

On Topological Properties of the Weak Topology of a Banach Space

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Received: February 11, 2015

Accepted: December 12, 2015

Being motivated by the famous Kaplansky theorem we study various sequential properties of a Banach space E and its closed unit ball B , both endowed with the weak topology of E . We show that B has the Pytkeev property if and only if E in the norm topology contains no isomorphic copy of ℓ_1 , while E has the Pytkeev property if and only if it is finite-dimensional. We extend Schlüchtermann and Wheeler's result from [27] by showing that B is a (separable) metrizable space if and only if it has countable cs^* -character and is a k -space. As a corollary we obtain that B is Polish if and only if it has countable cs^* -character and is Čech-complete, that supplements a result of Edgar and Wheeler [8].

Keywords: Weak topology, Banach space, $\aleph$ -space, k -space, cs^* -character

2000 Mathematics Subject Classification: Primary 46A03, 54E18; Secondary 54C35, 54E20

*The first named author was partially supported by ISF grant 441/11.

†The second named author was supported by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2013/058. Part of this paper has been prepared when he visited the Kurt Gödel Research Center for Mathematical Logic at the University of Vienna in October 2014.

‡The third-listed author was supported by the FWF grant I 1209-N25 as well as by the Austrian Academy of Sciences through the APART Program.

1. Introduction

Topological properties of a locally convex space (lcs) E endowed with the weak topology $\sigma(E, E')$, denoted by E_w for short, are of great importance and have been intensively studied from many years (see [18, 23] and references therein). Corson [7] started a systematic study of certain topological properties of the weak topology of Banach spaces. It is well known that a Banach space E in the weak topology is metrizable if and only if E is finite-dimensional. On the other hand, various topological properties generalizing metrizability have been studied intensively by topologists and analysts. Among the others let us mention the first countability, Fréchet–Urysohn property, sequentiality, k -space property, and countable tightness (see [9, 18]). It is well known that

$$\text{metric} \implies \text{first countably} \implies \text{Fréchet–Urysohn} \implies \text{sequential} \implies \begin{cases} k\text{-space} \\ \text{countably tight,} \end{cases}$$

and none of these implications can be reversed. Recall the following classical

Theorem 1.1 (Kaplansky). *If E is a metrizable lcs then E_w has countable tightness.*

The Kaplansky theorem can be strengthened by using other sequential concepts which are of great importance for the study of function spaces (see [2, 19]).

Following Arhangel'skii [2, II.2], we say that a topological space X has *countable fan tightness at a point* $x \in X$ if for each sets $A_n \subset X$, $n \in \mathbb{N}$, with $x \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$ there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, such that $x \in \overline{\bigcup_{n \in \mathbb{N}} F_n}$; X has *countable fan tightness* if X has countable fan tightness at each point $x \in X$. Clearly, if X has countable fan tightness, then X also has countable tightness.

Pytkeev [25] proved that every sequential space satisfies the following property, now known as the Pytkeev property, which is stronger than having countable tightness: A topological space X has the *Pytkeev property* if for any sets $A \subset X$ and each $x \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \dots$ of A such that each neighborhood of x contains some A_n .

A topological space X has the *Reznichenko property* (or is a *weakly Fréchet–Urysohn space*) if $x \in \overline{A} \setminus A$ and $A \subset X$ imply the existence of a countable infinite disjoint family $\mathcal{N}$ of finite subsets of A such that for every neighborhood U of x the family $\{N \in \mathcal{N} : N \cap U = \emptyset\}$ is finite. It is known that

$$\text{sequential} \implies \text{Pytkeev} \implies \text{Reznichenko} \implies \begin{cases} \text{countably} \\ \text{tight,} \end{cases}$$

and none of these implications is reversible (see [20, 25]).

For a Tychonoff topological space X we denote by $C_c(X)$ and $C_p(X)$ the space of all continuous real-valued functions on X endowed with the compact-open

topology and the topology of pointwise convergence, respectively. If X is a σ -compact space, then $C_p(X)$ has countable fan tightness by [2, II.2.2] and has the Reznichenko property by [19, Theorem 19]. If E is a metrizable lcs, then $X := (E', \sigma(E', E))$ is σ -compact by the Alaoglu–Bourbaki theorem. Since E_w embeds into $C_p(X)$, we notice the following generalization of the Kaplansky theorem.

Theorem 1.2. *Let E be a metrizable lcs (in particular, a Banach space). Then E_w has countable fan tightness and the Reznichenko property.*

On the other hand, infinite dimensional Banach spaces in the weak topology are never k -spaces.

Theorem 1.3 ([27]). *If E is a Banach space, then E_w is a k -space (in particular, sequential) if and only if E is finite dimensional.*

We prove another result of this type having in mind that the concepts of being a k -space and having the Pytkeev property are independent in general.

Theorem 1.4. *If E is a normed space, then E_w has the Pytkeev property if and only if E is finite dimensional.*

These results show that the question when a Banach space endowed with the weak topology is homeomorphic to a certain fixed model space from the infinite-dimensional topology is very restrictive and motivated specialists to detect the aforementioned properties only for some natural classes of subsets of E , e.g., balls or bounded subsets of E . By B_w we denote the closed unit ball B of a Banach space E endowed with the weak topology. It is well known that B_w is (separable) metrizable if and only if the dual space E' is norm separable. Schlüchtermann and Wheeler obtained in [27, Theorem 5.1] the following characterization.

