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# Hahn-Banach Theorem and Duality Theory on non-Archimedean Locally Convex Spaces

**Tomoki Mihara**
*Department of Mathematics, Tokyo Institute of Technology,*
*2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan*
*mihara.t.ab@m.titech.ac.jp*

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Let $k$ be a local field with valuation ring $O$ and residue field $\overline{k}$. We extend Hahn–Banach theorem for the class of seminormed $k$-vector spaces to several classes of locally convex spaces and subspaces over $k$, $O$, and $\overline{k}$. We establish continuous duality theory for several classes of locally convex spaces over $k$, $O$, and $\overline{k}$.

*Keywords:* $p$-adic analysis, Hahn–Banach theorem, duality theory

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## 1. Introduction

Let $k$ be a local field with valuation ring $O$ and residue field $\overline{k}$. This paper is devoted to two topics in non-Archimedean analysis: One is Hahn–Banach theorem, and the other one is duality theory. Throughout this paper, we abbreviate Hahn–Banach theorem to HBT. Let $\kappa \in \{k, \overline{k}, O\}$. For a pair $(M, M_0)$ of a topological $\kappa$-module $M$ and a $\kappa$-submodule $M_0 \subset M$, we say that *HBT holds for* $(M, M_0)$ if every continuous $\kappa$-linear homomorphism $M_0 \to \kappa$ admits a continuous $\kappa$-linear extension $M \to \kappa$. For a topological $\kappa$-module $M$, we say that *HBT holds for* $M$ if HBT holds for $(M, M_0)$ for any $\kappa$-submodule $M_0 \subset M$. By [7] Theorem 3, HBT holds for any seminormed $\kappa$-vector space for $\kappa \in \{k, \overline{k}\}$. We show that HBT holds also for any locally convex $\kappa$-vector space for $\kappa = k$ in Theorem 3.1, and for $\kappa = \overline{k}$ in Theorem 3.8. In particular, they are partial generalisations of [7] Theorem 3. On the other hand, HBT does not necessarily hold for a topological $O$-module. We introduce notions of a locally convex $O$-module and an adically saturated $O$-submodule in §3.3, and verify that HBT holds for any pair of a locally convex $O$-modules and a compact adically saturated $O$-submodule in Theorem 3.21.

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theory. Let $\mathrm{Ban}(O)$ denote the category of strictly Cartesian Banach $k$-vector spaces and submetric $k$-linear homomorphisms, $\mathrm{Ban}(k)$ denote the category of Banach $k$-vector spaces and bounded $k$-linear homomorphisms, $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)$ the category of compact Hausdorff linear topological $O$-modules and continuous $O$-linear homomorphisms, and $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)_k$ the localisation of $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)$ by $k$ in the sense of an enriched category. By [10] Theorem 4.6 and [12] Theorem 1.2, there is a contravariant functor $\mathrm{D}$ between $\mathrm{Ban}(O)$ and $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)$ (resp. $\mathrm{Ban}(k)$ and $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)_k$), and it gives a contravariant equivalence called *the Schikhof duality*. It can be interpreted as the restriction of a contravariant endomorphism $\mathbb{D}$ of the category $\mathrm{Mod}_{\mathrm{l}}(O)$ of linear topological $O$-modules and continuous $O$-linear homomorphisms to the full subcategories $\mathrm{LC}^{\mathrm{Ban}}(O)$ of Banach locally convex $O$-modules and $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)$ as is shown in Corollary 4.24. We also give a contravariant endomorphism $\mathbb{D}_{\overline{k}}$ of the full subcategory $\mathrm{LC}(\overline{k}) \subset \mathrm{Mod}_{\mathrm{l}}(O)$ of locally convex $\overline{k}$-vector spaces, whose restriction gives a contravariant equivalence between the full subcategories $\mathrm{LC}^{\mathrm{d}}(\overline{k})$ of discrete locally convex $\overline{k}$-vector spaces and $\mathrm{LC}^{\mathrm{ch}}(\overline{k})$ of compact locally convex $\overline{k}$-vector spaces in Corollary 4.11 so that $\mathbb{D}$ and $\mathbb{D}_{\overline{k}}$ are compatible with the reductions. Similar to many other duality between discreteness and compactness, we also aim to extends these duality to wider categories. For this purpose, we verify the reflexivity of compactly generated Hausdorff locally convex $k$-vector spaces in Theorem 4.2, compactly generated Hausdorff locally convex $\overline{k}$-vector spaces in Theorem 4.16, and compactly generated complete locally convex $O$-modules in Theorem 4.34. Finally, we prove the self-dualities of the full subcategories $\mathrm{LC}^{\sigma\mathrm{cf}}(\overline{k})$ of $\sigma$-compact Fréchet locally convex $\overline{k}$-vector spaces in Theorem 4.20 and $\mathrm{LC}^{\sigma\mathrm{cf}}(O)$ of Fréchet locally convex $O$-modules whose reductions are $\sigma$-compact in Theorem 4.32.

## 2. Preliminaries

We denote by Set the category of sets and maps. For a set $I$, we denote by $2_{\mathrm{f}}^I$ the set of finite subsets of $I$ directed by the inclusion relation. Let $k$ denote a local field, i.e. a complete discrete valuation field with finite residue field throughout this paper. We denote by $O$ the valuation ring of $k$, and by $\overline{k}$ the residue field of $O$ endowed with the discrete topology. We fix a uniformiser $\varpi_k$ of $k$, noting that no result in this paper depends on the choice of $\varpi_k$. For an $O$-module $M$ endowed with a topology (e.g. a topological $O$-module), we denote by $\mathscr{O}(M)$ (resp. $\mathscr{C}(M)$) the set of open (resp. closed) $O$-submodules of $M$, and by $\Pi_M$ the $O$-linear endomorphism $M \to M,\ m \mapsto \varpi_k m$. Let $\kappa \in \{k, \overline{k}, O\}$. We denote by $\mathrm{Mod}(\kappa)$ the category of topological $\kappa$-modules and continuous $\kappa$-linear homomorphisms. We will never consider a subcategory of $\mathrm{Mod}(O)$ which is not a full subcategory, and hence we simply say an isomorphism instead of an isomorphism in $\mathrm{Mod}(O)$. For an $M \in \mathrm{ob}(\mathrm{Mod}(k))$, we denote by $\mathscr{C}_k(M)$ the set of closed $k$-vector subspaces of $M$. Let $*$ be a formal symbol in $\{\mathrm{c}, \mathrm{h}, \mathrm{ch}, \mathrm{d}\}$. If $*$ is c (resp. h, ch, d), we denote by $\mathrm{Mod}^*(\kappa) \subset \mathrm{Mod}(\kappa)$ the full subcategory of compact (resp. Hausdorff, compact Hausdorff, discrete) topological $\kappa$-

modules. For an $(M_1, M_2) \in \mathrm{ob}(\mathrm{Mod}(O))^2$, we abbreviate $\mathrm{Hom}_{\mathrm{Mod}(O)}(M_1, M_2)$ to $\mathrm{Hom}(M_1, M_2)$. We regard $\mathrm{Mod}(k)$ (resp. $\mathrm{Mod}(\overline{k})$) as a full subcategory of $\mathrm{Mod}_{\mathrm{f}}(O)$ (resp. $\mathrm{Mod}(O)$) through the natural embedding, whose essential image consists of flat $L \in \mathrm{ob}(\mathrm{Mod}(O))$ with $\Pi_L \in \mathrm{Aut}_{\mathrm{Mod}(O)}(L)$ (resp. consists of $\mathscr{L} \in \mathrm{ob}(\mathrm{Mod}(O))$ with $\Pi_{\mathscr{L}} = 0$).

### 2.1. Linear Topological Modules

We recall a linear topological $O$-module (cf. [12] p. 2), and study several properties of it. A topological $O$-module $M$ is said to be *linear* if $\mathscr{O}(M)$ forms a fundamental system of neighbourhoods of $0 \in M$. We denote by $\mathrm{Mod}_{\mathrm{l}}(O) \subset \mathrm{Mod}(O)$ the full subcategory of linear topological $O$-modules, and put $\mathrm{Mod}_{\mathrm{l}}^{*}(O) := \mathrm{Mod}_{\mathrm{l}}(O) \cap \mathrm{Mod}^{*}(O)$ for an $* \in \{\mathrm{c}, \mathrm{h}, \mathrm{ch}\}$.

**Definition 2.1.** Every $O$-module $N$ forms a linear topological $O$-module with respect to *the adic topology on* $N$, i.e. the topology generated by the set of subsets of the form $n + \varpi_k^j N$ for an $(n, j) \in N \times \mathbb{N}$. We say that $N$ is *adically complete* if $N$ is complete with respect to the adic topology.

Let $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$. We put $M^{\mathrm{h}} := M / \bigcap_{M_0 \in \mathscr{O}(M)} M_0$. Since $\bigcap_{M_0 \in \mathscr{O}(M)} M_0$ coincides with the closure of $\{0\}$, the following holds:

**Proposition 2.2.** *The correspondence* $M \rightsquigarrow M^{\mathrm{h}}$ *gives a left adjoint functor* $\mathrm{Mod}_{\mathrm{l}}(O) \to \mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O)$ *of the inclusion* $\mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O) \hookrightarrow \mathrm{Mod}_{\mathrm{l}}(O)$.

For a category $C$, we denote by $\mathrm{Pro}_C$ (resp. by $\mathrm{Ind}_C$) the category of projective (resp. inductive) systems in $C$ and compatible systems of morphisms. Let $(M_i)_{i \in I}$ be a family in $\mathrm{Mod}_{\mathrm{l}}(O)$. Then $\prod_{i \in I} M_i$ lies in $\mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$, and satisfies the universality of the direct product in $\mathrm{Mod}_{\mathrm{l}}(O)$. Let $\leq$ be an order on $I$ for which $I$ is directed. Given a system $(\varphi_{i_1}^{i_2})_{i_1 \leq i_2} \in \prod_{i_1 \leq i_2} \mathrm{Hom}(M_{i_1}, M_{i_2})$ with $((M_i)_{i \in I}, (\varphi_{i_1}^{i_2})_{i_1 \leq i_2}) \in \mathrm{ob}(\mathrm{Pro}_{\mathrm{Mod}_{\mathrm{l}}(O)})$, we denote by $\varprojlim_{i \in I} M_i \subset \prod_{i \in I} M_i$ the projective limit of the underlying $O$-modules endowed with the relative topology. Then $\varprojlim_{i \in I} M_i$ lies in $\mathscr{C}(\prod_{i \in I} M_i)$, and satisfies the universality of the projective limit in $\mathrm{Mod}_{\mathrm{l}}(O)$. Given a system $(\varphi_{i_1}^{i_2})_{i_1 \leq i_2} \in \prod_{i_1 \leq i_2} \mathrm{Hom}(M_{i_1}, M_{i_2})$ with $((M_i)_{i \in I}, (\varphi_{i_1}^{i_2})_{i_1 \leq i_2}) \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{Mod}_{\mathrm{l}}(O)})$, we denote by $\varinjlim_{i \in I} M_i$ the inductive limit of the underlying $O$-modules of $M_i$ with $i \in I$ endowed with the inductive limit topology. To begin with, we verify that the inductive limit of linear topological $O$-modules forms a linear topological $O$-module, while the inductive limit of topological groups is not necessarily a topological group with respect to the inductive limit topology.

**Proposition 2.3.** *The universality of the inductive limit in* $\mathrm{Mod}_{\mathrm{l}}(O)$ *holds for* $\varinjlim_{i \in I} M_i$.

**Proof.** Put $M_\infty := \varinjlim_{i \in I} M_i$. For an $i_0 \in I$, let $\varphi_{i_0} \colon M_{i_0} \to M_\infty$ denote the canonical $O$-linear homomorphism, and put $I_{i_0} := \{i \in I \mid i \geq i_0\}$. For an $(i_1, i_2) \in I^2$ with $i_1 \leq i_2$, let $\varphi_{i_1}^{i_2} \colon M_{i_1} \to M_{i_2}$ denote the transition map. By the universality of the inductive limit topology, it suffices to show that $M_\infty$ forms a linear topological $O$-module. It is easy to see that an $O$-module $L$ endowed with a topology forms a linear topological $O$-module if and only if the set $\{l + L_0 \mid (l, L_0) \in L \times \mathscr{O}(L_0)\}$ forms an open basis. To begin with, we show that $m + M_{\infty,0}$ is open for any $(m, M_{\infty,0}) \in M_\infty \times \mathscr{O}(M_\infty)$. Take an $i_0 \in I$ with $m \in \varphi_{i_0}(M_{i_0})$ and an $m_0 \in M_{i_0}$ with $\varphi_{i_0}(m_0) = m$. For any $i \in I_{i_0}$, the continuity of the addition $M_i \times M_i \to M_i$ ensures that $\varphi_i^{-1}(m + M_{\infty,0}) = \varphi_{i_0}^{i}(m_0) + \varphi_i^{-1}(M_{\infty,0})$ is open subset of $M_i$. Therefore $m + M_{\infty,0}$ is open. Now we verify the assertion. Let $U \subset M_\infty$ be an open neighbourhood of an $m \in M_\infty$. Take an $i_0 \in I$ with $m \in \varphi_{i_0}(M_{i_0})$ and an $m_0 \in M_{i_0}$ with $\varphi_{i_0}(m_0) = m$. By the definition of the inductive limit topology, there is an $(M_{i,0})_{i \in I_{i_0}} \in \prod_{i \in I_{i_0}} \mathscr{O}(M_i)$ with $\varphi_{i_0}^{i}(m_0) + M_{i,0} \subset \varphi_i^{-1}(U)$ for each $i \in I_{i_0}$. Put $M_{\infty,0} := \sum_{i \in I_{i_0}} \varphi_i(M_{i,0})$. For any $i \in I_{i_0}$, we have $M_{i,0} \subset \varphi_i^{-1}(M_{\infty,0})$ and hence $\varphi_i^{-1}(M_{\infty,0}) \in \mathscr{O}(M_\infty)$. It implies $M_{\infty,0} \in \mathscr{O}(M_\infty)$, and $m + M_{\infty,0}$ is an open neighbourhood of $m \in M_\infty$ contained in $U$. Thus the set $\{m + M_{\infty,0} \mid (m, M_{\infty,0}) \in M_\infty \times \mathscr{O}(M_\infty)\}$ forms an open basis. $\square$

As a consequence, if $M_i$ is a locally convex $k$-vector space (cf. the beginning of §3.1) for any $i \in I$, then $\lim_{i \in I} M_i$ coincides with the one dealt with in [11] Ch. III §16 and [9] §3.4. We put $\varinjlim^{\mathrm{h}}_{i \in I} M_i := (\varinjlim_{i \in I} M_i)^{\mathrm{h}}$. Then Proposition 2.2 and Proposition 2.3 immediately imply the following:

**Corollary 2.4.** *The universality of the inductive limit in* $\mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O)$ *holds for* $\varinjlim^{\mathrm{h}}_{i \in I} M_i$.

In order to connect theory over $k$ to that over $O$, we introduce notions of the boundedness and a lattice. Let $M \in \mathrm{ob}(\mathrm{Mod}(O))$. An $S \subset M$ is said to be *bounded* if for any $M_0 \in \mathscr{O}(M)$, there is a $c \in O \setminus \{0\}$ with $c \cdot S \subset M_0$, and is said to be *totally bounded* if for any $M_0 \in \mathscr{O}(M)$, there is an $S_0 \in 2_{\mathrm{f}}^{S}$ with $S \subset S_0 + M_0$. We say that $M$ is *bounded* (resp. *totally bounded*) if $M$ itself is a bounded (resp. totally bounded) subset of $M$. In the case $M \in \mathrm{ob}(\mathrm{Mod}(k))$, the notion of the boundedness of a subset of $M$ is equivalent to that dealt with in [11] Ch. 1 §4 p. 17 and [9] Definition 3.6.1. Every compact subset is totally bounded by definition, and every totally bounded subset is bounded.

Let $W \in \mathrm{ob}(\mathrm{Mod}(k))$. A *lattice of* $W$ is a bounded $M \in \mathscr{C}(W)$ with $kM = W$. We note that there are several distinct definitions of a lattice of a $W \in \mathrm{ob}(\mathrm{Mod}(k))$. For example, a lattice in the sense in this paper is a bounded closed lattice in the sense in [11] Ch. 1 §2 p. 7, and is a bounded barrel in the sense in [9] Definition 7.1.1. In order to establish a general theory on continuous duals, we introduce a notion of a $\mathrm{b}_k$-class, which plays a similar role to the family $\mathscr{B}$ in [11] Ch. 1 §6 p. 29.

**Definition 2.5.** A $\mathrm{b}_k$-*class* is a covariant functor $\mathscr{B} \colon \mathrm{Mod}_{\mathrm{l}}(O) \to \mathrm{Set}$ satisfying the following three conditions: (i) For any $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$, $\mathscr{B}(M)$ is a set of $O$-submodules of $M$ containing $\{Om \mid m \in M\}$. (ii) For any $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$ and $(B_0, B_1) \in \mathscr{B}(M)^2$, there is a $B_2 \in \mathscr{B}(M)$ with $B_0 + B_1 \subset B_2$. (iii) For any $(M_1, M_2) \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))^2$ and $A \in \mathrm{Hom}(M_1, M_2)$, $\mathscr{B}(M_2)$ contains $\{A(B) \mid B \in \mathscr{B}(M_1)\}$, and $\mathscr{B}(A)(B)$ coincides with $A(B)$ for any $B \in \mathscr{B}(M_1)$.

For an $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$, we denote by $\mathrm{s}(M)$ (resp. $\mathrm{K}(M)$, $\mathrm{b}(M)$, $\mathrm{u}(M)$) the set of finitely generated (resp. compact, bounded, all) $O$-submodules of $M$. The correspondence $M \rightsquigarrow \mathrm{s}(M)$ (resp. $\mathrm{K}(M)$, $\mathrm{b}(M)$, $\mathrm{u}(M)$) gives a $\mathrm{b}_k$-class s (resp. K, b, u). Let $\mathscr{B}$ be a $\mathrm{b}_k$-class. Let $(M_1, M_2) \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))^2$. For $O$-submodules $M_{1,0} \subset M_1$ and $M_{2,0} \subset M_2$, we put $\mathrm{Hom}((M_1, M_{1,0}), (M_2, M_{2,0})) := \{A \in \mathrm{Hom}(M_1, M_2) \mid A(M_{1.0}) \subset M_{2,0}\}$. We denote by $\mathrm{Hom}(M_1, M_2)_{\mathscr{B}}$ the $O$-module $\mathrm{Hom}(M_1, M_2)$ endowed with the topology generated by the set of subsets of the form $A + \mathrm{Hom}((M_1, B), (M_2, M_{2,0}))$ for an $(A, B, M_{2,0}) \in \mathrm{Hom}(M_1, M_2) \times \mathscr{B}(M_1) \times \mathscr{O}(M_2)$. Then $\mathrm{Hom}(M_1, M_2)_{\mathscr{B}}$ forms a linear topological $O$-module.

