Optimization, Springer, Berlin (2009).

[4] F. Flores-Bazán, F. Flores-Bazán, S. Laengle: Characterizing efficiency on infinite-dimensional commodity spaces with ordering cones having possibly empty interior, J. Optim. Theory Appl. 164(2) (2015) 455–478.

[5] S.-M. Grad: Vector Optimization and Monotone Operators via Convex Duality, Springer, Berlin (2015).

[6] S.-M. Grad, E. L. Pop: Characterizing relative minimal elements via linear scalarization, in: Operations Research Proceedings 2013 (Rotterdam, 2013), D. Huisman et al. (ed.), Springer, Cham (2014) 153–159.

[7] S.-M. Grad, E. L. Pop: Vector duality for convex vector optimization problems by means of the quasi-interior of the ordering cone, Optimization 63(1) (2014) 21–37.

[8] L. M. Graña Drummond, F. M. P. Raupp, B. F. Svaiter: A quadratically convergent Newton method for vector optimization, Optimization 63(5) (2014) 661–677.

[9] I. A. Polyrakis: Demand functions and reflexivity, J. Math. Anal. Appl. 338(1) (2008) 695–704.

[10] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[11] B. F. Svaiter: Gauge functions for convex cones, J. Convex Analysis 21(3) (2014) 851–856.

[12] W. R. Zame: Competitive equilibria in production economies with an infinite-dimensional commodity space, Econometrica 55(5) (1987) 1075–1108.

[13] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).

[14] C. Zălinescu: On the use of the quasi-relative interior in optimization, Optimization 64(8) (2015) 1795–1823.