Theorem 1.5 ([27]). *The following conditions on a Banach space E are equivalent: (a) B_w is Fréchet–Urysohn; (b) B_w is sequential; (c) B_w is a k -space; (d) E contains no isomorphic copy of ℓ_1 .*

By Theorem 1.2, for every Banach space E , B_w has the Reznichenko property. We prove the following result which supplements Theorem 1.5.

Theorem 1.6. *For a Banach space E , B_w has the Pytkeev property if and only if E contains no isomorphic copy of ℓ_1 .*

So, if E is the James Tree space, then B_w is Fréchet–Urysohn but is not metrizable. On the other hand, B_w has countable (fan) tightness for every Banach space E . This fact and Theorems 1.5 and 1.6 motivate us to consider another natural generalizations of metrizability.

One of the most immediate extensions of the class of separable metrizable spaces is the class of $\aleph_0$ -spaces introduced by Michael in [22]. Following [22], a topological space X is an $\aleph_0$ -space if X possesses a countable k -network. A family $\mathcal{N}$ of subsets of X is a k -network if for any open subset $U \subset X$ and compact subset $K \subset U$ there exists a finite subfamily $\mathcal{F} \subset \mathcal{N}$ such that $K \subset \bigcup \mathcal{F} \subset U$. Schlüchtermann and Wheeler obtained the following theorem for B_w .

Theorem 1.7 ([27]). *The following conditions on a Banach space E are equivalent: (a) B_w is (separable) metrizable; (b) B_w is an $\aleph_0$ -space and a k -space.*

Having in mind the Nagata-Smirnov metrization theorem, O'Meara [21] introduced the class of $\aleph$ -spaces: A topological space X is called an $\aleph$ -space if X is regular and has a σ -locally finite k -network. Any metrizable space X is an $\aleph$ -space. A topological space X is an $\aleph_0$ -space if and only if X is a Lindelöf $\aleph$ -space ([12]). In [12, 13] it is shown that each $\aleph$ -space X has countable cs^* -character. Recall from [5] that a topological space X has *countable cs^* -character* if for each $x \in X$, there exists a countable family $\mathcal{D}$ of subsets of X , such that for each sequence in X converging to x and each neighborhood U of x , there is $D \in \mathcal{D}$ such that $D \subset U$ and D contains infinitely many elements of that sequence. The importance of this concept for the theory of Topological Vector Spaces may be explained by the following result obtained in [11]: If E is a Baire topological vector space (tvs) or a b -Baire-like lcs, then E is metrizable if and only if E has countable cs^* -character.

Tsaban and Zdomskyy [28] strengthened the Pytkeev property as follows. A topological space X has the *strong Pytkeev property* if for each $x \in X$, there exists a countable family $\mathcal{D}$ of subsets of X , such that for each neighborhood U of x and each $A \subset X$ with $x \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subset U$ and $D \cap A$ is infinite. Clearly, the strong Pytkeev property $\Rightarrow$ the Pytkeev property, however in general, the Fréchet–Urysohn property $\nRightarrow$ the strong Pytkeev property $\nRightarrow k$ -space (see [15]). For any Polish space X , the function space $C_c(X)$ has the strong Pytkeev property by [28]. This result has been extended to Čech-complete spaces X , see [15]. Moreover, if E is a strict (LM) -space, a sequential dual metric space, or a (DF) -space of countable tightness, then E has the strong Pytkeev property [15]. In particular, the space $D'(\Omega)$ of distributions over an open subset $\Omega \subset \mathbb{R}^n$ has the strong Pytkeev property while being not a k -space, see [15].

Being motivated by above facts, we generalize Theorem 1.7 as follows (the equivalence $(i) \Leftrightarrow (ii)$ is probably well known, but hard to locate and we propose an elementary proof of it below).

Theorem 1.8. *The following conditions on a Banach space E are equivalent:*

- (i) B_w is (separable) metrizable;
- (ii) B_w is first countable;

- (iii) B_w has the strong Pytkeev property;
- (iv) B_w is an $\aleph$ -space and a k -space;
- (v) B_w has countable cs^* -character and is a k -space;
- (vi) E_w is an $\aleph$ -space and B_w is a k -space;
- (vii) E_w has countable cs^* -character and B_w is a k -space;
- (viii) E_w has countable cs^* -character and contains no isomorphic copy of ℓ_1 ;
- (ix) E' is separable.

Note that the James Tree space and the sequence space $\ell_1(\mathbb{R})$ (which is an $\aleph$ -space in the weak topology, see [13]) show that Theorem 1.8 fails if one of the assumptions on E is dropped.

It is natural to ask about good conditions forcing metrizable B_w to be completely metrizable. Edgar and Wheeler proved the following characterization:

Theorem 1.9 ([8]). *Let E be a separable Banach space. Then the following conditions are equivalent: (1) B_w is completely metrizable; (2) B_w is a Polish space; (3) E has property (PC) and is an Asplund space; (4) B_w is metrizable, and every closed subset of it is a Baire space.*

Since a metrizable space X admits a complete metric generating its topology if and only if it is Čech-complete, and any Čech-complete space is a k -space, Theorem 1.8 immediately implies the following corollary which supplements the Edgar-Wheeler Theorem 1.9.