Let $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$ and $\kappa \in \{k, \overline{k}, O\}$ in the following in this subsection. We put $M_{\mathscr{B}}^{\mathbb{D}_\kappa} := \mathrm{Hom}(M, \kappa)_{\mathscr{B}}$. Since $\kappa$ lies in $\mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O))$, so is $M_{\mathscr{B}}^{\mathbb{D}_\kappa}$. The correspondence $M \rightsquigarrow M_{\mathscr{B}}^{\mathbb{D}_\kappa}$ gives a contravariant functor $(\cdot)_{\mathscr{B}}^{\mathbb{D}_\kappa} \colon \mathrm{Mod}_{\mathrm{l}}(O) \to \mathrm{Mod}_{\mathrm{l}}(O)$. We remark that $(\cdot)_{\mathrm{s}}^{\mathbb{D}_k}$ coincides with the weak dual functor $(\cdot)'_{\mathrm{s}}$ [11] Ch. 1 §9 p. 50 and [12] p. 4 and with the weak star dual functor $(\cdot)'_{\sigma}$ in [9] Definition 7.3.1, and $(\cdot)_{\mathrm{b}}^{\mathbb{D}_k}$ coincides with the strong dual functor $(\cdot)'_{\mathrm{b}}$ in [11] Ch. 1 §9 p. 50 and [9] Definition 7.3.1. We simply put $\mathbb{D}_\kappa := (\cdot)_{\mathrm{K}}^{\mathbb{D}_\kappa}$, $(\cdot)_{\mathscr{B}}^{\mathbb{D}} := (\cdot)_{\mathscr{B}}^{\mathbb{D}_O}$, and $\mathbb{D} := \mathbb{D}_O$. We mainly deal with $\mathbb{D}$, because it gives an extension of the Schikhof dual functors D in [11] §1 p. 2 and p. 3 as will be verified in Theorem 2.11.

**Definition 2.6.** Let $C$ be a category, and $D \colon C \to C$ a contravariant functor. A pair $(C_1, C_2)$ of full subcategories $C_1, C_2 \subset C$ closed under isomorphisms is said to be a $D$-*dual pair* if the essential images of $D|_{C_1}$ and $D|_{C_2}$ coincides with $C_2$ and $C_1$ respectively and there are natural equivalences $\mathrm{id}_{C_1} \Rightarrow D|_{C_2} \circ D|_{C_1}$ and $\mathrm{id}_{C_2} \Rightarrow D|_{C_1} \circ D|_{C_2}$. A full subcategory $C_0 \subset C$ is said to be $D$-*reflexive* if $(C_0, C_0)$ is a $D$-dual pair.

One of the main purpose of this paper is to discover a good $\mathbb{D}$-reflexive full subcategory of $\mathrm{Mod}_{\mathrm{l}}(O)$. For this sake, we study a basic property of $M_{\mathscr{B}}^{\mathbb{D}}$.

**Proposition 2.7.** *The continuous dual $M_{\mathscr{B}}^{\mathbb{D}}$ is adically complete, bounded, and flat.*

In order to prove Proposition 2.7, we verify a certain divisibility of homomorphisms.

**Proposition 2.8.** *Let* $(M_1, M_2) \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))^2$*. Suppose that* $\Pi_{M_2}$ *is a homeomorphism onto the image. Then the natural O-linear homomorphism* $\varpi_k^j \mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}(M_1, \varpi_k^j M_2)$ *is an isomorphism for any* $j \in \mathbb{N}$*.*

**Proof.** We denote by $\iota$ the embedding $\mathrm{Hom}(M_1, \varpi_k^j M_2) \hookrightarrow \mathrm{Hom}(M_1, M_2)$ induced by the inclusion $\varpi_k^j M_2 \hookrightarrow M_2$. Then the image of $\iota$ contains $\varpi_k^j \mathrm{Hom}(M_1, M_2)$. Let $A_1 \in \mathrm{Hom}(M_1, \varpi_k^j M_2)$. The assumption on $M_2$ ensures that $M_2$ is torsionfree. Therefore for each $m \in M_1$, there is a unique element $A_2 m \in M_2$ with $\varpi_k^j A_2 m = A_1 m$, and the map $A_2 \colon M_1 \to M_2,\ m \mapsto A_2 m$ is $O$-linear. The assumption on $M_2$ again ensures that the map $\Pi_{M_2}^j$ is a homeomorphism onto the image $\varpi_k^j M_2$. Therefore the continuity of $A_1$ implies that of $A_2$. We obtain $A_1 = \varpi_k^j A_2 \in \varpi_k^j \mathrm{Hom}(M_1, M_2)$. Thus the image of $\iota$ coincides with $\varpi_k^j \mathrm{Hom}(M_1, M_2)$. $\square$

**Proof of Proposition 2.7.** Since $O$ is bounded and torsionfree, so is $M_{\mathscr{B}}^{\mathbb{D}}$. Therefore $M_{\mathscr{B}}^{\mathbb{D}}$ is flat. Let $(w_i)_{i \in \mathbb{N}} \in \mathrm{Hom}(M, O)$ be a Cauchy sequence with respect to the adic topology. For any $m \in M$, $(w_i(m))_{i \in \mathbb{N}} \in O^{\mathbb{N}}$ is a Cauchy sequence on $O$ by the definition of the adic topology, and hence converges to a unique element $w(m) \in O$. We consider a map $w \colon M \to O,\ m \mapsto w(m)$. The continuity of the ring structure of $O$ ensure that $w$ is $O$-linear. Let $j \in \mathbb{N}$. Since $(w_i)_{i \in \mathbb{N}}$ is a Cauchy sequence with respect to the adic topology, there is an $i_0 \in \mathbb{N}$ such that $w_{i_1} - w_{i_2} \in \varpi_k^j \mathrm{Hom}(M, O)$ for any $(i_1, i_2) \in \mathbb{N}^2$ with $i_1, i_2 \geq i_0$. For any $m \in M$, we have $w_{i_1}(m) - w_{i_2}(m) \in \varpi_k^j O$ for any $(i_1, i_2) \in \mathbb{N}^2$ with $i_1, i_2 \geq i_0$, and hence $w_i(m) - w(m) \in \varpi_k^j O$ for any $i \in \mathbb{N}$ with $i \geq i_0$. We obtain $w^{-1}(\varpi_k^j O) = w_{i_0}^{-1}(\varpi_k^j O) \in \mathscr{O}(M)$ by the continuity of $w_{i_0}$. It implies that $w$ is continuous. Let $j \in \mathbb{N}$. By the argument above, there is an $i_0 \in \mathbb{N}$ such that $w_i(m) - w(m) \in \varpi_k^j O$ for any $i \in \mathbb{N}$ with $i \geq i_0$. By Proposition 2.8, we obtain $w_i - w \in \varpi_k^j \mathrm{Hom}(M, O)$ for any $i \geq i_0$. We conclude that $(w_i)_{i \in \mathbb{N}}$ converges to $w$ with respect to the adic topology. $\square$

## 2.2. Continuous Dual Functors

We study $\mathbb{D}$ and $\mathbb{D}_k$. To begin with, we compare them with the Schikhof dual functors. We denote by $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O) \subset \mathrm{Mod}_{\mathrm{l}}^{\mathrm{ch}}$ the full subcategory of compact Hausdorff flat linear topological $O$-modules (i.e. $\mathrm{Mod}_{\mathrm{comp}}^{\mathrm{fl}}(O)$ in [12] §1 p. 2).

**Remark 2.9.** An $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$ lies in $\mathrm{ob}(\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O))$ if and only if there is an isomorphism $M \cong O^I$ for some set $I$ by [4] $\mathrm{VII_B}$ 0.3.8.

A *Banach k-vector space* is a pair $(W, \| \cdot \|)$ of a $k$-vector space $W$ and a complete non-Archimedean norm $\| \cdot \|$ on $W$ (cf. [2] §2.8, [11] Ch. I §3 p. 9, and [9] §2). We denote by $\mathrm{Ban}(k)$ (resp. $\mathrm{Ban}(k)_{\mathrm{con}}$) the category of Banach $k$-vector spaces and bounded (resp. submetric) $k$-linear homomorphisms. Let $V = (W, \| \cdot \|) \in \mathrm{ob}(\mathrm{Ban}(k))$. We put $\underline{V} := W$ and $\|\underline{V}\| := \{\|v\| \mid v \in \underline{V}\}$. We endow $\underline{V}$ with the ultrametric $\underline{V} \times \underline{V} \to [0, \infty),\ (v_1, v_2) \mapsto \|v_1 - v_2\|$, and regard it as a topological $k$-vector space. We put $V(1) := \{v \in \underline{V} \mid \|v\| \leq 1\}$, which forms a lattice of $\underline{V}$. We say that $V$ is *strictly Cartesian* if $\|\underline{V}\| \subset |k|$. We denote by $\mathrm{Ban}(O) \subset \mathrm{Ban}(k)_{\mathrm{con}}$ the full subcategory of strictly Cartesian Banach $k$-vector spaces. We note that this definition looks different from [2] 2.5.1 Definition 1, but in fact is equivalent to it. For more detail, see the following:

**Remark 2.10.** For a set $I$, we denote by $\mathrm{C}_0(I,k) \subset k^I$ the $k$-vector subspace of $F = (F(i))_{i\in I} \in k^I$ with $\#\{i \in I \mid |F(i)| \geq \epsilon\} < \infty$ for any $\epsilon > 0$. Then $\mathrm{C}_0(I,k)$ forms a Banach $k$-vector space with respect to the complete non-Archimedean norm $\|\cdot\|_{\sup} \colon \mathrm{C}_0(I,k) \to [0,\infty)$, $F \mapsto \sup_{i\in I}|F(i)|$. A $V \in \mathrm{ob}(\mathrm{Ban}(k))$ lies in $\mathrm{ob}(\mathrm{Ban}(O))$ if and only if there is an isomorphism $V \cong (\mathrm{C}_0(I,k), \|\cdot\|_{\sup})$ in $\mathrm{Ban}(k)_{\mathrm{con}}$ for some set $I$ by the proof of [11] Proposition 10.2. We note that for any $V \in \mathrm{ob}(\mathrm{Ban}(k))$, there is an isomorphism $V \cong (\mathrm{C}_0(I,k), \|\cdot\|_{\sup})$ in $\mathrm{Ban}(k)$ for some set $I$ by [11] Proposition 10.2.

For an enriched category $C$ over $O$, we denote by $C_k$ the enriched category over $k$ obtained as the localisation of the enrichment of $C$ by $k$. We have natural equivalences $\mathrm{Ban}(O)_k \cong \mathrm{Ban}(k) \cong (\mathrm{Ban}(k)_{\mathrm{con}})_k$ by the proof of [12] Theorem 1.2. We denote by $\mathrm{D}$ the $O$-linear contravariant equivalences $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O) \leftrightarrow \mathrm{Ban}(O)$ and by $\mathrm{D}_k$ the $k$-linear contravariant equivalences $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)_k \leftrightarrow \mathrm{Ban}(k)$ obtained as the Schikhof dual functors for representations of the trivial group (cf. [10] Theorem 4.6 and [12] Theorem 1.2). We remark that there was the assumption $\mathrm{ch}(k) = 0$ in [12], but it can be removed. In [11], the categories were regarded as enriched categories over $\mathbb{Z}$, and hence could be localised only by $\mathbb{Q}$. In this paper, the categories are regarded as enriched categories over $O$, and hence can be localised by $k$. This is the reason why the assumption of $\mathrm{ch}(k)$ can be removed. The following ensures that $\mathbb{D}$ and $\mathbb{D}_k$ are generalisations of $\mathrm{D}$.

**Theorem 2.11.**

(*i*) *Let* $K \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O))$. *Then* $\mathrm{id}_{\mathrm{Hom}(K,k)} \colon K^{\mathbb{D}_k} \to \underline{(K^{\mathrm{D}})}$ *and* $\mathrm{id}_{\mathrm{Hom}(K,O)} \colon K^{\mathbb{D}} \cong K^{\mathrm{D}}(1)$ *are isomorphisms.*

(*ii*) *Let* $V \in \mathrm{ob}(\mathrm{Ban}(O))$. *Then the restriction map* $V^{\mathrm{D}} \to V(1)^{\mathbb{D}}$ *is an isomorphism, and the inclusion* $V^{\mathrm{D}} \hookrightarrow (\underline{V})^{\mathbb{D}_k}$ *is a homeomorphism onto a lattice.*

**Proof.** The assertion (*i*) follows from $K \in \mathrm{K}(K)$, because it implies $K^{\mathbb{D}} = K_{\mathrm{u}}^{\mathbb{D}}$ and $K^{\mathbb{D}_k} = K_{\mathrm{u}}^{\mathbb{D}_k}$. We verify the assertion (*ii*). The restriction map $\iota_1 \colon V^{\mathrm{D}} \to V(1)_{\mathrm{s}}^{\mathbb{D}}$ is bijective, because $V(1) \subset \underline{V}$ is open and $V$ is strictly Cartesian, and is a homeomorphism by the definition of the topology of pointwise convergence. Since the natural embedding $V(1)_{\mathrm{s}}^{\mathbb{D}} \hookrightarrow (\underline{V})_{\mathrm{s}}^{\mathbb{D}_k}$ is a homeomorphism onto the image, $\iota_1$ induces a homeomorphism $\iota_2 \colon V^{\mathrm{D}} \to (\underline{V})_{\mathrm{s}}^{\mathbb{D}_k}$ onto the image. On the other hand, we have $V(1)_{\mathrm{s}}^{\mathbb{D}} = V(1)^{\mathbb{D}}$, because $V(1)$ is adic. Therefore $\iota_1$ gives an isomorphism $V^{\mathrm{D}} \to V(1)^{\mathbb{D}}$. Let $\iota$ denote the natural embedding $V^{\mathrm{D}} \hookrightarrow (\underline{V})^{\mathbb{D}_k}$.

Let $K \subset \mathrm{K}(\underline{V})$. Since $K$ is bounded, there is a $j \in \mathbb{N}$ with $\varpi_k^j K \subset V(1)$. Since $\varpi_k^j K$ is compact, $(\varpi_k^j K)/(\varpi_k^j K \cap \varpi_k^j V(1))$ is a finite group. Take an $S \in 2_{\mathrm{f}}^{\varpi_k^j K}$ of representatives of $(\varpi_k^j K)/(\varpi_k^j K \cap \varpi_k^j V(1))$. Under the natural identification $\mathrm{Hom}((\underline{V}, V(1)), (k, O)) = V^{\mathrm{D}}$, we have $\iota^{-1}(\mathrm{Hom}((\underline{V}, K), (k, O))) = \iota_2^{-1}(\mathrm{Hom}((\underline{V}, \sum_{m \in S} Om), (k, \varpi_k^j O)))$. The right hand side is open by the continuity of $\iota_2$, and it implies the continuity of $\iota$. Since $k$ is Hausdorff, so is $(\underline{V})^{\mathbb{D}_k}$. Since $V^{\mathrm{D}}$ is compact and $(\underline{V})^{\mathbb{D}_k}$ is Hausdorff, $\iota$ is a homeomorphism onto the closed image, which is a lattice because every $m \in (\underline{V})^{\mathbb{D}_k}$ sends $V(1) \subset \underline{V}$ to a bounded $O$-submodule of $k$. □

Let $\kappa \in \{k, \overline{k}, O\}$, and fix a $\mathrm{b}_k$-class $\mathscr{B}$ in the following in this section. We study the relation between continuous duals and several operations in $\mathrm{Mod}_{\mathrm{l}}(O)$ such as the direct sum and the direct product. Let $((M_i)_{i \in I}, (\varphi_{i,j})_{i \leq j}) \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{Mod}_{\mathrm{l}}(O)})$. We put $M := \varinjlim_{i \in I} M_i$ and $M' := \varprojlim_{i \in I} (M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}}$. By Proposition 2.3, we have a natural $\kappa$-linear bijective homomorphism $\iota \colon M^{\mathbb{D}_\kappa}_{\mathscr{B}} \to M'$, which is continuous by the functoriality condition of $\mathscr{B}$.

**Proposition 2.12.** *Suppose that for any $B \in \mathscr{B}(M)$, there is a $B_{i_0} \in \mathscr{B}(M_{i_0})$ with $i_0 \in I$ whose image in $M$ contains $B$. Then $\iota$ is an isomorphism.*

**Proof.** Let $(B, \wp) \in \mathscr{B}(M) \times \mathscr{O}(\kappa)$. Take a $B_{i_0} \in \mathscr{B}(M_{i_0})$ with $i_0 \in I$ whose image in $M$ contains $B$. Then the image of $\mathrm{Hom}((M, B), (\kappa, \wp))$ in $(M_{i_0})^{\mathbb{D}_\kappa}_{\mathscr{B}}$ contains $\mathrm{Hom}((M_{i_0}, B_{i_0}), (\kappa, \wp))$. It implies $\mathrm{Hom}((M, B), (\kappa, \wp)) \in \mathscr{O}(M')$. Thus $\iota$ is open. □

**Corollary 2.13.** *Suppose that $I$ is countable, $M_i$ lies in $\mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O))$ for any $i \in I$, and $\varphi_{i,j}$ is injective for any $(i,j) \in I^2$ with $i \leq j$. Then there is a natural isomorphism $(\varinjlim_{i \in I} M_i)^{\mathbb{D}_\kappa} \cong \varprojlim_{i \in I} (M_i)^{\mathbb{D}_\kappa}$.*

In order to prove Corollary 2.13, we verify that $(M_i)_{i \in I}$ satisfies the assumption in the assertion of Proposition 2.12 in the case $\mathscr{B} = \mathrm{K}$.