Corollary 1.10. *The following conditions on a Banach space E are equivalent:*

- (i) B_w is Polish;
- (ii) B_w is a Čech-complete $\aleph$ -space;
- (iii) B_w has countable cs^* -character and is Čech-complete;
- (iv) E_w has countable cs^* -character and B_w is a Čech-complete space.

Note that the Čech-completeness of B_w in (iii) cannot be replaced by the weaker condition that every closed subset of B_w is a Baire space (as in Theorem 1.9 (4)) even if E_w is an $\aleph_0$ -space, see Example 4.10.

The paper is organized as follows. Section 2 deals with the (strong) Pytkeev property and the fan tightness. Applying these concepts we prove that any (DF) -space E is normable if and only if E has countable fan tightness, as well as the strong dual F of a strict (LF) -space has a countable fan tightness if and only if F is metrizable. Hence the space $D'(\Omega)$ of distributions over an open subset $\Omega \subset \mathbb{R}^n$ does not have countable fan tightness although it has countable tightness. Theorems 1.4 and 1.6 are proved in Section 3, and in Section 4 we prove Theorem 1.8. Also we provide some examples.

2. The strong Pytkeev property and the fan tightness in lcs

Below we provide a simple proof of the following result to keep the paper self-contained.

Proposition 2.1 ([4]). *The following assertions are equivalent for a topological space X :*

- (i) X is first countable.
- (ii) X has the strong Pytkeev property and countable fan tightness.

Proof. Let $\mathcal{N}_0$ be a family witnessing the strong Pytkeev property at $x \in X$. We claim that $\mathcal{N} := \{\bigcup \mathcal{N}' : \mathcal{N}' \in [\mathcal{N}_0]^{<\mathbb{N}}\}$ is a local base at x . If not, there exists open $U \ni x$ such that no element of $\mathcal{N}$ contained in U is a neighborhood of x . Let $\{N_i : i \in \mathbb{N}\}$ be the enumeration of all elements of $\mathcal{N}_0$ which are subsets of U . It follows from the above that x lies in the closure of $B_n := X \setminus \bigcup_{i \leq n} N_i$ for all n , and hence we can select a finite subset A_n of B_n such that $\bigcup_{n \in \mathbb{N}} A_n$ has x in its closure. But this is a contradiction because obviously no N_i can have infinite intersection with $\bigcup_{n \in \mathbb{N}} A_n$. $\square$

In [15, Question 5] we ask whether there exists a lcs E such that its dual E' has uncountable algebraic dimension and E_w has the strong Pytkeev property. Theorem 1.2 and Proposition 2.1 immediately imply a negative answer to this question for any metrizable lcs E .

Corollary 2.2. *Let E be a normed space. Then E_w has the strong Pytkeev property if and only if E is finite-dimensional.*

In [15] we provided large classes of lcs having the strong Pytkeev property including an important class of (DF) -spaces. Recall that a lcs E is a (DF) -space if E has a fundamental sequence of bounded sets and every bounded set in the strong dual $(E', \beta(E', E))$ of E which is the countable union of equicontinuous sets is itself equicontinuous. The strong dual $(E', \beta(E', E))$ of a metrizable lcs is a (DF) -space, see [23, Theorem 8.3.8]. Below we use the following result.

Fact 2.3 ([15]). *A (DF) -space E has countable tightness if and only if E has the strong Pytkeev property.*

Since a (DF) -space is normable if and only if it is metrizable, Fact 2.3 and Proposition 2.1 imply

Corollary 2.4. *A (DF) -space E is normable if and only if E has countable fan tightness.*

We use Fact 2.3 also to prove the following result which supplements [15, Theorem 3(iii)].

Proposition 2.5. *Let E be a strict (LF) -space and $E' := (E', \beta(E', E))$ its strong dual. Then E' has strong Pytkeev property if and only if E' has countable tightness.*

Proof. Assume that E' has countable tightness. Let E be a strict inductive limit of a sequence $\{E_n\}_n$ of Fréchet spaces. For each $n \in \mathbb{N}$, the strong dual $(E'_n)_\beta$ of E_n is a complete (DF) -space. Since E is the strict inductive limit, the space E'_β is linearly homeomorphic with the *projective limit* of the sequence $\{(E'_n)_\beta\}_n$ of (DF) -spaces, and moreover, E'_β can be continuously mapped onto each $(E'_n)_\beta$ by an open mapping, see [17]. Since E' has countable tightness, for every $n \in \mathbb{N}$, the quotient space $(E'_n)_\beta$ has countable tightness by [3, Proposition 3]. Hence all spaces $(E'_n)_\beta$ have the strong Pytkeev property by Fact 2.3. Therefore the product $\prod_n (E'_n)_\beta$ also has the strong Pytkeev property by [12]. Consequently E' has the strong Pytkeev property. $\square$

Propositions 2.1 and 2.5 imply

Corollary 2.6. *The strong dual F of a strict (LF) -space has countable fan tightness if and only if F is metrizable.*

We end this section with the next two examples.

Example 2.7. If $\Omega \subset \mathbb{R}^n$ is an open set, then the space of test functions $\mathfrak{D}(\Omega)$ is a complete Montel (LF) -space. As usually, $\mathfrak{D}'(\Omega)$ denotes its strong dual, the space of distributions. We proved in [15] that $\mathfrak{D}'(\Omega)$ has the strong Pytkeev property, but it is not a k -space. Now Proposition 2.1 implies that $\mathfrak{D}'(\Omega)$ does not have countable fan tightness.

Example 2.8. Let $E := C_c(X)$ be any Fréchet lcs which is not normed (for example $E := C_c(\mathbb{R})$). Then $(E', \beta(E', E))$ does not have countable fan tightness although it has countable tightness. Indeed, since $C_c(X)$ is quasi-normable, it is distinguished by [6, Corollary 1], and hence the space $E'_\beta := (E', \beta(E', E))$ is quasibarrelled. Then, by [18, Theorem 12.3], the space E'_β has countable tightness. Finally, Corollary 2.4 implies that E'_β does not have countable fan tightness.

3. Proofs of Theorems 1.4 and 1.6

Denote by S_E the unit sphere of a normed space E .

Lemma 3.1. *Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of unbounded subsets of a normed space E . Then there is $\chi \in S_E$ such that*

$$A_n \setminus \{x \in E : |\langle \chi, x \rangle| < 1/2\} \neq \emptyset, \text{ for every } n \in \mathbb{N}.$$

Proof. First we prove the following claim.

Claim. *There is a sequence $S = \{a_n\}_{n \in \mathbb{N}} \subset E$, where $a_n \in A_n$ for every $n \in \mathbb{N}$, and a sequence $T = \{\chi_n\}_{n \in \mathbb{N}} \subset S_{E'}$ such that*

$$|\langle \chi_n, a_i \rangle| > 1, \quad \text{for every } 1 \leq i \leq n < \infty. \quad (1)$$

We build S and T by induction. For $n = 1$ take arbitrarily $a_1 \in A_1$ such that $\|a_1\| > 1$ (note that A_1 is unbounded) and $\chi_1 \in S_{E'}$ such that $|\langle \chi_1, a_1 \rangle| = \|a_1\|$. Assume that for $n \in \mathbb{N}$, we found $a_i \in A_i$ and $\chi_k \in S_{E'}$ such that

$$|\langle \chi_k, a_i \rangle| > 1, \quad \text{for every } 1 \leq i \leq k \leq n. \quad (2)$$

We distinguish between two cases.

Case 1. The set $\langle \chi_n, A_{n+1} \rangle$ is an unbounded subset of $\mathbb{R}$. Then we choose $a_{n+1} \in A_{n+1}$ such that $|\langle \chi_n, a_{n+1} \rangle| > 1$ and set $\chi_{n+1} := \chi_n$. It is clear that (2) holds also for $n + 1$.

Case 2. $N := \sup\{|\langle \chi_n, a \rangle| : a \in A_{n+1}\} < \infty$. Since A_{n+1} is unbounded, there is $\eta \in S_E$ such that $\langle \eta, A_{n+1} \rangle$ is an unbounded subset of $\mathbb{R}$. Choose $0 < \lambda < 1$ such that

$$\lambda|\langle \chi_n, a_i \rangle| - (1 - \lambda)|\langle \eta, a_i \rangle| > 1, \quad \text{for every } 1 \leq i \leq n, \quad (3)$$

and choose $a_{n+1} \in A_{n+1}$ such that

$$(1 - \lambda)|\langle \eta, a_{n+1} \rangle| > N + 1. \quad (4)$$

Set $\xi := \lambda\chi_n + (1 - \lambda)\eta$. Then by (3), we have

$$|\langle \xi, a_i \rangle| \geq \lambda|\langle \chi_n, a_i \rangle| - (1 - \lambda)|\langle \eta, a_i \rangle| > 1, \quad \text{for every } 1 \leq i \leq n, \quad (5)$$

and, by (4),

$$|\langle \xi, a_{n+1} \rangle| \geq (1 - \lambda)|\langle \eta, a_{n+1} \rangle| - \lambda|\langle \chi_n, a_{n+1} \rangle| > 1. \quad (6)$$

Finally we set $\chi_{n+1} := \xi/\|\xi\| \in S_{E'}$. Since $\|\xi\| \leq 1$, (5) and (6) imply that χ_{n+1} and a_{n+1} satisfy (2). The claim is proved.

Since the unit ball $B_{E'}$ of the dual E' is compact in the weak* topology, we can find a cluster point $\chi \in B_{E'}$ of the sequence T defined in Claim. In particular, for every $i \in \mathbb{N}$ there is $n > i$ such that $|\langle \chi_n, a_i \rangle - \langle \chi, a_i \rangle| < 1/2$. Then (1) implies

$$|\langle \chi, a_i \rangle| > 1/2, \quad \text{for every } i \in \mathbb{N}. \quad (7)$$

Now (7) implies that $a_n \in A_n \setminus \{x \in E : |\langle \chi, x \rangle| < 1/2\}$ for every $n \in \mathbb{N}$, which proves the lemma. $\square$

We are at position to prove Theorem 1.4.

Proof of Theorem 1.4. Assume towards a contradiction that there is an infinite dimensional normed space E such that E_w has the Pytkeev property.

Claim. *For every subset $A \subset E_w$ with $0 \in \overline{A} \setminus A$, there is a bounded subset D of A such that $0 \in \overline{D}$.*

Indeed, suppose that there is a subset A of E_w with $0 \in \overline{A} \setminus A$ and such that $0 \notin \overline{A \cap nB}$ for every $n \in \mathbb{N}$. So $0 \in \bigcap_{n \in \mathbb{N}} \overline{A} \setminus nB$. Since E_w has countable fan tightness (see Theorem 1.2), there are finite subsets $F_n \subset A \setminus nB$ such that $0 \in \overline{\bigcup_{n \in \mathbb{N}} F_n}$. Set $F := \bigcup_{n \in \mathbb{N}} F_n$. Fix arbitrarily sequence $A_1, A_2, \dots$ of infinite subsets of F .