**Lemma 2.14.** *Let $(X_j)_{j \in J}$ be a countable inductive system of $\mathrm{T}_1$ topological spaces and continuous injective maps, and $K \subset \varinjlim_{j \in J} X_j$ a compact subspace. Then there is a $j \in J$ such that the image of $X_j$ contains $K$.*

The assertion of Lemma 2.14 for the case $J = \mathbb{N}$ follows from the fact that a compact topological space admits no closed discrete infinite subset. The assertion for the general case can be reduced to the case $J = \mathbb{N}$ by the following:

**Lemma 2.15.** *Every non-empty countable directed set admits either the greatest element or a cofinal infinite subset isomorphic to $\mathbb{N}$ as an ordered set.*

**Proof.** Let $J$ be a non-empty countable directed set without the greatest element. Take a bijective map $\varphi \colon \mathbb{N} \to J$. Since $\mathbb{N}$ is well-ordered, there is an infinite $J_0 \subset J$ such that $\varphi^{-1}|_{J_0}$ is ordered preserving. Then $J_0$ is cofinal and isomorphic to $\mathbb{N}$. $\square$

Let $(M_i)_{i \in I}$ be a family in $\mathrm{Mod}_{\mathrm{l}}(O)$. We denote by $\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}}$ the completion of $\bigoplus_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}}$ with respect to the adic topology, which is naturally embedded in $\prod_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}}$ by Proposition 2.7. We endow $\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}}$ with the topology generated by the set of subsets of the form $m + ((\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}}) \cap \prod_{i \in I} \mathrm{Hom}((M_i, B_i), (O, \varpi_k^j O)))$ for an $(m, (B_i)_{i \in I}, j) \in (\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}}) \times (\prod_{i \in I} \mathscr{B}(M_i)) \times \mathbb{N}$. We denote by $\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}_k}_{\mathscr{B}} \subset \prod_{i \in I}(M_i)^{\mathbb{D}_k}_{\mathscr{B}}$ the $k$-vector subspace generated by the image of $\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}}$ by the natural embedding $\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}} \hookrightarrow \prod_{i \in I}(M_i)^{\mathbb{D}}_{\mathscr{B}} \hookrightarrow \prod_{i \in I}(M_i)^{\mathbb{D}_k}_{\mathscr{B}}$. We endow $\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}_k}_{\mathscr{B}}$ with the topology generated by the set of subsets of the form $m + ((\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}_k}_{\mathscr{B}}) \cap \prod_{i \in I} \mathrm{Hom}((M_i, B_i), (k, \varpi_k^j O)))$ for an $(m, (B_i)_{i \in I}, j) \in (\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}_k}_{\mathscr{B}}) \times (\prod_{i \in I} \mathscr{B}(M_i)) \times \mathbb{Z}$. In the following in this subsection, assume $\kappa \in \{k, O\}$.

**Proposition 2.16.** *Suppose that for any $(B_i)_{i \in I} \in \prod_{i \in I} \mathscr{B}(M_i)$, there is a $B \in \mathscr{B}(\prod_{i \in I} M_i)$ with $\prod_{i \in I} B_i \subset B$. Then there is a natural isomorphism $\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}} \to (\prod_{i \in I} M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}}$.*

**Proof.** Put $M := \prod_{i \in I} M_i$ and $M' := \hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}}$. The $\kappa$-linear homomorphism $M^{\mathbb{D}_\kappa}_{\mathscr{B}} \to \prod_{i \in I}(M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}}$ induced by the zero-extensions $M_i \hookrightarrow M$ with $i \in I$ gives a $\kappa$-linear homomorphism $\iota \colon M^{\mathbb{D}_\kappa}_{\mathscr{B}} \to M'$ by Proposition 2.8. The injectivity of $\iota$ follows from the density of $\bigoplus_{i \in I} M_i$ in $M$. The surjectivity of $\iota$ follows from the fact that for any $\mu = (\mu_i)_{i \in I} \in M'$, $(\sum_{i \in I} \mu_i)(m) := \lim_{F \in 2^I_{\mathrm{f}}} \sum_{i \in F} \mu_i(m_i)$ converges in $\kappa$ for any $m = (m_i)_{i \in I} \in M$ by the definition of $M'$, and the map $\sum_{i \in I} \mu_i \colon M \to \kappa$ is an element of $M^{\mathbb{D}_\kappa}_{\mathscr{B}}$ with $\iota(\prod_{i \in I} \mu_i)$. The continuity of $\iota$ follows from the assumption in the assertion. The openness of $\iota$ follows from the functoriality condition of $\mathscr{B}$. $\square$

We note that we will verify a counterpart of Proposition 2.16 for the case $\kappa = \overline{k}$ in Lemma 4.7. The following is an immediate consequence of Proposition 2.16:

**Corollary 2.17.** *If $I$ is a finite set, then the identification $\hat{\bigoplus}_{i \in I}(M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}} \cong \prod_{i \in I}(M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}}$ induces an isomorphism $\prod_{i \in I}(M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}} \cong (\prod_{i \in I} M_i)^{\mathbb{D}_\kappa}_{\mathscr{B}}$.*

### 2.3. Topologies on Continuous Duals

We study the topologies on the continuous duals. Let $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$. We denote by $\mathscr{O}_{\mathscr{B}}(M)$ the set of $O$-submodules of $M$ of the form $\sum_{B \in \mathscr{B}} M_{B,0}$ for an $(M_{B,0})_{B \in \mathscr{B}(M)} \in \prod_{B \in \mathscr{B}(M)} \mathscr{O}(B)$, which contains $\mathscr{O}(M)$ by the equalities $M_0 = \sum_{B \in \mathscr{B}(M)}(B \cap M_0)$ with $M_0 \in \mathscr{O}(M)$. We say that $M$ is *$\mathscr{B}$-generated* if the equality $\mathscr{O}_{\mathscr{B}}(M) = \mathscr{O}(M)$ holds.

**Proposition 2.18.** *If $M$ is $\mathscr{B}$-generated, then $M_{\mathscr{B}}^{\mathbb{D}_\kappa}$ is complete.*

**Proof.** If $\mathscr{B} = \mathrm{u}$, then the assertion follows from the completeness of $\kappa$. In particular, $\prod_{B \in \mathrm{B}(M)} B_{\mathrm{u}}^{\mathbb{D}_\kappa}$ and hence $\varprojlim_{B \in \mathrm{B}(M)} B_{\mathrm{u}}^{\mathbb{D}_\kappa}$ are complete. Since $M$ is $\mathscr{B}$-generated, the restriction map $M_{\mathscr{B}}^{\mathbb{D}_\kappa} \to \varprojlim_{B \in \mathscr{B}(M)} B_{\mathrm{u}}^{\mathbb{D}_\kappa}$ is a homeomorphic $\kappa$-linear isomorphism. $\square$

We abbreviate $(\cdot)_{\mathscr{B}}^{\mathbb{D}_\kappa} \circ (\cdot)_{\mathscr{B}}^{\mathbb{D}_\kappa}$ to $(\cdot)_{\mathscr{B}}^{\mathbb{D}_\kappa \mathbb{D}_\kappa}$. For any $m \in M$, the map $M_{\mathscr{B}}^{\mathbb{D}_\kappa} \to \kappa$, $\mu \mapsto \mu(m)$ is continuous by $Om \in \mathscr{B}(M)$. We denote by $\eta_{M,\mathscr{B},\kappa} \colon M \to M_{\mathscr{B}}^{\mathbb{D}_\kappa \mathbb{D}_\kappa}$ the $O$-linear homomorphism given by setting $\eta_{M,\mathscr{B},\kappa}(m)(\mu) := \mu(m)$ for each $(m,\mu) \in M \times M_{\mathscr{B}}^{\mathbb{D}_\kappa}$. We simply put $\eta_{M,\kappa} := \eta_{M,\mathrm{K},\kappa}$ and $\eta_M := \eta_{M,O}$. This yields a natural transform $\eta_{\bullet,\mathscr{B},\kappa} \colon \mathrm{Hom}(O,\cdot) \Rightarrow \mathrm{Hom}(O,\cdot) \circ (\cdot)_{\mathscr{B}}^{\mathbb{D}_\kappa \mathbb{D}_\kappa}$, where $\mathrm{Hom}(O,\cdot) \colon \mathrm{Mod}_{\mathrm{l}}(O) \to \mathrm{Set}$ denotes the forgetful functor represented by $O$. We note that $\eta_{M,\mathscr{B},\kappa}$ does not necessarily continuous, and hence $\mathrm{Hom}(O,\cdot)$ can not be removed. We consider a sufficient condition for the continuity of $\eta_{M,\mathscr{B},\kappa}$.

**Definition 2.19.** We say that $\mathscr{B}$ is a $\mathrm{tb}_k$*-class* if $\mathscr{B}(M)$ is a set of totally bounded $O$-submodules for any $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$.

For example, s and K are $\mathrm{tb}_k$-classes, but u is not. Suppose that $\mathscr{B}$ is a $\mathrm{tb}_k$-class in the following in this subsection. We remark that if $M$ is s-generated, then every lattice of $M$ is open. Therefore the following is a partial generalisation of [11] Proposition 6.15:

**Proposition 2.20.** *If $M$ is $\mathscr{B}$-generated, then $\eta_{M,\mathscr{B},\kappa}$ is continuous.*

In order to prove Proposition 2.20, it suffices to show the following:

**Lemma 2.21 (Banach–Steinhaus theorem for $(\cdot)_{\mathscr{B}}^{\mathbb{D}_\kappa}$).** *Suppose that $M$ is Hausdorff (resp. complete). Then every equicontinuous (resp. closed equicontinuous) subset of* $\mathrm{Hom}(M,\kappa)$ *is totally bounded (resp. compact) as a subset of $M_{\mathscr{B}}^{\mathbb{D}_\kappa}$. In addition if $M$ is $\mathscr{B}$-generated, then a subset of* $\mathrm{Hom}(M,\kappa)$ *is totally bounded (resp. compact) as a subset of $M_{\mathscr{B}}^{\mathbb{D}_\kappa}$ if and only if it is equicontinuous (resp. closed and equicontinuous).*

**Proof.** Let $E \subset M_{\mathscr{B}}^{\mathbb{D}_\kappa}$. Put $M_0^{(\wp)} := \bigcap_{\mu \in E} \mu^{-1}(\wp)$ for a $\wp \in \mathscr{O}(\kappa)$. Suppose that $E$ is equicontinuous. Let $(B,\wp_0) \in \mathscr{B}(M) \times \mathscr{O}(\kappa)$. Since $B$ is totally bounded and $M_0^{(\wp_0)} \in \mathscr{O}(M)$, there is a $B_0 \in 2_{\mathrm{f}}^B$ with $B \subset \bigcup_{m \in B_0} m + M_0^{(\wp_0)}$. If $\kappa = O$ or $\kappa = \overline{k}$, then $\kappa/\wp_0$ is a finite set, and hence so is $\iota_{B,\wp_0}(E)$. Suppose $\kappa = k$. Since $B_0$ is a finite set and $M_0^{(\wp_0)}$ is an open neighbourhood of $0 \in M$, there is a $j_1 \in \mathbb{N}$ with $\{\varpi_k^{j_1} m \mid m \in B_0\} \subset B \cap M_0^{(\wp_0)}$. In particular, we have $\sum_{\mu \in E} \mu(B_0) \subset \sum_{\mu \in E} \mu(B) \subset \varpi_k^{-j_1} \wp_0$. Since $\varpi_k^{-j_1} \wp_0/\wp_0$ is a finite set, so is $\iota_{B,\wp_0}(E)$. Take a complete representative $E_{B,\wp_0} \in 2_{\mathrm{f}}^E$ of $\iota_{B,\wp_0}(E)$. By the choice of $E_{B,\wp_0}$, we have $E \subset \bigcup_{\mu \in E_{B,\wp_0}} \mu + \mathrm{Hom}((M,B),(\kappa,\wp_0))$. It implies that $E$

is totally bounded. Suppose that $M$ is $\mathscr{B}$-generated. The assertion for the compactness follows from that for the total boundedness by Proposition 2.18 and [6]. Suppose that $E$ is totally bounded. Let $(B, \wp_0) \in \mathscr{B}(M) \times \mathscr{O}(M)$. Take an $E_{B,\wp_0} \in 2^E_{\mathrm{f}}$ with $E \subset \bigcup_{\mu \in E_{B,\wp_0}} \mu + \mathrm{Hom}((M,B),(\kappa,\wp_0))$, which ensures $B \cap M_0^{(\wp_0)} = \bigcap_{\mu \in E_{B,\wp_0}} (\mu|_B)^{-1}(\wp_0) \in \mathscr{O}(B)$. Therefore we obtain $M_0^{(\wp_0)} \in \mathscr{O}(M)$ because $M$ is $\mathscr{B}$-generated. Thus $E$ is equicontinuous. □

In order to study the duality with respect to $\mathbb{D}$, we prepare a sufficient condition for the openness of $\eta_{M,\mathscr{B},\kappa}$. We say that $M$ is $(\mathscr{B},\kappa)$-*admissible* if $M$ is $\mathscr{B}$-generated and the set $\mathscr{O}_{\mathscr{B}^{\mathbb{D}_\kappa}}(M) := \left\{ \bigcap_{\mu \in B} \mu^{-1}(\wp) \subset M \ \middle|\middle| \ (B,\wp) \in \mathscr{B}(M^{\mathbb{D}_\kappa}_{\mathscr{B}}) \times \mathscr{O}(\kappa) \right\}$ is coinitial in $\mathscr{O}(M)$.

**Theorem 2.22.** *If $M$ is $(\mathscr{B},\kappa)$-admissible, then $\eta_{M,\mathscr{B},\kappa}$ is a quotient map onto the image.*

**Proof.** By Proposition 2.20, it suffices to verify that $\eta_{M,\mathscr{B},\kappa}$ is an open map onto the image. Let $M_0 \in \mathscr{O}(M)$. Since $M$ is $(\mathscr{B},\kappa)$-admissible, there is a $(B,\wp) \in \mathscr{B}(M^{\mathbb{D}_\kappa}_{\mathscr{B}}) \times \mathscr{O}(\kappa)$ with $\bigcap_{\mu \in B} \mu^{-1}(\wp) \subset M_0$. We have $\eta_{M,\mathscr{B},\kappa}(\bigcap_{\mu \in B} \mu^{-1}(\wp)) = \eta_{M,\mathscr{B},\kappa}(M) \cap \mathrm{Hom}((M^{\mathbb{D}_\kappa}_{\mathscr{B}}, B), (\kappa,\wp)) \in \mathscr{O}(\eta_{M,\mathscr{B},\kappa}(M))$, and hence $\eta_{M,\mathscr{B},\kappa}(M_0) \in \mathscr{O}(\eta_{M,\mathscr{B},\kappa}(M))$. Thus $\eta_{M,\mathscr{B},\kappa}$ is an open map onto the image. □

In the study of the compact-open topology, the class of compactly generated topological spaces plays an important role. We introduce a counterpart in $\mathrm{Mod}_{\mathrm{l}}(O)$.

**Definition 2.23.** We put $M_{\mathrm{K}} := \varinjlim_{K \in \mathrm{K}(M)} K$. We say that $M$ is *compactly generated* if the canonical continuous bijective $O$-linear homomorphism $\mathrm{K}(\mathrm{id}_M) \colon M_{\mathrm{K}} \to M$ is a homeomorphism, and is *Fréchet* if $M$ is first countable and complete. We denote by $\mathrm{Mod}_{\mathrm{l}}^{\mathrm{cgh}}(O) \subset \mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O)$ the full subcategory of compactly generated Hausdorff linear topological $O$-modules, and by $\mathrm{Mod}_{\mathrm{l}}^{\mathrm{f}}(O) \subset \mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O)$ the full subcategory of Fréchet linear topological $O$-modules.

We study several criteria for when $M$ is compactly generated. First, a complete linear topological $O$-module lies in $\mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}^{\mathrm{cgh}}(O))$ if and only if its underlying topological space is compactly generated by the proof of [12] Lemma 1.5 i, and hence the following holds:

**Proposition 2.24.** *The category $\mathrm{Mod}_{\mathrm{l}}^{\mathrm{f}}(O)$ is a full subcategory of $\mathrm{Mod}_{\mathrm{l}}^{\mathrm{cgh}}(O)$.*

Secondly, the definition of $\mathscr{O}_{\mathrm{K}}(M)$ yields the following criterion:

**Proposition 2.25.** *Suppose $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O))$. Then $M$ lies in $\mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}^{\mathrm{cgh}}(O))$ if and only if $M$ is $\mathrm{K}$-generated.*

Thirdly, the functoriality condition of K and Proposition 2.2 ensure the following:

**Proposition 2.26.** *For any* $(K_i)_{i \in I} \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{Mod}_1^{\mathrm{c}}(O)})$, $(\varinjlim_{i \in I} K_i)^{\mathrm{h}}$ *lies in* $\mathrm{ob}(\mathrm{Mod}_1^{\mathrm{cgh}}(O))$.

Fourthly, we obtain a sufficient condition using the $\sigma$-compactness.

**Proposition 2.27.** *For any* $((K_i)_{i \in I}, (\varphi_i^j)_{i \leq j}) \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{Mod}_1^{\mathrm{ch}}(O)})$ *such that $I$ is countable and $\varphi_i^j$ is injective for any $(i,j) \in I^2$ with $i \leq j$,* $\varinjlim_{i \in I} K_i$ *lies in* $\mathrm{ob}(\mathrm{Mod}_1^{\mathrm{cgh}}(O))$.

**Proof.** Every compact Hausdorff topological space is $\mathrm{T}_1$ and normal. The inductive limit of $\mathrm{T}_1$ topological spaces and continuous injective maps is again $\mathrm{T}_1$ by the definition of the inductive limit topology. The countable inductive limit of normal topological spaces and homeomorphisms onto the closed images is again normal by Urysohn's lemma, Tietze extension theorem, and Lemma 2.15. A normal topological space is Hausdorff if and only if it is $\mathrm{T}_1$ by definition. Thus $M$ is a normal Hausdorff topological space. Thus the assertion follows from Proposition 2.26. $\square$

We give a basic property of a compactly generated linear topological $O$-module.

**Proposition 2.28.** *Suppose that $M$ is $\sigma$-compact and lies in* $\mathrm{ob}(\mathrm{Mod}_1^{\mathrm{cgh}}(O))$. *For any countable $I \subset \mathrm{K}(M)$ directed by the inclusion relation with $\bigcup_{K \in I} K = M$, the inclusions $K \hookrightarrow M$ with $K \in I$ induce an isomorphism $\varinjlim_{K \in I} K \to M$.*

We note that in addition if $M$ is complete, then $\mathrm{K}(M)$ admits a countable $I \subset \mathrm{K}(M)$ directed by the inclusion relation with $\bigcup_{K \in I} K = M$, because the closed $O$-submodule of a complete linear topological $O$-module generated by a compact subset is again compact by a similar argument to the proof of [12] Lemma 1.5 i. In order to verify Proposition 2.28, we show variants of Baire category theorem.