By the construction of F , all sets A_n are unbounded. Lemma 3.1 implies that there is a weakly open neighborhood U of zero such that $A_n \setminus U$ is not empty for every $n \in \mathbb{N}$. Thus E_w does not have the Pytkeev property. This contradiction proves the claim.

Now Theorem 14.3 of [18] and Claim imply that E_w is a Fréchet-Urysohn space, and hence, by Theorem 1.3, E is finite dimensional. This contradiction shows that E_w does not have the Pytkeev property. $\square$

To prove Theorem 1.6 we define the following subset of S_{ℓ_1}

$$A := \{a = (a(i)) \in S_{\ell_1} \mid \exists m, n \in \mathbb{N} : a(m) = -a(n) = 1/2, \text{ and } a(i) = 0 \text{ otherwise}\}. \quad (8)$$

Lemma 3.2. $0 \in \overline{A}$.

Proof. Let U be a neighborhood of 0 of the canonical form

$$U = \{x \in \ell_1 : |\langle \chi_k, x \rangle| < \epsilon, \text{ where } \chi_k \in S_{\ell_\infty} \text{ for } 1 \leq k \leq s\}.$$

Let I be an infinite subset of $\mathbb{N}$ such that, for every $1 \leq k \leq s$, either $\chi_k(i) > 0$ for all $i \in I$, or $\chi_k(i) = 0$ for all $i \in I$, or $\chi_k(i) < 0$ for all $i \in I$. Take a natural number $N > 1/\epsilon$. Since I is infinite, by induction, one can find $m, n \in \mathbb{N}$ satisfying the following condition: for every $1 \leq k \leq s$ there is $0 < t_k \leq N$ such that

$$\frac{t_k - 1}{N} \leq \min \{|\chi_k(m)|, |\chi_k(n)|\} \leq \max \{|\chi_k(m)|, |\chi_k(n)|\} \leq \frac{t_k}{N}. \quad (9)$$

Set $a = (a(i)) \in A$, where $a(m) = -a(n) = 1/2$, and $a(i) = 0$ otherwise. Then, by the construction of I and (9), we obtain $|\chi_k(a)| < 1/N < \epsilon$ for every $1 \leq k \leq s$. Thus $a \in U$, and hence $0 \in \overline{A}$. $\square$

Proposition 3.3. *The unit ball B_w of ℓ_1 in the weak topology does not have the Pytkeev property.*

Proof. It is enough to show that for any sequence $A_1, A_2, \dots$ of infinite subsets of the set A defined in (8) there is a neighborhood of 0 which does not contain A_i for every $i \in \mathbb{N}$.

Fix arbitrarily $a_1 \in A_1$. So $a_1(m_1) = -a_1(n_1) = 1/2$ for some $m_1, n_1 \in \mathbb{N}$. Set $\chi(m_1) = -\chi(n_1) = 1$ and $D_1 = \text{supp}(a_1) = \{m_1, n_1\}$. Then

$$\left| \sum_{i \in D_1} \chi(i) a_1(i) \right| = |\chi(m_1) a_1(m_1) + \chi(n_1) a_1(n_1)| = 1.$$

Since A_2 is infinite, we can choose $a_2 \in A_2$ such that $\text{supp}(a_2) = \{m_2, n_2\} \not\subset D_1$. If $\text{supp}(a_2) \cap D_1 = \emptyset$, we set $\chi(m_2) = -\chi(n_2) = 1$. If $m_2 \in D_1$, we set $\chi(n_2) = -\chi(m_2)$, and if $n_2 \in D_1$, we set $\chi(m_2) = -\chi(n_2)$. Then

$$\left| \sum_{i \in D_2} \chi(i) a_2(i) \right| = |\chi(m_2) a_2(m_2) + \chi(n_2) a_2(n_2)| = 1.$$

Put $D_2 = \text{supp}(a_2) \cup D_1$. Suppose we found $a_k \in A_k$ for $1 \leq k \leq s$ and defined $D_s = \cup_{k \leq s} \text{supp}(a_k)$ and $\chi(i) = \pm 1$ for $i \in D_s$ such that

$$\left| \sum_{i \in D_s} \chi(i) a_k(i) \right| = 1, \quad \text{for every } 1 \leq k \leq s. \quad (10)$$

Since A_{s+1} is infinite and D_s is finite, we can choose $a_{s+1} \in A_{s+1}$ such that

$$\text{supp}(a_{s+1}) = \{m_{s+1}, n_{s+1}\} \not\subset D_s.$$

If $\text{supp}(a_{s+1}) \cap D_s = \emptyset$, we set $\chi(m_{s+1}) = -\chi(n_{s+1}) = 1$. If $m_{s+1} \in D_s$, we set $\chi(n_{s+1}) = -\chi(m_{s+1})$, and if $n_{s+1} \in D_s$, we set $\chi(m_{s+1}) = -\chi(n_{s+1})$. So $|\chi(m_{s+1}) a_{s+1}(m_{s+1}) + \chi(n_{s+1}) a_{s+1}(n_{s+1})| = 1$. Put $D_{s+1} = \text{supp}(a_{s+1}) \cup D_s$. In particular, (10) holds also for $s+1$. Put $D = \cup_{s \in \mathbb{N}} D_s$.