**Lemma 2.29.** *Let $M \in \mathrm{ob}(\mathrm{Mod}_1(O))$. Suppose that $M$ is completely metrisable or locally compact Hausdorff. Then for any countable $I \subset \mathscr{C}(M)$ directed by the inclusion relation with $M = \sum_{M' \in I} M'$, there is an $M' \in I$ with $M' \in \mathscr{O}(M)$.*

**Proof.** Since $I$ is directed, we have $M = \sum_{M' \in I} M' = \bigcup_{M' \in I} M'$. Since $M$ is completely metrisable or locally compact Hausdorff, $M$ is Čech-complete by [5] Theorem 4.3.26 and [5] §3.9 p. 192. Therefore there is an $M' \in I$ such that $M'$ contains a non-empty open subset by Baire category theorem for a Čech-complete topological space (cf. [1] 59 and [5] 3.9.3 Theorem). By the continuity of the maps $M \to M,\ m \mapsto m - m_0$ with $m_0 \in M'$, $M'$ is open. $\square$

**Lemma 2.30.** *Let $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O))$ and $K \in \mathrm{K}(M)$. For any countable $I \subset \mathscr{C}(M)$ directed by the inclusion relation with $K \subset \sum_{M' \in I} M'$, there is an $M' \in I$ with $K \subset M'$.*

**Proof.** Since $I$ is directed, we have $K = K \cap (\sum_{M' \in I} M') = \sum_{M' \in I}(K \cap M')$. Applying Lemma 2.29 to $K$ and $\{K \cap M' \mid M' \in I\}$, we obtain an $M'_0 \in I$ with $K \cap M'_0 \in \mathscr{O}(K)$. Since $K$ is compact, $K/(K \cap M'_0)$ is a finite set. Take a complete representative $S \subset K$ of the canonical projection $K \twoheadrightarrow K/(K \cap M'_0)$. Take an $M' \in I$ with $M'_0 \subset M'$ and $S \subset M'$. It satisfies $K \subset \sum_{m \in S} m + (K \cap M'_0) \subset M'$. □

**Proof of Proposition 2.28.** Since $M$ is Hausdorff, every $K \in \mathrm{K}(M)$ is closed. Let $K_0 \in \mathrm{K}(M)$. We have $K_0 = \sum_{K \in I}(K_0 \cap K)$ by $M = \bigcup_{K \in I} K$, and hence there is a $K \in I$ with $K_0 \subset K$ by Lemma 2.30. Therefore $I$ is cofinal in $\mathrm{K}(M)$. It implies the assertion. □

## 3. Hahn–Banach Theorem for Locally Convex Spaces

We verify HBT for several classes over $\kappa \in \{k, \overline{k}, O\}$.

### 3.1. Over a Local Field

We put $\mathrm{LC}(k) := \mathrm{Mod}(k) \cap \mathrm{Mod}_{\mathrm{l}}(O)$ and $\mathrm{LC}^{\mathrm{h}}(k) := \mathrm{LC}(k) \cap \mathrm{Mod}^{\mathrm{h}}(k)$. An $L \in \mathrm{ob}(\mathrm{Mod}(k))$ is said to be a *locally convex $k$-vector space* if $L$ lies in $\mathrm{ob}(\mathrm{LC}(k))$ (cf. [11] Ch. I §4 p. 13 and [9] Definition 3.3.7).

**Theorem 3.1 (HBT for $\mathrm{LC}(k)$).** *Let $L \in \mathrm{ob}(\mathrm{LC}(k))$. For any $k$-vector subspace $L_1 \subset L$, the restriction map $\mathrm{Hom}(L, k) \to \mathrm{Hom}(L_1, k)$ is surjective.*

We remark that the assertion of Theorem 3.1 for the case where $L$ is seminormed follows from [7] Theorem 3 and [11] Corollary 9.3, that for the case where $L$ is separable follows from [9] Corollary 4.2.6, and that for the case where $L_1$ is of finite dimension follows from [9] Theorem 4.4.3 (1) and [9] Theorem 4.4.5. In order to prove Theorem 3.1, we verify a certain extension property of open $O$-submodules and recall a basic relation between a seminorm and an open $O$-submodule.

**Proposition 3.2.** *For any $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$ with an $O$-submodule $M' \subset M$, the map $\mathscr{O}(M) \to \mathscr{O}(M')$, $M_0 \mapsto M' \cap M_0$ is surjective.*

**Proof.** Let $M'_0 \in \mathscr{O}(M')$. Take an $M_0 \in \mathscr{O}(M)$ with $M' \cap M_0 \subset M'_0$, and put $M_1 := M_0 + M'_0 \in \mathscr{O}(M)$. We have $M'_0 \subset M' \cap M_1$. Let $m \in M' \cap M_1$. Take an $(m_0, m'_0) \in M_0 \times M'_0$ with $m = m_0 + m'_0$. We have $m_0 \in M' \cap M_0 \subset M'_0$ by $m \in M'$ and $m'_0 \in M'_0 \subset M'$. Thus $m = m_0 + m'_0 \in M'_0 + M'_0 = M'_0$. We conclude $M' \cap M_1 = M'_0$. □

For an $L \in \mathrm{ob}(\mathrm{Mod}(k))$, a *continuous seminorm on* $L$ is a non-Archimedean seminorm $|\cdot|$ (cf. [11] Ch. 1 §2 p. 6 and [9] Definition 3.1.1) on the underlying $k$-vector space of $L$ with $\{l \in L \mid |l| \leq 1\} \in \mathscr{O}(L)$.

**Proposition 3.3.** *Let $L \in \mathrm{ob}(\mathrm{Mod}(k))$ and $L_1 \in \mathscr{O}(L)$. Then the map $|\cdot|_{L_1} \colon L \to [0,\infty)$, $l \mapsto |l|_{L_1} := \inf\{|c| \mid c \in k, l \in cL_1\}$ is a continuous seminorm on $L$ with $\{l \in L \mid |l|_{L_1} \leq 1\} = L_1$.*

**Proof.** We have $L_1 \subset \{l \in L \mid |l|_{L_1} \leq 1\}$ by definition, and hence $|\cdot|_{L_1}$ is a continuous seminorm (cf. [11] Ch. 1 §2 p. 8 and [9] Theorem 3.1.9. (ii)). Since $|k|$ is closed in $[0,\infty)$, we have $|l|_{L_1} = \min\{|c| \mid c \in k^\times, c^{-1}l \in L_1\}$ for any $l \in L$. $\square$

**Proof of Theorem 3.1.** Let $\lambda_0 \in \mathrm{Hom}(L_1,k)$. We have $\lambda_0^{-1}(O) \in \mathscr{O}(L_1)$, and hence there is an $L' \in \mathscr{O}(L)$ with $L_1 \cap L' = \lambda_0^{-1}(O)$ by Proposition 3.2. Then $|\cdot|_{L'}$ forms a continuous seminorm on $L$ with $\{l \in L \mid |l|_{L'} \leq 1\} = L'$ by Proposition 3.3. Thus the assertion follows from [11] Proposition 9.2 by $|\lambda_0(l)| \leq |l|_{L'}$ for any $l' \in L_1$. $\square$

As in real analysis, HBT is deeply related to the reflexivity in the following sense:

**Corollary 3.4.** *Let $\mathscr{B}$ be a $\mathrm{b}_k$-class. Then $\eta_{L,\mathscr{B},k}$ is injective for any $L \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(k))$.*

In order to prove Corollary 3.4, we prepare the following two structure theorems:

**Proposition 3.5.** *Let $K \in \mathrm{ob}(\mathrm{Mod}^{\mathrm{h}}(O))$. Suppose that $K$ is finitely generated as an $O$-module. Then for any $S \in 2^K_{\mathrm{f}}$ generating $K$, the map $\iota_S \colon O^S \twoheadrightarrow K$, $(c_m)_{m \in S} \mapsto \sum_{m \in S} c_m m$ is a quotient map, and hence $K$ lies in $\mathrm{ob}(\mathrm{Mod}^{\mathrm{ch}}_{\mathrm{l}}(O))$.*

**Proposition 3.6.** *Let $W \in \mathrm{ob}(\mathrm{Mod}^{\mathrm{h}}(k))$. Suppose that $W$ is of finite dimension as a $k$-vector space. Then for any $k$-linear basis $S \in 2^W_{\mathrm{f}}$, the map $\iota_{S,k} \colon k^S \to W$, $(c_w)_{w \in S} \mapsto \sum_{w \in S} c_w w$ is an isomorphism, and hence $W$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(k))$.*

**Proof of Proposition 3.5.** The continuity of $\iota_S$ follows from that of the $O$-module structure of $K$. Since $O^S/\ker(\iota_S)$ is compact and $K$ is Hausdorff, the assertion holds. $\square$

**Proof of Proposition 3.6.** The continuity of $\iota_{S,k}$ follows from that of the $k$-vector space structure of $W$. Put $K := \iota_{S,k}(O^S)$. By Proposition 3.5, $\iota_{S,k}|_{O^S} = \iota_S \colon O^S \to W_0$ is a homeomorphism. By $O^S \in \mathscr{O}(k^S)$, it suffices to verify $K \in \mathscr{O}(W)$. Assume $K \notin \mathscr{O}(W)$. Then by the continuity of the maps $K \to$

$K\colon w \mapsto w - w_0$ with $w_0 \in K$, $K$ is not a neighbourhood of 0 in $W$. Therefore there is a net $(x_i)_{i \in I}$ on $W$ lying in $W \backslash K$ converging to 0. For an $i \in I$, we denote by $r_i \in \mathbb{N} \backslash \{0\}$ the smallest integer with $\varpi_k^{r_i} x_i \in K$. Then $(\varpi_k^{r_i})_{i \in I}$ is a net on $O$. Since $O$ is compact, there is a convergent subnet $(\varpi_k^{r_{i_j}})_{j \in J}$ with an order preserving cofinal map $i_\bullet\colon J \to I\colon j \mapsto i_j$. By the continuity of the scalar multiplication $O \times W \to W$, $(\varpi_k^{r_{i_j}} x_{i_j})_{j \in J}$ is a net on $W$ converges to 0. On the other hand, $(\varpi_k^{r_{i_j}} x_{i_j})_{j \in J}$ lies in $K \backslash \varpi_k K$. Since $O^S \backslash \varpi_k O^S$ is compact and $W$ is Hausdorff, $K \backslash \varpi_k K = \iota_{S,k}(O^S \backslash \varpi_k O^S)$ is a closed subspace of $W$. It contradicts $0 \notin K \backslash \varpi_k K$. We conclude $K \in \mathscr{O}(W)$. $\square$

**Proof of Corollary 3.4.** Let $l \in \ker(\eta_{L,\mathscr{B},k})$. Assume that $l \neq 0$. By Proposition 3.6, $\iota_{\{l\},k}\colon k \to kl$ is an isomorphism. By Theorem 3.1, there is an $\lambda \in \mathrm{Hom}(L,k)$ with $\lambda|_{kl} = (\iota_{\{l\},k})$, i.e. $\lambda(l) = 1$. It contradicts $\lambda(l) = \eta_{L,\mathscr{B},k}(l)(\lambda) = 0$, and hence we obtain $l = 0$. It implies $\ker(\eta_{L,\mathscr{B},k}) = \{0\}$. $\square$

**Corollary 3.7.** *Let $L \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(k))$. For any $k$-vector subspace $W \subset L$ of finite dimension as a $k$-vector space, there is a $W^\perp \in \mathscr{C}_k(L)$ such that the addition $\iota\colon W \times W^\perp \to L$ is bijective, the projection $L \twoheadrightarrow W$ induced by $\iota^{-1}$ is a quotient map, and the other projection $L \twoheadrightarrow W^\perp$ induced by $\iota^{-1}$ is continuous.*

**Proof.** Take a $k$-linear basis $S \subset W$. Then $\iota_{S,k}\colon k^S \to W$ is an isomorphism by Proposition 3.6. For an $l \in S$, we denote by $\mathrm{pr}_l\colon k^S \to k$ the $l$-th projection. By Theorem 3.1, there is an $\lambda_l \in \mathrm{Hom}(L,k)$ with $\lambda_l|_W = \mathrm{pr}_l \circ \iota_{S,k}^{-1}$ for each $l \in S$. Put $\lambda := \prod_{l \in S} \lambda_l\colon L \to k^S$ and $W^\perp := \ker(\lambda)$. Since $\lambda$ is continuous, $W^\perp$ lies in $\mathscr{C}_k(L)$. Since $\iota_{S,k}^{-1}$ is bijective, we obtain $L = W \oplus W^\perp$ as a $k$-vector space. Let $\iota\colon W \times W^\perp \to L$ denote the addition as in the assertion. Since $W^\perp$ is closed, $L/W^\perp$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(k))$. Therefore $\iota^{-1}$ induces an isomorphism $L/W^\perp \cong W$ by Proposition 3.6. It implies that the projection $L \twoheadrightarrow W$ induced by $\iota^{-1}$ is a quotient map, which coincides with $\iota_{S,k} \circ \lambda$. The other projection $L \twoheadrightarrow W^\perp$ induced by $\iota^{-1}$ coincides with $\mathrm{id}_L - \iota_{S,k} \circ \lambda$, and hence is continuous. $\square$

### 3.2. Over the Residue Field

Imitating the definition of the notion of a locally convex $k$-vector space, we introduce a counterpart over $\overline{k}$. We put $\mathrm{LC}(\overline{k}) := \mathrm{Mod}(\overline{k}) \cap \mathrm{Mod}_{\mathrm{l}}(O)$. An $\mathscr{L} \in \mathrm{ob}(\mathrm{Mod}(\overline{k}))$ is said to be a *locally convex $\overline{k}$-vector space* if $\mathscr{L}$ lies in $\mathrm{ob}(\mathrm{LC}(\overline{k}))$. For a formal symbol $* \in \{\mathrm{c}, \mathrm{h}, \mathrm{ch}, \mathrm{d}\}$, we put $\mathrm{LC}^*(\overline{k}) := \mathrm{LC}(\overline{k}) \cap \mathrm{Mod}^*(O)$.

**Theorem 3.8 (HBT for $\mathrm{LC}(\overline{k})$).** *Let $\mathscr{L} \in \mathrm{ob}(\mathrm{LC}(\overline{k}))$ and $\mathscr{W} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{d}}(\overline{k}))$. For any $\overline{k}$-vector subspace $\mathscr{L}_1 \subset \mathscr{L}$, the restriction map $\mathrm{Hom}(\mathscr{L},\mathscr{W}) \to \mathrm{Hom}(\mathscr{L}_1,\mathscr{W})$ is surjective.*

**Proof.** Let $\mathscr{A} \in \mathrm{Hom}(\mathscr{L}_1,\mathscr{W})$. Since $\mathscr{W}$ is discrete, $\ker(\mathscr{A})$ is an open $\overline{k}$-vector subspace of $\mathscr{L}_1$. By Proposition 3.2, there is an $\mathscr{L}_0 \in \mathscr{O}(\mathscr{L})$ with $\mathscr{L}_1 \cap \mathscr{L}_0 = \ker(\mathscr{A})$. We have natural embeddings $\mathscr{L}_1/\ker(\mathscr{A}) \hookrightarrow \mathscr{L}/\mathscr{L}_0$

and $\mathscr{L}_1/\ker(\mathscr{A}) \hookrightarrow \mathscr{W}$ of discrete $\overline{k}$-vector spaces. Take a direct summand $\overline{\mathscr{W}}_1 \subset \mathscr{L}/\mathscr{L}_0$ of $\mathscr{L}_1/\ker(\mathscr{A})$ as a $\overline{k}$-vector space. Then the composite $\mathscr{L} \twoheadrightarrow (\mathscr{L}/\mathscr{L}_0)/\overline{\mathscr{W}}_1 \cong \mathscr{L}_1/\ker(\mathscr{A}) \hookrightarrow \mathscr{W}$ extends $\mathscr{A}$, and is continuous because its kernel contains $\mathscr{L}_0 \in \mathscr{O}(\mathscr{L})$. □

We remark that $\overline{k}$ can be regarded as a trivial valuation field, and hence as a spherically complete valuation field. Therefore Theorem 3.8 for special cases follows from the same references right after Theorem 3.1 other than [11] Corollary 9.3, where the valuation is assumed to be non-trivial. Let $\mathscr{B}$ be a $\mathrm{b}_k$-class. Let $\mathscr{L} \in \mathrm{ob}(\mathrm{LC}(\overline{k}))$ in the following in this subsection. Theorem 3.8 immediately implies the following two corollaries in a similar way as the proof of Corollary 3.4 and Corollary 3.7, because every Hausdorff finite set is discrete:

**Corollary 3.9.** *If $\mathscr{L} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$, then $\mathscr{B}_{\mathscr{L},\mathscr{B},\overline{k}}$ is injective.*

**Corollary 3.10.** *For any $\overline{k}$-vector subspace $\mathscr{L}_1 \subset \mathscr{L}$, there is an $\mathscr{L}_1^{\perp} \in \mathscr{C}(\mathscr{L})$ with $\mathscr{L} = \mathscr{W}_1 \oplus \mathscr{W}_1^{\perp}$ as a $\overline{k}$-vector space.*

Theorem 3.8 gives the following structure theorem for $\mathrm{ob}(\mathrm{LC}^{\mathrm{ch}}(\overline{k}))$:

**Corollary 3.11.** *A $\mathscr{K} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{ch}}(\overline{k}))$ if and only if there is an $S \subset \mathrm{Hom}(\mathscr{K},\overline{k})$ such that $\prod_{w \in S} w \colon \mathscr{K} \to \overline{k}^S$ is an isomorphism.*

We remark that Corollary 3.11 immediately follows from [4] $\mathrm{VII_B}$ 0.3.8 if we use a notion of a completed tensor product for topological $\overline{k}$-vector spaces. We give an alternative proof with no use of completed tensor products. By Corollary 3.9, we have $\bigcap_{w \in \mathrm{Hom}(\mathscr{K},\overline{k})} \ker(w) = \{0\}$. Therefore it suffices to show the following:

**Lemma 3.12.** *Let $\mathscr{K} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{c}}(\overline{k}))$. Let $I_1$ be a set. For any $(w_i)_{i \in I_1} \in \mathrm{Hom}(\mathscr{K},\overline{k})^{I_1}$ with $\bigcap_{i \in I_1} \ker(w_i) = \{0\}$, there is an $I_0 \subset I_1$ such that $\prod_{i \in I_0} w_i \colon \mathscr{K} \to \overline{k}^{I_0}$ is an isomorphism.*

**Proof.** We denote by $\mathscr{I}$ the set of subsets $I \subset I_1$ such that $\prod_{i \in I} w_i \colon \mathscr{K} \to \overline{k}^I$ is surjective. Then $\mathscr{I}$ is directed by inclusions, and contains $\emptyset$. Let $\mathscr{I}_0 \subset \mathscr{I}$ be a totally ordered subset. Put $I := \bigcup_{I_0 \in \mathscr{I}_0} I_0$. We verify the surjectivity of $\prod_{i \in I} w_i$. Since $\mathscr{K}$ is compact and $\overline{k}^I$ is Hausdorff, it suffices to show that the image of $\prod_{i \in I} w_i$ is dense in $\overline{k}^I$. Let $(c_i)_{i \in I} \in \overline{k}^I$ and $F \in 2^I_{\mathrm{f}}$. We construct an $m \in \mathscr{K}$ with $w_i(m) = c_i$ for any $i \in F$. Since $F$ is a finite set, there is an $I_0 \in \mathscr{I}_0$ with $F \subset I_0$. Since $\prod_{i \in I_0} w_i$ is surjective, there is an $m \in \mathscr{K}$ with $w_i(m) = c_i$ for any $i \in I_0$. Therefore the image of $\prod_{i \in I} w_i$ is dense in $\overline{k}^I$. We obtain $I \in \mathscr{I}_0$. It implies that $\mathscr{I}$ is a non-empty inductive ordered set. By Zorn's lemma, $\mathscr{I}$ admits a maximal element $I_0$. We verify the injectivity of $\prod_{i \in I_0} w_i$. Assume $\ker(\prod_{i \in I_0} w_i) \neq \{0\}$. Take an $m \in \ker(\prod_{i \in I_0} w_i) \setminus \{0\}$. Since $\bigcap_{i \in I_1} \ker(w_i) = \{0\}$, there is an $i_1 \in I_1$ with $w_{i_1}(m) \neq 0$. Put $I := I_0 \sqcup \{i_1\} \subset I_1$.

Let $(c_i)_{i \in I} \in \overline{k}^I$. By the surjectivity of $\prod_{i \in I_0} w_i$, there is an $m' \in \mathscr{K}$ with $(w_i(m'))_{i \in I_0} = (c_i)_{i \in I_0}$. Then the image of $m' + w_{i_1}(m)^{-1}(c_{i_1} - w_{i_1}(m'))m$ by $\prod_{i \in I} w_i$ coincides with $(c_i)_{i \in I}$. We obtain $I \in \mathscr{I}$, and it contradicts the maximality of $I_0$. It implies $\ker(\prod_{i \in I_0} w_i) = \{0\}$. Since $\mathscr{K}$ is compact and $\overline{k}^{I_0}$ is Hausdorff, $\prod_{i \in I_0} w_i$ is an isomorphism. $\square$

In order to apply Theorem 3.8 to a study of a direct sum decomposition in $\mathrm{LC}(\overline{k})$, we recall a general fact on the quotient topology.

**Lemma 3.13.** *Let $G$ be a topological group, $H \subset G$ a closed subgroup, and $K \subset G$ a compact subset which forms a complete system of representatives of the canonical projection $G \twoheadrightarrow G/H$. Then the multiplication $K \times H \to G$ is a homeomorphism.*

**Proof.** Let $\iota \colon K \times G \to G$ denote the multiplication. Then $\iota$ coincides with composite of the homeomorphism $K \times G \to K \times G$, $(k, g) \mapsto (k, kg)$ and the second projection $\mathrm{pr}_2 \colon K \times G \to G$. Since $\iota$ is continuous, so is $\iota|_{K \times H}$. Since $\mathrm{pr}_2$ is closed by the compactness of $K$, and hence so is $\iota$. Since $H$ is closed, so is $\iota|_{K \times H}$. $\square$

**Proposition 3.14.** *For any compact Hausdorff $\overline{k}$-vector subspace $\mathscr{K} \subset \mathscr{L}$, there is a $\mathscr{K}^{\perp} \in \mathscr{C}(\mathscr{L})$ such that the addition $\mathscr{K} \times \mathscr{K}^{\perp} \to \mathscr{L}$ is an isomorphism.*

**Proof.** By Corollary 3.11, there is an $S \subset \mathrm{Hom}(\mathscr{K}, \overline{k})$ such that $\prod_{w \in S} w \colon \mathscr{K} \to \overline{k}^S$ is an isomorphism. Every $w \in S$ extends to an $\lambda_w \in \mathrm{Hom}(\mathscr{L}, \overline{k})$ by Theorem 3.8. Then $\mathscr{K}^{\perp} := \ker(\prod_{w \in S} \lambda_w)$ is a desired one by Lemma 3.13. $\square$

**Corollary 3.15.** *If $\mathscr{L} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$, then for any $\mathscr{L}_1 \in \mathscr{C}(\mathscr{L})$, the restriction map $\mathscr{L}^{\mathbb{D}_{\overline{k}}} \to \mathscr{L}_1^{\mathbb{D}_{\overline{k}}}$ is a quotient map.*

**Proof.** We denote by $\pi \colon \mathscr{L}^{\mathbb{D}_{\overline{k}}} \to \mathscr{L}_1^{\mathbb{D}_{\overline{k}}}$ the restriction map, which is a continuous surjective map by Theorem 3.8. It suffices to verify that $\pi$ is open. Let $\mathscr{K} \in \mathrm{K}(\mathscr{L})$. Since $\mathscr{L}$ is Hausdorff, so is $\mathscr{K}$. Put $\mathscr{K}_0 := \mathscr{L}_1 \cap \mathscr{K}$. By $\mathscr{L}_1 \in \mathscr{C}(\mathscr{L})$ and $\mathscr{K} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{c}}(\overline{k}))$, we have $\mathscr{K}_0 \in \mathrm{ob}(\mathrm{LC}^{\mathrm{c}}(\overline{k}))$. Let $\overline{\lambda}_0 \in \mathrm{Hom}((\mathscr{L}_1, \mathscr{K}_0), (\overline{k}, \{0\})))$. Take a $\lambda_0 \in \mathscr{L}^{\mathbb{D}_{\overline{k}}}$ with $\pi(\lambda_0) = \overline{\lambda}_0$. We have $\lambda_0(l) = \overline{\lambda}_0(l) = 0$ for any $l \in \mathscr{K}_0$. By Proposition 3.14, there is a $\mathscr{K}_0^{\perp} \in \mathscr{C}(\mathscr{K})$ such that the addition $\mathscr{K}_0 \times \mathscr{K}_0^{\perp} \to \mathscr{K}$ is an isomorphism. Since $\mathscr{K}$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{ch}}(\overline{k}))$, so does $\mathscr{K}_0^{\perp}$. We have $\mathscr{L}_1 \cap \mathscr{K}_0^{\perp} = \{0\}$ by $\mathscr{L}_1 \cap \mathscr{K} = \mathscr{K}_0$. By Lemma 3.13, the addition $\mathscr{K}_0^{\perp} \times \mathscr{L}_1 \to \mathscr{K}_0^{\perp} + \mathscr{L}_1 \subset \mathscr{L}$ is an isomorphism. Therefore the map $\overline{\lambda}_1 \colon \mathscr{K}_0^{\perp} + \mathscr{L}_1 \to \overline{k} \colon l_1 + l_2 \mapsto \lambda(l_2)$ is continuous. By Theorem 3.8, $\overline{\lambda}$ extends to an $\lambda_1 \in \mathscr{L}^{\mathbb{D}_{\overline{k}}}$. We have $\pi(\lambda_0 - \lambda_1) = \overline{\lambda}_0$, and $\lambda_0 - \lambda_1 \in \mathrm{Hom}((\mathscr{L}, \mathscr{K}), (\overline{k}, \{0\})))$. It implies $\mathrm{Hom}((\mathscr{L}_1, \mathscr{K}_0), (\overline{k}, \{0\}))) \subset \pi(\mathrm{Hom}((\mathscr{L}, \mathscr{K}), (\overline{k}, \{0\})))$. Thus $\pi$ is open. $\square$

### 3.3. Over the Valuation Ring

An $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$ is said to be a *locally convex $O$-module* if $M$ is bounded and $\Pi_M$ is a homeomorphism onto the closed image. We denote by $\mathrm{LC}(O) \subset \mathrm{Mod}_{\mathrm{l}}(O)$ the full subcategory of locally convex $O$-modules, and put $\mathrm{LC}^{\mathrm{h}}(O) := \mathrm{LC}(O) \cap \mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O)$. An $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$ lies in $\mathrm{ob}(\mathrm{LC}(O))$ if and only if $M$ is bounded and flat and $\Pi_M$ is a closed map. To begin with, we study a relation between locally convexity over $k$ and $O$. Let $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$. We put $M_k := \varinjlim((M)_{i \in \mathbb{N}}, (\Pi_M^{j-i})_{i \leq j}) \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$. We denote by $\iota_{M,j} \in \mathrm{Hom}(M, M_k)$ the $j$-th canonical morphism for a $j \in \mathbb{N}$.

**Proposition 3.16.** *The correspondence $M \rightsquigarrow M_k$ gives a left adjoint functor $\mathrm{Mod}_{\mathrm{l}}(O) \to \mathrm{LC}(k)$ of the inclusion $\mathrm{LC}(k) \hookrightarrow \mathrm{Mod}_{\mathrm{l}}(O)$.*

**Proof.** The adjoint property of $M_k$ follows from Proposition 2.3. Therefore it suffice to verify that $\Pi_{M_k}$ is open by the characterisation of the essential image of the embedding $\mathrm{Mod}(k) \hookrightarrow \mathrm{Mod}(O)$ in the beginning of §2. Let $L_0 \in \mathscr{O}(M_k)$. We have $\iota_{M,j+1}^{-1}(L_0) \subset \iota_{M,j}^{-1}(\varpi_k L_0)$ and hence $\iota_{M,j}^{-1}(\varpi_k L_0) \in \mathscr{O}(M)$ by $\iota_{M,j+1}^{-1}(L_0) \in \mathscr{O}(M)$ for any $j \in \mathbb{N}$. Therefore we obtain $\varpi_k L_0 \in \mathscr{O}(M_k)$. Thus $\Pi_{M_k}$ is open. □

**Proposition 3.17.** *For any $L \in \mathrm{ob}(\mathrm{LC}(k))$, every lattice of $L$ lies in $\mathrm{ob}(\mathrm{LC}(O))$. Conversely, for any $M \in \mathrm{ob}(\mathrm{LC}(O))$, $M_k$ lies in $\mathrm{ob}(\mathrm{LC}(k))$ and $\iota_{M,0}$ is a homeomorphism onto a lattice.*

**Proof.** The first assertion follows from the continuity of the scalar multiplication. The flatness of $M$ ensures the injectivity of $\iota_{M,j}$ for any $j \in \mathbb{N}$. By Proposition 3.16, $M_k$ lies in $\mathrm{ob}(\mathrm{LC}(k))$. By $M \in \mathrm{ob}(\mathrm{LC}(O))$, $\varpi_k^j M$ is closed in $M$, and $\Pi_M^j \colon M \hookrightarrow M$ is a homeomorphic $O$-linear isomorphism onto $\varpi_k^j M$ for any $j \in \mathbb{N}$. We have $\iota_{M,j}^{-1}(\iota_{M,0}(M)) = \varpi_k^j M$ for any $j \in \mathbb{N}$, and hence $\iota_{M,0}(M)$ is closed in $M_k$. Since $M$ is bounded, so is $\iota_{M,0}(M)$. Since $\iota_{M,0}(M)$ generates $M_k$ as a $k$-vector space, $\iota_{M,0}(M)$ is a lattice of $M_k$. Therefore it suffices to verify that $\iota_{M,0}$ is an open map onto $\iota_{M,0}(M)$. Let $M_0 \in \mathscr{O}(M)$. Applying Proposition 3.2 to $\Pi_l$ in a repetitive way, we obtain an $(M_j)_{j \in \mathbb{N}} \in \mathscr{O}(M)^{\mathbb{N}}$ with $\varpi_k M \cap M_{j+1} = \varpi_k M_j$. The family $(\iota_{M,j}(M_j))_{j \in \mathbb{N}}$ forms an increasing sequence with $(\iota_{M,j_1}(M) \cap \iota_{M,j_2}(M_{j_2}) = \iota_{M,j_1}(M_{j_1})$ for any $(j_1, j_2) \in \mathbb{N} \times \mathbb{N}$ with $j_1 \leq j_2$. Therefore we get $\iota_{M,0}(M_0) = \iota_{M,0}(M) \cap \sum_{j \in \mathbb{N}} \iota_{M,j}(M_j) \in \mathscr{O}(\iota_{M,0}(M))$. Thus $\iota_{M,0}$ is an open map onto $\iota_{M,0}(M)$. □

Proposition 3.2, Proposition 3.18, and the last assertion of Proposition 3.17 immediately imply the following:

**Corollary 3.18.** *The correspondence $M \rightsquigarrow M_k$ gives a left adjoint functor $\mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O) \to \mathrm{LC}^{\mathrm{h}}(k)$ of the inclusion $\mathrm{LC}^{\mathrm{h}}(k) \hookrightarrow \mathrm{Mod}_{\mathrm{l}}^{\mathrm{h}}(O)$.*

The following helps us to use Theorem 2.11 in a study of $\mathrm{LC}(O)$:

**Proposition 3.19.** *The category* $\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)$ *is a full subcategory of* $\mathrm{LC}(O)$.

**Proof.** Let $K \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O))$. Since $K$ is compact, it is bounded. Since $K$ is flat, the map $\Pi_K$ is injective. Since $K$ is compact and Hausdorff, $\Pi_K$ is a homeomorphism onto the closed image. Thus $K$ lies in $\mathrm{ob}(\mathrm{LC}(O))$. □

Let $N$ be an $O$-module. An $O$-submodule $N_0 \subset N$ is said to be *adically saturated* if $N_0 \cap \varpi_k N = \varpi_k N_0$. We denote by $\mathrm{AS}(N)$ the set of adically saturated $O$-submodules. We put $\mathrm{ASK}(M) := \mathrm{AS}(M) \cap \mathrm{K}(M)$ and $\mathrm{AS}\mathscr{C}(M) := \mathrm{AS}(M) \cap \mathscr{C}(M)$. Now we state HBT's over $O$ with respect to $\mathrm{Hom}(\cdot, O)$. We have to assume several conditions on submodules unlike the result in [8] Theorem 6.25 for $\mathrm{Hom}(\cdot, k/O)$. Let $M \in \mathrm{ob}(\mathrm{LC}(O))$ and $K \in \mathrm{AS}\mathscr{C}(M)$ in the following in this subsection.

**Theorem 3.20 (HBT for $\mathrm{LC}(O)$).** *The coimage of the restriction map* $\mathrm{Hom}(M, O) \to \mathrm{Hom}(K, O)$ *is a torsion $O$-module.*

**Theorem 3.21 (HBT for $\mathrm{LC}^{\mathrm{h}}(O)$).** *If* $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(O))$ *and* $K \in \mathrm{ASK}(M)$, *then the restriction map* $\mathrm{Hom}(M, O) \to \mathrm{Hom}(K, O)$ *is surjective.*

We emphasise that Theorem 3.20 and Theorem 3.21, which are HBT's over $O$, do not directly follow from known HBT's over $k$ such as [7] Theorem 3. The following is a key proposition for HBT's over $O$:

**Proposition 3.22.** *The embedding* $\iota_{K,M} \colon K_k \hookrightarrow M_k$ *induced by the inclusion* $K \hookrightarrow M$ *is a homeomorphism onto the closed image.*

In order to verify Proposition 3.22, we prepare the following compatibility of the relative topologies:

**Lemma 3.23.** *Let* $(X_i)_{i \in I}$ *be an inductive system of topological spaces, and* $Y \subset \varinjlim_{i \in I} X_i$ *a closed subspace. Then the canonical map* $\varinjlim_{i \in I}(X_i \cap Y) \to Y$ *is a homeomorphism, where* $X_i \cap Y$ *denotes the preimage of* $Y$ *in* $X_i$ *for an* $i \in I$.

The assertion immediately follows from the fact that the preimage of a closed subspace of $\varinjlim_{i \in I}(X_i \cap Y)$ in $X_{i_0} \cap Y$ is closed in $X_{i_0}$ for any $i_0 \in I$.

**Proof of Proposition 3.22.** By Proposition 3.17, $\iota_{M,j}$ is a homeomorphism onto the image for any $j \in \mathbb{N}$. By $K \in \mathrm{AS}(M)$, we have $\iota_{M,j}^{-1}(k\iota_{M,0}(K)) = K \in \mathscr{C}(M)$ for any $j \in \mathbb{N}$, and hence $k\iota_{M,0}(K) \in \mathscr{C}_k(M_k)$. Therefore $\iota_{K,M}$ is a homeomorphism onto $k\iota_{M,0}(K)$ is a homeomorphism by Lemma 3.23. □

**Proof of Theorem 3.20.** By Proposition 3.22, $\iota_{K,M} \colon K_k \to k\iota_{M,0}(K)$ is an isomorphism. Let $w \in \mathrm{Hom}(K, O)$. By Theorem 3.1, $w_k \circ \iota_{K,M}$ extends to a $\lambda \in \mathrm{Hom}(M_k, k)$. The boundedness of $M$ ensures that of $\lambda(\iota_{M,0}(M))$. Therefore there is a $j \in \mathbb{N}$ with $\lambda(\iota_{M,0}(M)) \subset \varpi_k^{-j} O$. Then $\mu := (\varpi_k^j \lambda) \circ \iota_{M,0}$ lies in $\mathrm{Hom}(M, O)$ and satisfies $\mu|_K = \varpi_k^j w$. $\square$

In order to verify Theorem 3.21, we prepare the two more lemmata.

**Lemma 3.24.** *Let $K_1 \in \mathrm{K}(M)$. Then every $K_0 \in \mathscr{O}(K_1)$ satisfies $K_0 + \varpi_k^j M \in \mathscr{O}(K_1 + \varpi_k^j M)$ for any $j \in \mathbb{N}$.*

**Proof.** Put $\mathscr{K} := (K_1 + \varpi_k^j M)/(K_1 \cap \varpi_k^j M)$. Since $K_1$ is compact, so is its image $\overline{K}_1$ in $\mathscr{K}$. Since $\varpi_k^j M$ is closed by $M \in \mathrm{ob}(\mathrm{LC}(O))$, so is its image $\overline{M}$ in $\mathscr{K}$ by $K \cap \varpi_k^j M \subset \varpi_k^j M$. Therefore the addition $A \colon K_1 \times \varpi_k^j M \twoheadrightarrow K + \varpi_k^j M$ induces an isomorphism $\iota \colon \overline{K}_1 \times \overline{M} \to \mathscr{K}$ by Lemma 3.13. Since the image $\overline{K}_0$ of $K_0$ in $\overline{K}_1$ is open by $K_0 \subset \{m \in K_1 \mid m + (K_1 \cap \varpi_k^j M) \in \overline{K}_0\}$, so is $\iota(\overline{K}_0 \times \overline{M})$ in $\mathscr{K}$. Its preimage in $K_1 + \varpi_k^j M$ coincides with $K_0 + \varpi_k^j M$, and hence we obtain $K_0 + \varpi_k^j M \in \mathscr{O}(K_1 + \varpi_k^j M)$. $\square$

In the following in this subsection, we assume $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(O))$ and $K \in \mathrm{K}(M)$.