Set $\chi = (\chi(i)) \in S_{\ell_\infty}$, where $\chi(i) = \chi(n_s)$ if $i = n_s$, $\chi(i) = \chi(m_s)$ if $i = m_s$ for some $s \in \mathbb{N}$, and $\chi(i) = 0$ if $i \notin D$. Then (10) implies that $|\chi(a_s)| = 1$ for every $s \in \mathbb{N}$. Finally we set $U = \{x \in \ell_1 : |\langle \chi, x \rangle| < 1/2\}$. Then $a_s \in A_s \setminus U$ for every $s \in \mathbb{N}$. Thus B_w does not have the Pytkeev property. $\square$

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. If B_w has the Pytkeev property, then E contains no isomorphic copy of ℓ_1 by Proposition 3.3. The converse assertion follows from Theorem 1.5. $\square$

4. Proof of Theorem 1.8

We need the following fact which is similar to Proposition 7.7 of [22].

Proposition 4.1. *Assume that a regular space X is covered by an increasing sequence $\{A_n\}_{n \in \mathbb{N}}$ of closed subsets.*

- (i) *If all A_n are $\aleph$ -spaces (resp. an $\aleph_0$ -spaces), and if each compact $K \subset X$ is covered by some A_n , then X is an $\aleph$ -space (resp. an $\aleph_0$ -space, respectively).*
- (ii) *If all A_n have countable cs^* -character and each convergent sequence $S \subset X$ is covered by some A_n , then X has countable cs^* -character.*

Proof. (i): Assume that all A_n are $\aleph$ -spaces and let $\mathcal{D}_n = \bigcup_{k \in \mathbb{N}} \mathcal{D}_{n,k}$ be a σ -locally finite k -network for A_k . It is easy to see that the family $\mathcal{D} := \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \mathcal{D}_{n,k}$ is σ -locally finite in X and covers X . If $K \subset U$ with K compact and U open in X , take A_n such that $K \subset A_n$ and choose a finite subfamily $\mathcal{F} \subset \mathcal{D}_n \subset \mathcal{D}$ such that $K \subset \bigcup \mathcal{F} \subset A_n \cap U \subset U$. This means that $\mathcal{D}$ is a k -network for X , and hence X is an $\aleph$ -space. The case A_n are $\aleph_0$ -spaces is proved analogously.

(ii): Fix $x \in X$, an open neighborhood U of x and a sequence $S = \{s_n\}_{n \in \mathbb{N}}$ converging to x . Set $I(x) := \{n \in \mathbb{N} : x \in A_n\}$ and let $\mathcal{D}_n(x)$ be a countable cs^* -network at x in A_n , $n \in I(x)$. Set $\mathcal{D}(x) = \bigcup \{\mathcal{D}_n(x) : n \in I(x)\}$. Take $n \in I(x)$ such that $S \subset A_n$. Then, by definition, we can find $D \in \mathcal{D}_n(x) \subset \mathcal{D}(x)$ such that $x \in D \subset U$ and $\{n \in \mathbb{N} : s_n \in D\}$ is infinite. Thus $\mathcal{D}(x)$ is a countable cs^* -network at x . $\square$

Corollary 4.2. *Let A be a closed subset of a tvs E .*

- (i) *If every compact subset $K \subset E$ is contained in some nA , then E is an $\aleph$ -space (resp. an $\aleph_0$ -space) if and only if so is A .*
- (ii) *If every convergent sequence $S \subset E$ is contained in some nA , then E has countable cs^* -character if and only if so does A .*

Proof. Clearly $E = \bigcup_{n \in \mathbb{N}} nA$. Since A and nA are topologically isomorphic for every $n \in \mathbb{N}$, the assertion follows from Proposition 4.1. $\square$

Since any weakly compact subset of a Banach space is contained in nB for some $n \in \mathbb{N}$, the last corollary implies (cf. [27, Theorem 4.2])

Corollary 4.3.

- (i) *A Banach space E is a weakly $\aleph$ -space (resp. a weakly $\aleph_0$ -space) if and only if so is B_w .*
- (ii) *A Banach space E has countable cs^* -character if and only if so does B_w .*

Any first countable topological group is metrizable, but there are first-countable non-metrizable compact spaces. However, for the unit ball of Banach spaces we have the following result which supplements Theorem 1.5 (probably known but hard to locate).

Proposition 4.4. *Let E be a Banach space. Then B_w is metrizable if and only if it is first countable.*

Proof. Assume that B_w is first countable. Denote by $\mathcal{W}$ the standard group uniformity

$$\{(x, y) \in E^2 : x - y \in U, \text{ where } 0 \in U \in \sigma(E, E')\}$$

on E generating the topology $\sigma(E, E')$. Since B_w is homeomorphic to the ball $2B_w = B_w - B_w$ of radius 2, the set $2B_w$ is also first countable. So there exists a sequence $\{U_n\}_{n \in \mathbb{N}} \subset \sigma(E, E')$ of open symmetric neighborhoods of 0, such that $\{U_n \cap 2B_w : n \in \mathbb{N}\}$ is a countable base at 0. Let us show that the family $\{W_n : n \in \mathbb{N}\}$, where $W_n = \{(x, y) \in B \times B : x - y \in U_n\}$, is a countable base of the uniformity $\mathcal{W}|_B$. Indeed, fix arbitrarily $W = \{(x, y) \in B \times B : x - y \in U\} \in \mathcal{W}|_B$ with $0 \in U \in \sigma(E, E')$. Choose $n \in \mathbb{N}$ such that $U_n \cap 2B \subset U \cap 2B$. Then, for every $(x, y) \in W_n$, we have $x - y \in U_n$ and also $x - y \in 2B$. So $x - y \in U_n \cap 2B \subset U \cap 2B$, and hence $(x, y) \in W$.