**Lemma 3.25.** *For any $L_0 \in \mathscr{O}(K_k)$, $\iota_{K,M}(L_0) + \iota_{M,0}(M) \subset M_k$ lies in $\mathscr{O}(\iota_{K,M}(K_k) + \iota_{M,0}(M))$.*

**Proof.** Put $M := \iota_{K,M}(K_k) + \iota_{M,0}(M)$ and $M_0 := \iota_{K,M}(L_0) + \iota_{M,0}(M)$. Let $j \in \mathbb{N}$. By $K \in \mathrm{AS}(M)$, we have $\iota_{M,j}^{-1}(M) = K + \varpi_k^j M$ and $\iota_{M,j}^{-1}(M_0) = \iota_{K,j}^{-1}(L_0) + \varpi_k^j M$. It implies $\iota_{M,j}^{-1}(M_0) \in \mathscr{O}(\iota_{M,j}^{-1}(M))$ by Lemma 3.24. Therefore it suffices to show $M \in \mathscr{C}(M_k)$ by Lemma 3.23. Let $j \in \mathbb{N}$ again. By $M \in \mathrm{ob}(\mathrm{LC}(O))$, we have $\varpi_k^j M \in \mathscr{C}(M)$. Since $M$ is Hausdorff, so is $M/\varpi_k^j M$. Since $K$ is compact, so is its image $\overline{K}$ in $M/\varpi_k^j M$. Since $M/\varpi_k^j M$ is Hausdorff, we obtain $\overline{K} \in \mathscr{C}(M/\varpi_k^j M)$, and hence $\iota_{M,j}^{-1}(M) = K + \varpi_k^j M = \{l \in M \mid l + \varpi_k^j M \in \overline{K}\} \in \mathscr{C}(M)$. It implies $M \in \mathscr{C}(M_k)$. $\square$

**Proof of Theorem 3.21.** Let $w \in \mathrm{Hom}(K, O)$. By Lemma 3.25, $\iota_{K,M}(w_k^{-1}(O)) + \iota_{M,0}(M) \subset M_k$ lies in $\mathscr{O}(\iota_{K,M}(K_k) + \iota_{M,0}(M))$. By $K \in \mathrm{AS}(M)$, we have $\iota_{K,M}^{-1}(\iota_{M,0}(M)) = \iota_{K,0}(K)$, and hence $\iota_{K,M}^{-1}(\iota_{K,M}(w_k^{-1}(O)) + \iota_{M,0}(M)) = w_k^{-1}(O)$. By Proposition 3.2, there is an $L_0 \in \mathscr{O}(M_k)$ with $(\iota_{K,M}(K_k) + \iota_{M,0}(M)) \cap L_0 = \iota_{K,M}(w_k^{-1}(O)) + \iota_{M,0}(M)$. In particular, we have $\iota_{M,0}(M) \subset L_0$ and $\iota_{K,M}^{-1}(L_0) = w_k^{-1}(O)$. It implies $|w_k(l)| \leq |\iota_{K,M}(l)|_{L_0}$ for any $l \in K_k$ by Proposition 3.3. Therefore there is a $\lambda \in \mathrm{Hom}(M_k, k)$ with $\lambda \circ \iota_{M,K} = w_k$ and $|\lambda(l)| \leq |l|_{L_0}$ for any $l \in M_k$ by Proposition 3.22 and HBT for seminormed $k$-vector spaces ([7] Theorem 3 and [11] Proposition 9.2). By $\iota_{M,0}(M) \subset L_0$, we obtain $\lambda(\iota_{M,0}(m)) \in O$ for any $m \in M$. Thus $\mu := \lambda \circ \iota_{M,0}$ lies in $\mathrm{Hom}(M, O)$ and satisfies $\mu|_K = w$. $\square$

Theorem 3.21 immediately implies the following two corollaries in a similar way as the proof of Corollary 3.4 and Proposition 3.14 by Remark 2.9:

**Corollary 3.26.** *Let $\mathscr{B}$ be a* $\mathrm{b}_k$*-class. Then* $\eta_{M,\mathscr{B},O}$ *is injective.*

**Corollary 3.27.** *There is a* $K^\perp \in \mathrm{AS}\mathscr{C}(M)$ *such that the addition* $K \times K^\perp \to M$ *is an isomorphism.*

We study a compatibility between the continuous duals and the reduction. For an $N \in \mathrm{ob}(\mathrm{Mod}(O))$, we put $N_{\overline{k}} := N/\varpi_k N \in \mathrm{Mod}(\overline{k})$. By the definition of $\mathrm{LC}(\overline{k})$, the correspondence $N \rightsquigarrow N_{\overline{k}}$ gives a left adjoint functor $\mathrm{Mod}_{\mathrm{l}}(O) \to \mathrm{LC}(\overline{k})$ of the inclusion $\mathrm{LC}(\overline{k}) \hookrightarrow \mathrm{Mod}_{\mathrm{l}}(O)$. We put $\mathrm{LC}^{\mathrm{f}}(\kappa) := \mathrm{Mod}_{\mathrm{l}}^{\mathrm{f}}(O) \cap \mathrm{LC}^{\mathrm{h}}(\kappa)$ for a $\kappa \in \{k, \overline{k}, O\}$.

**Theorem 3.28.** *If* $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{f}}(O))$*, then the natural map* $(M^{\mathbb{D}})_{\overline{k}} \to (M_{\overline{k}})^{\mathbb{D}_{\overline{k}}}$ *is an isomorphism.*

**Proof.** The canonical projection $\pi_1 \colon M \twoheadrightarrow M_{\overline{k}}$ induces a $\pi_1^{\mathbb{D}_{\overline{k}}} \in \mathrm{Hom}((M_{\overline{k}})^{\mathbb{D}_{\overline{k}}}, M^{\mathbb{D}_{\overline{k}}})$. The surjectivity of $\pi$ ensures the injectivity of $\pi_1^{\mathbb{D}_{\overline{k}}}$, and the universality of the quotient ensures the surjectivity of $\pi_1^{\mathbb{D}_{\overline{k}}}$. By the definition of the compact-open topology, $\pi_1^{\mathbb{D}_{\overline{k}}}$ is open, and hence is an isomorphism. The canonical projection $\pi_2 \colon O \twoheadrightarrow \overline{k}$ induces a $\pi_3 \in \mathrm{Hom}(M^{\mathbb{D}}, M^{\mathbb{D}_{\overline{k}}})$. By the universality of the quotient, $\pi_3$ induces a $\pi_4 \in \mathrm{Hom}((M^{\mathbb{D}})_{\overline{k}}, M^{\mathbb{D}_{\overline{k}}})$. Since $\pi_2$ is open, so are $\pi_3$ and hence $\pi_4$ by the definition of the compact-open topology. Proposition 2.8 ensures the injectivity of $\pi_4$, and hence is a homeomorphism onto the image. Let $\overline{\lambda} \in (M_{\overline{k}})^{\mathbb{D}_{\overline{k}}}$. If $\overline{\lambda} = 0$, then we have $\overline{\lambda} = \pi_4(0) \in \mathrm{im}(\pi_4)$. Suppose $\overline{\lambda} \neq 0$. Take an $m_0 \in M$ with $(\overline{\lambda} \circ \pi_1)(m_0) = 1 \in \overline{k}$. Put $M_0 := \ker(\overline{\lambda} \circ \pi_1) \in \mathscr{O}(M)$. We have $Om_0 + M_0 = M$ by the choice of $m_0$. By Proposition 3.2 and Proposition 3.17, there is an $M_1 \in \mathscr{O}(M_k)$ with $\iota_{M,0}^{-1}(M_1) = M_0$. We have $\varpi_k M \subset M_0$, and hence $\iota_{M,0}(M) \subset \varpi_k^{-1} M_1$. We have $\{l \in M_k \mid |l|_{\varpi_k^{-1} M_1} \leq 1\} = \varpi_k^{-1} M_1$ by Proposition 3.3. Therefore by Proposition 3.5 and HBT for seminormed $k$-vector spaces ([11] Proposition 9.2), there is a $\mu_k \in (M_k)^{\mathbb{D}_k}$ with $\mu_k(m_0) = 1$ and $|\mu_k(l)| \leq |l|_{\varpi_k^{-1} M_1}$ for any $l \in M_k$. By $\iota_{M,0}(M) \subset \varpi_k^{-1} M_1$ and $M_0 \subset M_1$, we have $\mu_k(\iota_{M,0}(M)) \subset O$ and $\iota_{M,0}(M_0) \subset \varpi_k O$. Therefore $\mu := \mu_k \circ \iota_{M,0}$ lies in $\mathrm{Hom}(M, O)$, and satisfies $\mu(m_0) = 1$ and $\mu(M_0) \subset \varpi_k O$. In particular, we have $\overline{\lambda} = \pi_4(\mu) \in \mathrm{im}(\pi_4)$. It implies that $\pi_4$ is surjective, and hence is an isomorphism. We obtain an isomorphism $\pi_1^{\mathbb{D}_{\overline{k}}} \circ \pi_4^{-1} \colon (M^{\mathbb{D}})_{\overline{k}} \to (M_{\overline{k}})^{\mathbb{D}_{\overline{k}}}$. □

## 4. Reflexivity for Locally Convex Spaces

We give several criteria for the reflexivity with respect to the continuous duals. As a result, we obtain good full subcategories of $\mathrm{Mod}_{\mathrm{l}}(O)$ with self-duality.

### 4.1. Over a Local Field

We study the reflexivity for $\mathbb{D}_k$. Let $L \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(k))$ in this subsection.

**Theorem 4.1.** *If* $L$ *admits a compact lattice, then* $\eta_{L,k}$ *is bijective.*

**Proof.** Let $K \in \mathrm{K}(L)$ be a lattice. Then we have $K \in \mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O)$. Let $\eta\colon K \to K^{\mathrm{DD}}$ denote the isomorphism given by the Schikhof duality (cf. [12] Theorem 1.2), by $\iota\colon K \hookrightarrow L$ the inclusion, and by $\varphi\colon K^{\mathrm{DD}} \hookrightarrow K^{\mathbb{D}_k\mathbb{D}_k}$ the homeomorphism onto the lattice $\eta_{K,k}(K)$ given by Theorem 2.11. Then $\iota_{K,L}$ is bijective and $\iota^{\mathbb{D}_k}$ is injective by $kK = L$. Since $\iota^{\mathbb{D}_k\mathbb{D}_k} \circ \varphi \circ \eta_k\colon K_k \to L^{\mathbb{D}_k\mathbb{D}_k}$ coincides with $\eta_{L,k} \circ \iota_{K,L}$, it suffices to verify the bijectivity of $\iota^{\mathbb{D}_k\mathbb{D}_k}$. By the completeness of $k$, it suffices to show that $\iota^{\mathbb{D}_k}$ is a homeomorphism onto the dense image. Let $(K_0, \wp) \in \mathrm{K}(L) \times \mathscr{O}(k)$. Since $\Pi_L^{-j}|_K$ is a continuous map from a compact space to a Hausdorff space, we have $\varpi^{-j}K \in \mathscr{C}(L)$ for any $j \in \mathbb{N}$. Therefore there is a $j \in \mathbb{N}$ with $K_0 \subset \varpi_k^{-j}K$ by Lemma 2.30. We obtain $\iota^{\mathbb{D}_k}(L^{\mathbb{D}_k}) \cap \mathrm{Hom}((L,K),(k,\varpi_k^j\wp)) \subset \iota^{\mathbb{D}_k}(\mathrm{Hom}((L,K_0),(k,\wp)))$. It implies $\iota^{\mathbb{D}_k}$ is a homeomorphism onto the image. Let $W \subset K^{\mathbb{D}_k}$ denote the closure of $\iota^{\mathbb{D}_k}(L^{\mathbb{D}_k})$. Assume $W \neq K^{\mathbb{D}_k}$. Take a $w \in K^{\mathbb{D}_k} \setminus W$. Put $L_1 := W + kw \subset K^{\mathbb{D}_k}$. Since $W \subset K^{\mathbb{D}_k}$ is closed, $K^{\mathbb{D}_k}/W$ is Hausdorff. Therefore the image of $L_1$ in $K^{\mathbb{D}_k}/W$ is closed by Proposition 3.6. It implies $L_1 \in \mathscr{C}(K^{\mathbb{D}_k})$ by $W \subset L_1$. By Theorem 2.11 $(i)$ and Banach's open mapping theorem (cf. [3] Theorem I.3.3/1), $\mathrm{id}_W + \iota_{\{w\},k}\colon W \times k \to L_1$ is an isomorphism. By Theorem 3.1, there is an $\omega \in K^{\mathbb{D}_k\mathbb{D}_k}$ with $\omega(w_0 + cw) = c$ for any $(w_0, c) \in W \times k$. We have $\omega \in \ker(\iota^{\mathbb{D}_k\mathbb{D}_k}) \setminus \{0\}$. Since $\varphi(K^{\mathrm{DD}})$ is a lattice of $K^{\mathbb{D}_k\mathbb{D}_k}$, there is a $c \in k^\times$ with $c\omega \in \varphi(K^{\mathrm{DD}})$. Put $m := \eta^{-1}(\varphi^{-1}(c\omega)) \in K$. We have $\lambda(m) = c^{-1}\iota^{\mathbb{D}_k\mathbb{D}_k}(\omega)(\lambda) = 0$ for any $\lambda \in L^{\mathbb{D}_k}$, and hence $\eta_{L,k}(m) = 0$. By Corollary 3.4, we obtain $m = 0$. This contradicts $\eta_{K,k}(m) = c\omega \neq 0$. It implies $W = K^{\mathbb{D}_k}$. Therefore $\iota^{\mathbb{D}_k}(L^{\mathbb{D}_k})$ is dense in $K^{\mathbb{D}_k}$. $\square$

We put $\mathrm{LC}^{\mathrm{cgh}}(\kappa) := \mathrm{LC}(\kappa) \cap \mathrm{Mod}_1^{\mathrm{cgh}}(O) \subset \mathrm{LC}^{\mathrm{h}}(\kappa)$ for a $\kappa \in \{k, \overline{k}, O\}$. We show the natural equivalence $\mathrm{id}_{\mathrm{LC}^{\mathrm{cgh}}(k)} \cong \mathbb{D}_k\mathbb{D}_k|_{\mathrm{LC}^{\mathrm{cgh}}(k)}$.

**Theorem 4.2.** *If $L$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{cgh}}(k))$, $\eta_{L,k}$ is an isomorphism.*

We note that Theorem 4.2 does not follow from [11] Proposition 13.7 ii, because the equality $\mathrm{s}(L) = \mathrm{K}(L)$ rarely holds. In order to verify Theorem 4.2, we show that Theorem 2.22 is applicable to $L$.

**Lemma 4.3.** *If $L$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{cgh}}(k))$, then $L$ is $(\mathrm{K}, k)$-admissible.*

**Proof.** By Proposition 2.25, $L$ is K-generated. Let $L_1 \in \mathscr{O}(L)$. By [9] Theorem 3.4.10, there is a $V \in \mathrm{ob}(\mathrm{Ban}(O))$ with an $A \in \mathrm{Hom}(L, \underline{V})$ with $L_0 := A^{-1}(V(1)) \subset L_1$. By Theorem 2.11 $(ii)$, the natural embedding $V^{\mathrm{D}} \hookrightarrow (\underline{V})^{\mathbb{D}_k}$ is a homeomorphism onto a lattice $K \in \mathrm{K}((\underline{V})^{\mathbb{D}_k})$. Put $E := A^{\mathbb{D}_k}(K) \in \mathrm{K}(L^{\mathbb{D}_k})$. By the proof of [12] Theorem 1.2, we have $V(1) = \{v \in \underline{V} \mid (m(v))_{m \in V^{\mathrm{D}}} \in O^{V^{\mathrm{D}}}\}$, and hence $L_0 = \bigcap_{\lambda \in E} \lambda^{-1}(O) \in \mathscr{O}_{\mathrm{K}^{\mathbb{D}_k}}(L)$. $\square$

**Proof of Theorem 4.2.** By Theorem 2.22, Corollary 3.4, and Lemma 4.3, it suffices to verify the surjectivity of $\eta_{L,k}$. Let $\ell \in L^{\mathbb{D}_k\mathbb{D}_k}$. We have $\ell^{-1}(O) \in \mathscr{O}(L^{\mathbb{D}_k})$. Take a $K \in k\mathrm{K}(L)$ with $\mathrm{Hom}((L,K),(k,O)) \subset \ell^{-1}(O)$. For any

$\lambda \in L^{\mathbb{D}_k}$ with $\lambda|_{kK} = 0$, we have $\lambda \in \bigcap_{j \in \mathbb{N}} \mathrm{Hom}((L, K), (k, \varpi_k^j O)) \subset \ker(\ell)$ by Proposition 2.8. Therefore $\ell$ induces a $k$-linear map $\ell_0 \colon (kK)^{\mathbb{D}_k} \to k$ through the restriction map $L^{\mathbb{D}_k} \to (kK)^{\mathbb{D}_k}$ by Theorem 3.1. We have $\mathrm{Hom}((kK, K), (k, O)) \subset \ell_0^{-1}(O)$ by the choice of $K$. It implies the continuity of $\ell_0$. By Theorem 4.1, there is a unique $l \in kK$ with $\eta_{kK,k}(l) = \ell_0$. In particular, we have $\ell(\lambda) = \lambda(l)$ for any $\lambda \in L^{\mathbb{D}_k}$. Thus we obtain $\ell = \eta_{L,k}(l)$. $\square$

A $((V_i)_{i \in I}, (\varphi_{i,j})_{i \leq j}) \in \mathrm{ob}(\mathrm{Pro}_{\mathrm{Ban}(O)})$ is said to be *good* if $I$ is countable and $\varphi_{i,j}(V_j)$ is dense in $V_i$ for any $(i,j) \in I^2$ with $i \leq j$. We recall that $L$ is Fréchet if and only if $L$ is isomorphic to $\varprojlim_{i \in I} \underline{V}_i$ for a good $(V_i)_{i \in I} \in \mathrm{ob}(\mathrm{Pro}_{\mathrm{Ban}(O)})$. The following is a consequence of Corollary 2.13, Proposition 2.24, Proposition 2.27, and Theorem 4.2:

**Corollary 4.4.** *The natural map $\varinjlim_{i \in I} (\underline{V}_i)^{\mathbb{D}_k} \to (\varprojlim_{i \in I} \underline{V}_i)^{\mathbb{D}_k}$ is an isomorphism for any good $(V_i)_{i \in I} \in \mathrm{ob}(\mathrm{Pro}_{\mathrm{Ban}(O)})$.*

We note that Corollary 4.4 does not follow from [11] Proposition 16.5 or [11] Proposition 16.10. Indeed, the equality $\mathrm{K}(\underline{V}_i) = \mathrm{b}(\underline{V}_i)$ holds if and only if $\underline{V}_i$ is of finite dimension as a $k$-vector space, and we never assume the compactness of transition maps.

### 4.2. Over the Residue Field

We study the reflexivity for $\mathbb{D}_{\overline{k}}$. Let $(\mathscr{L}_i)_{i \in I}$ be a family in $\mathrm{LC}(\overline{k})$. We endow $\bigoplus_{i \in I} \mathscr{L}_i$ with the topology generated by the set of subsets of the form $l + \bigoplus_{i \in I} \mathscr{L}_{i,0}$ for an $(l, (\mathscr{L}_{i,0})_{i \in I}) \in (\bigoplus_{i \in I} \mathscr{L}_i) \times (\prod_{i \in I} \mathscr{O}(\mathscr{L}_i))$. By the definition of the topology of $\bigoplus_{i \in I} \mathscr{L}_i$ and Proposition 2.3, we obtain the following:

**Proposition 4.5.** *The universality of the direct sum in $\mathrm{LC}(\overline{k})$ holds for $\bigoplus_{i \in I} \mathscr{L}_i$, and the natural map $\varinjlim_{F \in 2^I_{\mathrm{f}}} \prod_{i \in F} \mathscr{L}_i \to \bigoplus_{i \in I} \mathscr{L}_i$ is an isomorphism. In addition, for any order $\leq$ on $I$ for which $I$ is directed and for any system $(\varphi_{i_1}^{i_2})_{i_1 \leq i_2} \in \prod_{i_1 \leq i_2} \mathrm{Hom}(\mathscr{L}_{i_1}, \mathscr{L}_{i_2})$ with $((\mathscr{L}_i)_{i \in I}, (\varphi_{i_1}^{i_2})_{i_1 \leq i_2}) \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{LC}(\overline{k})})$, the natural map $\bigoplus_{i \in I} \mathscr{L}_i \twoheadrightarrow \varinjlim_{i \in I} \mathscr{L}_i$ is a quotient map.*

First, we state the duality between $\prod$ and $\bigoplus$ for $\mathbb{D}_{\overline{k}}$ as an analogue of [11] Proposition 9.10 and [11] Proposition 9.11 for $(\cdot)^{\mathbb{D}_k}_{\mathrm{b}}$ and $(\cdot)^{\mathbb{D}_k}_{\mathrm{s}}$.

**Theorem 4.6.** *Suppose that $\eta_{\mathscr{L}_i, \overline{k}}$ is an isomorphism for any $i \in I$. Then $\eta_{\bigoplus_{i \in I} \mathscr{L}_i, \overline{k}}$ and $\eta_{\prod_{i \in I} \mathscr{L}_i, \overline{k}}$ are isomorphisms.*

In order to prove Theorem 4.6, it suffices to verify the following two lemmata:

**Lemma 4.7.** *Let $\mathscr{B}$ be a $\mathrm{b}_k$-class, and $(M_i)_{i \in I}$ a family in $\mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$. Suppose that for any $(B_i)_{i \in I} \in \prod_{i \in I} \mathscr{B}(M_i)$, there is a $B \in \mathscr{B}(\prod_{i \in I} M_i)$ with $\prod_{i \in I} B_i \subset B$. Then the natural map $\bigoplus_{i \in I} (M_i)^{\mathbb{D}_{\overline{k}}}_{\mathscr{B}} \to (\prod_{i \in I} M_i)^{\mathbb{D}_{\overline{k}}}_{\mathscr{B}}$ is an isomorphism.*

**Lemma 4.8.** *If $\mathscr{L}_i \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$ for any $i \in I$, then there is a natural isomorphism $(\bigoplus_{i \in I} \mathscr{L}_i)^{\mathbb{D}_{\overline{k}}} \to \prod_{i \in I} \mathscr{L}_i^{\mathbb{D}_{\overline{k}}}$.*

Lemma 4.7 is an immediate consequence of Proposition 4.5. In order to verify Lemma 4.8, it suffices to show the following by Proposition 2.12 and Proposition 4.5:

**Lemma 4.9.** *If $\mathscr{L}_i \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$ for any $i \in I$, then for any $\mathscr{K} \in \mathrm{K}(\bigoplus_{i \in I} \mathscr{L}_i)$, there is a pair $(F, \mathscr{K}_F)$ of an $F \in 2^I_{\mathrm{f}}$ and a $\mathscr{K}_F \in \mathrm{K}(\bigoplus_{i \in F} \mathscr{L}_i)$ such that $\mathscr{K}$ coincides with the image of $\mathscr{K}_F$.*

**Proof.** For an $I_0 \subset I$, put $\mathscr{L}_{I_0} := \bigoplus_{i \in I_0} \mathscr{L}_i$. Let $I_0 \subset I$. The zero-extension $\iota_{I_0} \colon \mathscr{L}_{I_0} \hookrightarrow \mathscr{L}_I$ is a homeomorphism onto the closed image, because $\mathscr{L}_i$ is $\mathrm{T}_1$ for any $i \in I \backslash I_0$. It implies $\iota_{I_0}^{-1}(\mathscr{K}) \in \mathrm{K}(\mathscr{L}_{I_0})$. Assume that there is no pair $(F, \mathscr{K}_1)$ of an $F \in 2^I_{\mathrm{f}}$ and a $\mathscr{K}_F \in \mathrm{K}(\mathscr{L}_F)$ with $\mathscr{K} = \iota_F(\mathscr{K}_F)$. Then no $F \in 2^I_{\mathrm{f}}$ satisfies $\mathscr{K} \subset \iota_F(\mathscr{L}_F)$. Take sequences $F_\bullet \colon \mathbb{N} \to 2^I_{\mathrm{f}}$, $n \mapsto F_n$ and $l_\bullet \colon \mathbb{N} \to \mathscr{K}$, $n \mapsto l_n$ in an inductive way so that $F_n \subset F_{n+1}$, $l_n \in \iota_{F_n}(\mathscr{L}_{F_n})$, and $l_{n+1} \notin \iota_{F_n}(\mathscr{L}_{F_n})$ for any $n \in \mathbb{N}$. Then $l_\bullet$ is injective. Put $I_0 := \bigcup_{n \in \mathbb{N}} F_n$ and $D := \{l_n \mid n \in \mathbb{N}\} \subset \mathscr{K}$. Since $l_\bullet$ is injective, $D$ is an infinite set. By the definition of $(F_\bullet, l_\bullet)$, we have $D \subset \iota_{i_0}(\mathscr{L}_{I_0})$ and $D \cap \iota_{F_n}(\mathscr{L}_{F_n}) \subset \{l_i \mid i \in F_n\} \in 2^{\mathscr{K}}_{\mathrm{f}}$ for any $n \in \mathbb{N}$. Since $\mathscr{L}_{F_n}$ is $\mathrm{T}_1$ for any $n \in \mathbb{N}$ and $\{F_n \mid n \in \mathbb{N}\}$ is cofinal in $2^{I_0} \cap 2^I_{\mathrm{f}}$, $D$ is closed and discrete by Proposition 4.5. It contradicts that $D$ is an infinite set of $\mathscr{K}$. Thus there is a pair $(F, \mathscr{K}_F)$ of an $F \in 2^I_{\mathrm{f}}$ and a $\mathscr{K}_F \in \mathrm{K}(\mathscr{L}_F)$ with $\mathscr{K} = \iota_F(\mathscr{K}_F)$. $\square$

This completes the proof of Theorem 4.6. As a consequence, the full subcategory of $\mathrm{LC}(\overline{k})$ generated by $\overline{k}$ closed under isomorphisms, direct sums, and direct products is $\mathbb{D}_{\overline{k}}$-reflexive. For example, $\overline{k}^{\mathbb{N}}, \overline{k}^{\oplus \mathbb{N}}, (\overline{k}^{\mathbb{N}})^{\oplus \mathbb{N}}, (\overline{k}^{\oplus \mathbb{N}})^{\mathbb{N}}, \ldots$ are reflexive with respect to $\mathbb{D}_{\overline{k}}$. We remark that $\overline{k}^{\mathbb{N}}$ is compact, $\overline{k}^{\oplus \mathbb{N}}$ is discrete, and the others are not compact or discrete. By Corollary 3.11 and the freeness of a vector space, Theorem 4.6 implies the following:

**Corollary 4.10.** *For any $\mathscr{L} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{ch}}(\overline{k}))$ (resp. $\mathscr{L} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{d}}(\overline{k}))$), $\eta_{\mathscr{L}, \overline{k}}$ is an isomorphism, and $\mathscr{L}^{\mathbb{D}_{\overline{k}}}$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{d}}(\overline{k}))$ (resp. $\mathrm{ob}(\mathrm{LC}^{\mathrm{ch}}(\overline{k}))$).*

**Corollary 4.11.** *The pair $(\mathrm{LC}^{\mathrm{ch}}(\overline{k}), \mathrm{LC}^{\mathrm{d}}(\overline{k}))$ forms a $\mathbb{D}_{\overline{k}}$-dual pair.*

Secondly, we state the duality between $\varinjlim$ and $\varprojlim$ for $\mathbb{D}_{\overline{k}}$ as an analogue of [11] Proposition 16.5 and [11] Corollary 16.6 for $(\cdot)^{\mathbb{D}_k}_{\mathrm{b}}$.

**Theorem 4.12.** *Let $(\mathscr{L}_i)_{i \in I} \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{LC}(\overline{k})})$. Suppose that for any $K \in \mathrm{K}(\varinjlim_{i \in I} \mathscr{L}_i)$, there is a pair $(i_0, K_{i_0})$ of an $i_0 \in I$ and a $K_{i_0} \in \mathrm{K}(\mathscr{L}_{i_0})$ such that the image of $K_{i_0}$ in $\varinjlim_{i \in I} \mathscr{L}_i$ contains $K$, and that $\eta_{\mathscr{L}_i, \overline{k}}$ is a quotient map for any $i \in I$. Then $\eta_{\varinjlim_{i \in I} \mathscr{L}_i, \overline{k}}$ is a quotient map with $\ker(\eta_{\varinjlim_{i \in I} \mathscr{L}_i, \overline{k}}) = \bigcap_{\mathscr{L} \in \mathscr{O}(\varinjlim_{i \in I} \mathscr{L}_i)} \mathscr{L}$. In addition if $\eta_{\mathscr{L}_i, \overline{k}}$ is an isomorphism for any $i \in I$ and*

$\varinjlim_{i \in I} \mathscr{L}_i$ *lies in* $\mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$, *then* $\eta_{\varinjlim_{i \in I} \mathscr{L}_i, \overline{k}}$ *is an isomorphism.*

In order to verify Theorem 4.12, it suffices to show the following two lemmata:

**Lemma 4.13.** *For any* $(\mathscr{L}_i)_{i \in I} \in \mathrm{ob}(\mathrm{Pro}_{\mathrm{LC}^{\mathrm{h}}(\overline{k})})$, *the natural map* $\varinjlim_{i \in I} \mathscr{L}_i^{\mathbb{D}_{\overline{k}}} \to (\varprojlim_{i \in I} \mathscr{L}_i)^{\mathbb{D}_{\overline{k}}}$ *is a quotient map.*

**Lemma 4.14.** *For any* $\mathscr{L} \in \mathrm{ob}(\mathrm{LC}(\overline{k}))$, *the equality* $\ker(\eta_{\mathscr{L},\mathscr{B},\overline{k}}) = \bigcap_{\mathscr{L}_0 \in \mathscr{O}(\mathscr{L})} \mathscr{L}_0$ *holds.*

Lemma 4.14 immediately follows from Proposition 2.2 and Corollary 3.9. Lemma 4.13 follows from Proposition 2.3, Corollary 3.15, Proposition 4.5, and Lemma 4.7 by a routine argument. We note that the natural bijective map $(\mathscr{L}^{\mathrm{h}})^{\mathbb{D}_{\overline{k}}} \to \mathscr{L}^{\mathbb{D}_{\overline{k}}}$ is not necessarily an isomorphism, but the preimage of $\ker(\eta_{\mathscr{L}^{\mathrm{h}},\mathscr{B},\overline{k}})$ in $\mathscr{L}$ coincides with $\ker(\eta_{\mathscr{L},\mathscr{B},\overline{k}})$. This completes the proof of Theorem 4.12. The following is an immediate consequence of Lemma 2.14 and Theorem 4.12:

**Corollary 4.15.** *Let* $((\mathscr{L}_i)_{i \in I}, (\varphi_{i,j})_{i \leq j}) \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{LC}^{\mathrm{h}}(\overline{k})})$. *Suppose that* $I$ *is countable and* $\varphi_{i,j}$ *is injective for any* $(i,j) \in I^2$ *with* $i \leq j$. *If* $\eta_{\mathscr{L}_i,\overline{k}}$ *is an isomorphism for any* $i \in I$ *and* $\varinjlim_{i \in I} \mathscr{L}_i$ *lies in* $\mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$, *then* $\eta_{\varinjlim_{i \in I} \mathscr{L}_i, \overline{k}}$ *is an isomorphism.*

We note that the natural bijective map $(\varinjlim_{i \in I}{}^{\mathrm{h}} \mathscr{L}_i)^{\mathbb{D}_{\overline{k}}} \to (\varinjlim_{i \in I} \mathscr{L}_i)^{\mathbb{D}_{\overline{k}}}$ is not necessarily an isomorphism. Therefore the full subcategory of $\mathrm{LC}^{\mathrm{h}}(\overline{k})$ generated by $\overline{k}$ closed under isomorphisms, inductive limits, and projective limits seems not to be $\mathbb{D}_{\overline{k}}$-reflexive. The following is an immediate consequence of Corollary 4.10 and Theorem 4.12:

**Theorem 4.16.** *For any* $\mathscr{L} \in \mathrm{LC}^{\mathrm{cgh}}(\overline{k})$, $\eta_{\mathscr{L},\overline{k}}$ *is an isomorphism.*

It can be seen that every $\mathscr{L} \in \mathrm{LC}^{\mathrm{cgh}}(\overline{k})$e is $(\mathrm{K}, \overline{k})$-admissible in a way similar to the proof of Lemma 4.3. Therefore Theorem 4.16 can be verified in an alternative way similar to the proof of Theorem 4.2. We have the following corollaries of Theorem 4.16:

**Corollary 4.17.** *Let* $(\mathscr{W}_i)_{i \in I} \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{LC}^{\mathrm{d}}(\overline{k})})$. *Then* $\varinjlim_{i \in I} \mathscr{W}_i$ *lies in* $\mathrm{ob}(\mathrm{LC}^{\mathrm{d}}(\overline{k}))$, *and the natural map* $(\varinjlim_{i \in I} \mathscr{W}_i)^{\mathbb{D}_{\overline{k}}} \to \varprojlim_{i \in I} (\mathscr{W}_i)^{\mathbb{D}_{\overline{k}}}$ *is an isomorphism.*

**Corollary 4.18.** *Let* $(\mathscr{K}_i)_{i \in I} \in \mathrm{ob}(\mathrm{Pro}_{\mathrm{LC}^{\mathrm{ch}}(\overline{k})})$. *Then* $\varprojlim_{i \in I} \mathscr{K}_i$ *lies in* $\mathrm{ob}(\mathrm{LC}^{\mathrm{ch}}(\overline{k}))$, *and the natural map* $\varinjlim_{i \in I} (\mathscr{K}_i)^{\mathbb{D}_{\overline{k}}} \to (\varprojlim_{i \in I} \mathscr{K}_i)^{\mathbb{D}_{\overline{k}}}$ *is an isomorphism.*

**Corollary 4.19.** *Let* $((\mathscr{W}_i)_{i \in I}, (\varphi_{i,j})_{i \leq j}) \in \mathrm{ob}(\mathrm{Pro}_{\mathrm{LC}^{\mathrm{d}}(\overline{k})})$. *Suppose that* $I$ *is* *countable and $\varphi_{i,j}$ is surjective for any $(i,j) \in I^2$ with $i \leq j$. Then $\eta_{\varprojlim_{i \in I} \mathscr{W}_i, \overline{k}}$ and the natural map $\varinjlim_{i \in I} \mathscr{W}_i^{\mathbb{D}_{\overline{k}}} \to (\varprojlim_{i \in I} \mathscr{W}_i)^{\mathbb{D}_{\overline{k}}}$ are isomorphisms.*

Corollary 4.17 immediately follows from Corollary 4.10 and Theorem 4.12. Corollary 4.18 immediately follows from Corollary 4.10 and Corollary 4.17. Corollary 4.19 follows from Proposition 2.27, Corollary 4.10, Corollary 4.14, Corollary 4.10, Lemma 4.13, and Theorem 4.12 by a routine argument. Now we give a criterion of the reflexivity. We denote by $\mathrm{LC}^{\sigma\mathrm{cf}}(\overline{k}) \subset \mathrm{LC}^{\mathrm{f}}(\overline{k})$ the full subcategory of $\sigma$-compact Fréchet locally convex $\overline{k}$-vector spaces.

**Theorem 4.20.** *The full subcategory $\mathrm{LC}^{\sigma\mathrm{cf}}(\overline{k}) \subset \mathrm{Mod}_{\mathrm{l}}(O)$ is $\mathbb{D}_{\overline{k}}$-reflexive.*

**Proof.** It suffices to verify that for any $\mathscr{L} \in \mathrm{ob}(\mathrm{LC}^{\sigma\mathrm{cf}}(\overline{k}))$, $\eta_{\mathscr{L},\overline{k}}$ is an isomorphism and $\mathscr{L}^{\mathbb{D}_{\overline{k}}}$ again lies in $\mathrm{ob}(\mathrm{LC}^{\sigma\mathrm{cf}}(\overline{k}))$. By Proposition 2.24 and Theorem 4.16, $\eta_{\mathscr{L},\overline{k}}$ is an isomorphism. Since $\mathscr{L}$ is $\sigma$-compact and complete, there is a $((\mathscr{K}_i)_{i \in I}, (\varphi_{i,j})_{i \leq j}) \in \mathrm{ob}(\mathrm{Ind}_{\mathrm{LC}^{\mathrm{ch}}(\overline{k})})$ such that $\mathscr{L}$ is isomorphic to $\varinjlim_{i \in I} \mathscr{K}_i$, $I$ is countable, and $\varphi_{i,j}$ is injective for any $(i,j) \in I^2$ with $i \leq j$ by Proposition 2.28. It implies that $\mathscr{L}^{\mathbb{D}_{\overline{k}}}$ is isomorphic to $\varprojlim_{i \in I} \mathscr{K}_i^{\mathbb{D}_{\overline{k}}}$ by Corollary 2.13, and hence is Fréchet. Since $\mathscr{L}$ is first countable, $\mathscr{O}(\mathscr{L})$ admits a countable coinitial subset $J$. Since $\mathscr{L}$ is complete, the natural map $\mathscr{L} \to \varprojlim_{\mathscr{L}_0 \in J} \mathscr{L}/\mathscr{L}_0$ is an isomorphism. We obtain an isomorphism $\varinjlim_{\mathscr{L}_0 \in J} (\mathscr{L}/\mathscr{L}_0)^{\mathbb{D}_{\overline{k}}} \to \mathscr{L}^{\mathbb{D}_{\overline{k}}}$, and hence $\mathscr{L}^{\mathbb{D}_{\overline{k}}}$ is $\sigma$-compact by Corollary 4.10 and Lemma 4.13. □

We note that Theorem 4.20 is an analogue of [11] Corollary 16.6 and [9] Corollary 7.4.30, but they are not applicable to $\mathrm{LC}(\overline{k})$ or $\mathbb{D}_{\overline{k}}$.