By Theorem 8.1.21 of [9], the uniformity $\mathcal{W}|_B$ is generated by a metric ρ on B . So the topologies τ_ρ and τ_B on B induced by the metric ρ and $\mathcal{W}|_B$ coincide (see [9, 4.1.11, 4.2.6 and 8.1.18]). Taking into account that τ_B and $\sigma(E, E')|_B$ also coincide (see [9, 8.1.17]), we obtain that B_w is metrizable. The converse assertion is trivial. $\square$

Recall that a topological space X has the *property* (α_4) if for any $\{x_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x \in X$, $m \in \mathbb{N}$, there exists a sequence $(m_k)_k$ of distinct natural numbers and a sequence $(n_k)_k$ of natural numbers such that $\lim_k x_{m_k, n_k} = x$.

The following result is similar to [14, Lemma 3.2] and is essentially a corollary of the proof of [24, Theorem 4]. However, the latter is devoted to general (i.e., not necessary Hausdorff) topological groups which makes big parts of its proof not relevant for us, and hence we believe that a streamlined proof of the proposition below is still of some value.

Proposition 4.5. *Any convex and Fréchet-Urysohn subset F of a topological vector space (tvs) E has the property (α_4) .*

Proof. Let $(x_{m,n})_{n \in \mathbb{N}}$ be a sequence of elements of F convergent to $x \in F$ for all $m \in \mathbb{N}$. Denote by J the set of all $m \in \mathbb{N}$ for which the family $\{n \in \mathbb{N} : x = x_{m,n}\}$ is nonempty. If J is infinite the assertion is trivial. If J is finite, without loss of generality we may additionally assume that $x \neq x_{m,n}$ for all $m, n \in \mathbb{N}$. Two cases are possible.

(i): The set $I := \{m \in \mathbb{N} : \exists n(m) \in \mathbb{N} (x = 1/2(x_{m,n(m)} + x_{1,m}))\}$ is infinite. Since $x_{m,n(m)} = 2x - x_{1,m}$ for all $m \in I$ and $(x_{1,m})_{m \in I}$ converges to x , so does the sequence $(x_{m,n(m)})_{m \in I}$.

(ii): The set I defined above is finite, say $\max(I) = q$. Set

$$X = \{1/2(x_{m,n} + x_{1,m}) : n \in \mathbb{N}, m > q\},$$

and observe that $X \subset F \setminus \{x\}$ and $x \in \overline{X}$. Since F has the Fréchet-Urysohn property, there exists a sequence $(1/2(x_{m_k, n_k} + x_{1, m_k}))_{k \in \mathbb{N}}$ of elements of X converging to x . It follows from the assumption $x \notin \{x_{1, m} : m \in \mathbb{N}\}$ that for every $m \in \mathbb{N}$ there are at most finitely many k such that $m_k = m$, and therefore passing to a subsequence if necessary we may assume that $m_{k+1} > m_k$ for all k . Thus $(x_{1, m_k})_{k \in \mathbb{N}}$ converges to x , and hence so does $(x_{m_k, n_k})_{k \in \mathbb{N}}$.

In any of these cases there exists a sequence which converges to x and meets $(x_{m, n})_{n \in \mathbb{N}}$ for infinitely many m , which completes our proof. $\square$

Corollary 4.6. *A convex subset D of a tvs E is metrizable if and only if D is both a Fréchet-Urysohn space and an $\aleph$ -space.*

Proof. This follows from Proposition 4.5 and the fact that a topological space which is an $\aleph$ -space and Fréchet-Urysohn with property (α_4) is metrizable, see [13, Theorem 2.2]. $\square$

Next proposition is crucial for the proof of Theorem 1.8.

Proposition 4.7. *Let E be a Banach space. If B_w is a k -space and has countable cs^* -character, then it is metrizable.*

Proof. In the proof all subspaces X of E are considered with the weak topology $\sigma(E, E')|_X$. By Theorem 1.5 we can assume that B_w is Fréchet-Urysohn. By Proposition 4.4, it is enough to prove that B_w is first countable.

Fix $x \in B$, a neighborhood U of x in B_w and a countable cs^* -network $\mathcal{N}$ at x in B_w . Let $\{N_i : i \in \mathbb{N}\}$ be an enumeration of all elements of $\mathcal{N}$ which are subsets of U . We claim that there exists m such that $\bigcup_{i \leq m} N_i$ is a neighborhood of x in B_w . If not, for every $m \in \mathbb{N}$ there exists a sequence $\{x_{m, n} : n \in \mathbb{N}\} \subset B \setminus \bigcup_{i \leq m} N_i$ converging to x . Since B_w is a bounded subset of E_w we apply Proposition 4.5 to find a sequence $(m_k)_k$ of distinct natural numbers and a sequence $(n_k)_k$ of natural numbers such that $\lim_k x_{m_k, n_k} = x$. It follows that there exists $i \in \mathbb{N}$ such that the intersection $C := N_i \cap \{x_{m_k, n_k} : k \in \mathbb{N}\}$ is infinite. On the other hand, $x_{m_k, n_k} \notin N_i$ for all $k > i$ because $m_k > i$, and hence C is finite. This contradiction shows that $\{\bigcup \mathcal{N}' : \mathcal{N}' \in [\mathcal{N}]^{<\aleph}\}$ is a countable base at x in B_w . $\square$

Now we prove Theorem 1.8.

Proof of Theorem 1.8. $(i) \Rightarrow (ii) \Rightarrow (iii)$ is clear. $(iii) \Leftrightarrow (i)$ follows from Theorem 1.2 and Propositions 2.1 and 4.4. The separable case and $(i) \Leftrightarrow (ix)$ are well known, see [10]. $(vii) \Leftrightarrow (viii)$ follows from Theorem 1.5. $(i) \Rightarrow (iv)$ is clear, and $(iv) \Rightarrow (v)$ and $(vi) \Rightarrow (vii)$ follow from the fact that every $\aleph$ -space

has countable cs^* -character (see [12, Corollary 3.8]). (iv) $\Leftrightarrow$ (vi) and (v) $\Leftrightarrow$ (vii) follow from Corollary 4.3. Finally, (v) $\Leftrightarrow$ (i) follows from Proposition 4.7. $\square$

The following corollary generalizes Corollary 5.6 of [13].

Corollary 4.8. *Let E be a Banach space not containing ℓ_1 . Then E' is separable if and only if E_w has countable cs^* -character.*

In [13, Corollary 5.3] we proved that a reflexive Fréchet space is a weakly $\aleph$ -space if and only if E' is separable. For reflexive Banach spaces we strengthen this result as follows.

Corollary 4.9. *Let E be an infinite dimension reflexive Banach space. Then E_w does not have the strong Pytkeev property. Moreover, the following conditions are equivalent: (i) E_w has countable cs^* -character; (ii) B_w has countable cs^* -character; (iii) E' is separable; (iv) B_w is Polish.*

Proof. The space E_w does not have the strong Pytkeev property by Corollary 2.2. By Theorem B of [8], B_w is Čech-complete, and thus the second assertion follows from Theorem 1.8 and the fact that any metrizable separable Čech-complete space is Polish. $\square$

For example, for $1 < p < \infty$, the reflexive space $\ell_p(\Gamma)$ has countable cs^* -character in the weak topology if and only if Γ countable, which extends Example 3.1 of [13]. Corollary 4.9 also shows that for the strong Pytkeev property there does not exist a result analogous to Corollary 4.3. Indeed, if $E = \ell_2$, then B_w has the strong Pytkeev property because it is Polish, but E_w does not have the strong Pytkeev property.

Below we consider examples clarifying relations between the notions from Theorems 1.8 and 1.9 and other notions considered in [8].

Example 4.10. Let $E = \ell_1$. Then E is a weakly $\aleph_0$ -space by [22, 7.10] and B_w is almost Čech-complete [8, 6(9)]. So every closed subset of B_w is also almost Čech-complete, hence Baire (see [1]). Another argument: The original norm of E has the Kadec-Klee property, i.e. the weak and the norm topologies coincide on the unit sphere of E ; now apply [10, Proposition 12.56]. Since $\ell'_1 = \ell_\infty$ is not separable, B_w is not metrizable, and hence B_w is not a k -space by Theorem 1.8. In particular, the conditions “ B_w is a k -space” in Theorem 1.7 and “ B_w is metrizable” in Theorem 1.9 cannot be removed. Note that E has RNP and (PC) but it is not a Godefroy space (see [8]).

Example 4.11. Let $E = c_0$. As $c'_0 = \ell_1$ is separable, E is a weakly $\aleph_0$ -space by [14] and B_w is metrizable (hence a k -space). The ball B_w is not Čech-complete by [8]. Note that (see [8]) E does not have RNP and (PC) and is not a Godefroy space, but E is Asplund.

Recall that a topological space X is called *cosmic*, if X is a regular space with a countable network (a family $\mathcal{N}$ of subsets of X is called a *network* in X if, whenever $x \in U$ with U open in X , then $x \in N \subset U$ for some $N \in \mathcal{N}$). A space X is called a σ -space if it is regular and has a σ -locally finite network.

Remark 4.12. Let E be the James Tree space. Then E_w is a cosmic space as the continuous image of a separable metrizable space E (see [22]). Since E' is not separable, E' has uncountable cs^* -character. So one cannot replace the countability of cs^* -character of E_w or B_w in Theorem 1.8 by E_w or B_w being cosmic.

For a Banach space E , Corson [7] proved that E_w is paracompact if and only if E_w is Lindelöf, and Reznichenko [26] proved that E_w is Lindelöf if and only if E_w is normal. On the other hand, we proved in [12] that a topological space X is cosmic (resp. an $\aleph_0$ -space) if and only if it is a Lindelöf σ -space (resp. a Lindelöf $\aleph$ -space). In particular, these results imply

Proposition 4.13. *For a Banach space E , the following conditions are equivalent: (a) E_w is an $\aleph_0$ -space; (b) E_w is a paracompact $\aleph$ -space; (c) E_w is a Lindelöf $\aleph$ -space; (d) E_w is a normal $\aleph$ -space.*

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