### 4.3. Over the Valuation Ring

We study the reflexivity for $\mathbb{D}$. An $M \in \mathrm{ob}(\mathrm{Mod}(O))$ is said to be *adic* if its topology coincides with the adic topology. An $M \in \mathrm{ob}(\mathrm{LC}(O))$ is said to be *Banach* if $M$ is complete and adic. We denote by $\mathrm{LC}^{\mathrm{Ban}}(O) \subset \mathrm{LC}^{\mathrm{f}}(O)$ the full subcategory of Banach locally convex $O$-modules.

**Remark 4.21.** For an $M \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$, the following are equivalent:

(i) The module $M$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{Ban}}(O))$.
(ii) The module $M$ is isomorphic to $V(1)$ for a $V \in \mathrm{Ban}(O)$.

To begin with, we show that $\mathbb{D}$ induces endomorphisms of $\mathrm{LC}(O)$ and $\mathrm{LC}^{\mathrm{h}}(O)$.

**Proposition 4.22.** *Let $\mathscr{B}$ be a $\mathrm{b}_k$-class. For any $M_1 \in \mathrm{ob}(\mathrm{Mod}_{\mathrm{l}}(O))$ and $M_2 \in \mathrm{ob}(\mathrm{LC}(O))$ (resp. $\mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(O))$), $\mathrm{Hom}(M_1, M_2)_{\mathscr{B}}$ lies in $\mathrm{ob}(\mathrm{LC}(O))$ (resp. $\mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(O))$).*

**Proof.** The boundedness (resp. Hausdorffness) of $M_2$ ensures that of

$\mathrm{Hom}(M_1, M_2)_{\mathscr{B}}$. By $M_2 \in \mathrm{ob}(\mathrm{LC}(O))$, $\Pi_{M_2} \colon M_2 \to \varpi_k M_2$ induces an isomorphism $\Pi \colon \mathrm{Hom}(M_1, M_2)_{\mathscr{B}} \to \mathrm{Hom}(M_1, \varpi_k M_2)_{\mathscr{B}}$. We denote by $\iota \colon \mathrm{Hom}(M_1, \varpi_k M_2)_{\mathscr{B}} \hookrightarrow \mathrm{Hom}(M_1, M_2)_{\mathscr{B}}$ the natural embedding. Since $\Pi_{\mathrm{Hom}(M_1,M_2)_{\mathscr{B}}}$ is the composite of $\Pi$ and $\iota$, it suffices to verify that $\iota$ is closed. Let $C \subset \mathrm{Hom}(M_1, \varpi_k M_2)_{\mathscr{B}}$ be a closed subset. Let $(A_i)_{i \in I}$ be a Cauchy net on $\mathrm{Hom}(M_1, M_2)_{\mathscr{B}}$ lying in $\iota(C)$ converging to an $A \in \mathrm{Hom}(M_1, M_2)_{\mathscr{B}}$. For any $m \in M_1$, $(A_i m)_{i \in I} \in (\varpi_k M_2)^I$ converges to $Am \in M_2$ by $Om \in \mathscr{B}(M_1)$, and hence $Am \in \varpi_k M_2$ because $M_2$ is a locally convex $O$-module. It implies $A \in \iota(\mathrm{Hom}(M_1, \varpi_k M_2)_{\mathscr{B}})$. The injectivity of $\iota$ ensures the unique existence of a net $(A'_i)_{i \in I}$ on $\mathrm{Hom}(M_1, \varpi_k M_2)_{\mathscr{B}}$ and an $A' \in \mathrm{Hom}(M_1, \varpi_k M_2)_{\mathscr{B}}$ with $\iota(A'_i) = A_i$ for any $i \in I$ and $\iota(A') = A$. Let $B \in \mathscr{B}(M_1)$ and $M_{2,0} \in \mathscr{O}(\varpi_k M_2)$. By Proposition 3.2, there is an $M_{2,1} \in \mathscr{O}(M_2)$ with $\varpi_k M_2 \cap M_{2,1} = M_{2,0}$. Since $(A_i)_{i \in I}$ converges to $A$, there is an $i_0 \in I$ such that $\iota(A'_i - A') = A_i - A \in \mathrm{Hom}((M_1, B), (M_2, M_{2,1}))$ for any $i \in \mathbb{N}$ with $i \geq i_0$. It implies $A'_i - A' \in \mathrm{Hom}((M_1, B), (\varpi_k M_2, M_{2,0}))$ for any $i \in \mathbb{N}$ with $i \geq i_0$. Therefore $(A'_i)_{i \in I}$ converges to $A'$. Since $C$ is closed, $A'$ lies in $C$, and so does $A = \iota(A')$ in $\iota(C)$. Thus $\iota(C)$ is closed. We conclude that $\iota$ is closed. □

We obtain the following analogues of Corollary 4.10 and Corollary 4.11 by Proposition 2.8, Remark 2.9, Remark 2.10, Theorem 2.11, Proposition 3.19, Remark 4.21, and Proposition 4.22 by a routine argument:

**Corollary 4.23.** *For any $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{ch}}(O))$ (resp. $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{Ban}}(O))$), $\eta_M$ is an isomorphism, and $M^{\mathbb{D}}$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{Ban}}(O))$ (resp. $\mathrm{ob}(\mathrm{LC}^{\mathrm{ch}}(O))$).*

**Corollary 4.24.** *The pair $(\mathrm{Mod}^{\mathrm{ch}}_{\mathrm{fl}}(O), \mathrm{LC}^{\mathrm{Ban}}(O))$ is a $\mathbb{D}$-dual pair.*

In order to study the reflexivity for $\mathbb{D}$, we observe a compatibility between the continuous duals and the localisation as an analogue of Theorem 3.28.

**Theorem 4.25.** *Let $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{f}}(O))$. Then the natural map $(M^{\mathbb{D}})_k \to (M_k)^{\mathbb{D}_k}$ is an isomorphism, and for any $U \in \mathscr{O}((M^{\mathbb{D}})_k)$ with $\iota_{M^{\mathbb{D}},0}(M^{\mathbb{D}}) \subset U$, there is a $K \in \mathrm{K}(M_k)$ with $K \subset \iota_{M,0}(M)$ and $\mathrm{Hom}((M_k, K), (k, O)) \subset U$.*

In order to verify Theorem 4.25, we prepare several lemmata.

**Lemma 4.26.** *Let $M \in \mathrm{ob}(\mathrm{LC}(O))$. Then the natural map $(M^{\mathbb{D}})_k \to (M_k)^{\mathbb{D}_k}$ is bijective.*

The injectivity follows from that of the natural map $\iota_{M^{\mathbb{D}},j}(M^{\mathbb{D}}) \to \mathrm{Hom}(M, \iota_{O,j}(O))$ with $j \in \mathbb{N}$, and the surjectivity is essentially verified in the proof of Theorem 3.20.

**Lemma 4.27.** *Let $I$ be a set. Let $M \in \mathrm{ob}(\mathrm{Mod}^{\mathrm{h}}_{1}(O))$ and $(m_i)_{i \in I} M^I$. The map $O^{\oplus I} \to M$, $(c_i)_{i \in I} \mapsto \sum_{i \in I} c_i m_i$ continuously extends to $O^I$ through the natural embedding $O^{\oplus I} \hookrightarrow O^I$ if and only if for any $(c_i)_{i \in I} \in O^I$, the net $(\sum_{i \in F} c_i m_i)_{F \in 2^I_{\mathrm{f}}}$ on $M$ converges.*

The necessary implication follows from the definition of the direct product topology. The sufficient implication follows from the Hausdorffness and the continuity of the $O$-module structure of $M$.

**Lemma 4.28.** *Let $M \in \mathrm{ob}(\mathrm{Mod}_1^{\mathrm{f}}(O))$ and $M_0 \in \mathscr{C}(M)$ with $\varpi_k M \subset M_0$. For any $\mathscr{K} \in \mathrm{K}(M/M_0)$, there is a $K \in \mathrm{K}(M)$ such that the image $\overline{K}$ of $K$ in $M/M_0$ coincides with $\mathscr{K}$ and satisfies $K \cap M_0 = K \cap \varpi_k M = \varpi_k K$.*

**Proof.** By $M \in \mathrm{ob}(\mathrm{Mod}_1^{\mathrm{h}}(O))$, $M_0 \in \mathscr{C}(M)$, $\varpi_k M \subset M_0$, we have $M/M_0 \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$, and hence $\mathscr{K} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$. By Corollary 3.11, there is an $\mathscr{I} \subset \mathscr{K}$ such that the map $\overline{\iota} \colon \overline{k}^{\oplus \mathscr{I}} \to \mathscr{K} \colon (\overline{c}_{\overline{m}})_{\overline{m} \in \mathscr{I}} \mapsto \sum_{\overline{m} \in \mathscr{I}} \overline{c}_{\overline{m}} \overline{m}$ extends to an isomorphism $\overline{k}^{\mathscr{I}} \to \mathscr{K}$. Since $M$ is first countable, so is $M/M_0$. It implies that $\mathscr{K}$ is first countable, and hence $\mathscr{I}$ is countable. Since $M$ is first countable, $\mathscr{O}(M)$ admits a countable coinitial subset $J$. First, suppose that $\mathscr{I}$ is a finite set or $J$ admits the smallest element. If $\mathscr{I}$ is a finite set, then so is $\mathscr{K}$. If $J$ admits the smallest element, then $M$ is discrete, and so is $M/M_0$. Therefore $\mathscr{K}$ is a finite set in both cases. Take a complete system $F \subset M$ of representatives of a $\overline{k}$-linear basis $\mathscr{F} \subset \mathscr{K}$, which is $O$-linearly independent. Then the $O$-submodule $K \subset M$ generated by $F$ is a finitely generated free $O$-module. By Proposition 3.5, $\iota_F \colon O^F \to K$ is an isomorphism. Therefore $K$ lies in $\mathrm{K}(M)$, and the image $\overline{K}$ of $K$ in $M/M_0$ coincides with $\mathscr{K}$. Since the kernel of the canonical projection $O^F \twoheadrightarrow \overline{k}^{\mathscr{F}}$ is $\varpi_k O^F$, we have $K \cap M_0 = \varpi_k K$. By $K \cap \varpi_k M \subset K \cap M_0$ and $\varpi_k K \subset K \cap \varpi_k M$, we obtain $K \cap \varpi_k M = \varpi_k K$.

Secondly, suppose that both of $\mathscr{I}$ is an infinite set, and $J$ does not admit the smallest element. Then we may assume that $J$ is isomorphic to $\mathbb{N}$ as an ordered set by Lemma 2.15. We denote by $M^{(\bullet)} \colon \mathbb{N} \to J,\ n \mapsto M^{(n)}$ the unique order-preserving bijective map. Replacing $J$ by $J \cup \{M\}$ if necessary, we may assume $M^{(0)} = M$. For an $n \in \mathbb{N}$, we denote by $\overline{M}^{(n)}$ the image of $M^{(n)}$ in $M/M_0$, which lies in $\mathscr{O}(M/M_0)$. Since $J$ is coinitial in $\mathscr{O}(M)$, so is $\{\overline{M}^{(n)} \mid n \in \mathbb{N}\}$ in $\mathscr{O}(M/M_0)$. Since $\mathscr{K}$ is Hausdorff, we have $\mathscr{K} \cap \bigcap_{n \in \mathbb{N}} \overline{M}^{(n)} = \{0\}$ by Proposition 2.2. Therefore for any $\overline{m} \in \mathscr{I}$, $\{n \in \mathbb{N} \mid \overline{m} \in \overline{M}^{(n)}\}$ is a finite set containing 0, and we denote by $n_{\overline{m}} \in \mathbb{N}$ its greatest element. For any $\overline{m} \in \mathscr{I}$, we have $m \in \overline{M}^{(n_{\overline{m}})}$, and hence there is an $m \in M^{(n_{\overline{m}})}$ with $m + M_0 = \overline{m}$. Take a complete system $I \subset M$ of representatives of $\mathscr{I}$ satisfying $m \in M^{(n_{m+M_0})}$ for any $m \in I$. By Lemma 4.27, $\{\overline{m} \in \mathscr{I} \mid n_{\overline{m}} \leq n\}$ is a finite set, and hence so is $\{m \in I \mid n_{m+M_0} \leq n\}$ for any $n \in \mathbb{N}$. Therefore the completeness of $M$ ensures that the map $O^{\oplus I} \to M,\ (c_m)_{m \in I} \mapsto \sum_{m \in I} c_m m$ extends to an $\iota \in \mathrm{Hom}(O^I, M)$ by Lemma 4.27. Put $K := \iota(O^I)$. Since $O^I$ is compact, we have $K \in \mathrm{K}(M)$. Let $\sigma \colon I \to \mathscr{I}$ denote the canonical bijective map. For any $\overline{m} \in \mathscr{K}$, we have $\iota([\overline{\iota}^{-1}(\overline{m})]) \in K$ and $\iota([\overline{\iota}^{-1}(\overline{m})]) + M_0 = \overline{m}$, where $[\cdot] \colon \overline{k}^{\mathscr{I}} \hookrightarrow O^I$ denotes an embedding given by a fixed Teichmüller embedding $\overline{k} \hookrightarrow O$ and $\sigma$. Therefore $\overline{K}$ contains $\mathscr{K}$. Since the image of $O^{\oplus I}$ in $O^I$ is dense, $\overline{K}$ in $M/M_0$ coincides with $\mathscr{K}$. Let $(c_i)_{i \in I} \in \ker(\iota)$. We have $(c_{\sigma^{-1}(\overline{i})} + \varpi_k O)_{\overline{i} \in \mathscr{I}} = 0 \in \overline{k}^{\mathscr{I}}$, and hence

$c_i \in \varpi_k O$ for any $i \in I$. It implies $\ker(\iota) \in \varpi_k O^I$. Since $K$ is torsionfree, we obtain $\ker(\iota) = \{0\}$. Therefore $\iota$ is injective. The assertion holds by the same argument as above. □

**Lemma 4.29.** *Let $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(O))$ and $K \in \mathrm{K}(M)$. Suppose that $K$ is first countable. Then there is a $(K_0, K_0^{\perp}) \in \mathrm{ASK}(M) \times \mathrm{K}(M)$ the addition $K_0 \times \varpi_k K_0^{\perp} \to K$ is an isomorphism.*

**Proof.** By $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(O))$, we have $\varpi_k M \in \mathscr{C}(M)$ and $M_{\overline{k}} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$. Since $K$ is compact, so is its image $\overline{K}$ in $M_{\overline{k}}$. Since $K$ is compact and $M_{\overline{k}}$ is Hausdorff, the natural map $K/(K \cap \varpi_k M) \to \overline{K}$ is an isomorphism. Applying Lemma 4.28 to $(M, M_0, \mathscr{K}) = (M, \varpi_k M, \overline{K})$, we obtain a $K_0 \in \mathrm{K}(K)$ with $K + \varpi_k M =\subset K_0 + \varpi_k M$ and $K_0 \cap \varpi_k M = \varpi_k K_0$. In particular, we have $K_0 \in \mathrm{ASK}(M)$, and hence $K_0 \cap \varpi_k M \in \mathrm{ASK}(\varpi_k M)$ by the flatness of $M$. Therefore there is a $K_1 \in \mathrm{AS}\mathscr{C}(K \cap \varpi_k M)$ such that the addition $(K_0 \cap \varpi_k M) \times K_0^{\perp} \to K \cap \varpi_k M$ is an isomorphism by Corollary 3.27. Put $K_0^{\perp} := \Pi_M^{-1}(K_1)$. By $M \in \mathrm{ob}(\mathrm{LC}(O))$ and $K_1 \subset \varpi_k M$, we have $K_0^{\perp} \in \mathrm{K}(M)$. We obtain $K = K_0 + \varpi_k K_0^{\perp}$ by $K \subset K_0 + \varpi_k M$, and $K_0 \cap \varpi_k K_0^{\perp} = \{0\}$ by $K_1 \subset K \cap \varpi_k M$. Therefore the addition $K_0 \times \varpi_k K_0^{\perp} \to K$ is an isomorphism by Lemma 3.13. □

**Proof of Theorem 4.25.** Let $\iota\colon (M^{\mathbb{D}})_k \to (M_k)^{\mathbb{D}_k}$ denote the natural map, which is bijective by Lemma 4.26. It suffices to verify that $\iota$ is an isomorphism. Let $K \in \mathrm{K}(M_k)$. By Proposition 3.17, $\iota_{M,0}(M)$ is a lattice of $M_k$. Therefore there is a $j_1 \in \mathbb{N}$ with $K \subset \varpi_k^{-j} \iota_{M,0}(M) = \iota_{M,j}(M)$ by Lemma 2.30 and Corollary 3.18. Put $K_0 := \iota_{M,j}^{-1}(K) \in \mathrm{K}(M)$. We have $\iota^{-1}(\mathrm{Hom}((M_k, K), (k, O))) = \sum_{j_2 \in \mathbb{N}} \iota_{M^{\mathbb{D}}, j_2}(\mathrm{Hom}((M, K_0), (O, \varpi_k^{j_1+j_2} O))) \in \mathscr{O}((M^{\mathbb{D}})_k)$. Therefore $\iota$ is continuous. Let $U \in \mathscr{O}((M^{\mathbb{D}})_k)$. By Proposition 2.7, there is an $N \in \mathbb{N}$ with $\varpi_k^N \iota_{M^{\mathbb{D}},0}(M^{\mathbb{D}}) \subset U$. Put $K_0 := \{0\} \in \mathrm{K}(M)$. We have $\iota_{M^{\mathbb{D}},0}(\mathrm{Hom}((M, K_0), (O, O))) \subset \varpi_k^{-N} U$. By induction on $j \in \mathbb{N}$, we construct a sequence $(K_j)_{j \in \mathbb{N}} \in \mathrm{ASK}(M)^{\mathbb{N}}$ with $\mathrm{Hom}((M, \bigoplus_{j=0}^{j_1} \varpi_k^j K_j), (O, \varpi_k^{j_1} O) \subset \iota_{M^{\mathbb{D}}, j_1}^{-1}(\varpi_k^{-N} U)$ for any $j_1 \in \mathbb{N}$ such that the addition $\bigoplus_{j \in \mathbb{N}} \varpi_k^j K_j \to M$ induces a homeomorphism $\prod_{j \in \mathbb{N}} \varpi_k^j K_j \hookrightarrow M$ onto the closed image. Let $j_1 \in \mathbb{N}$. Suppose that there is a sequence $(K_j)_{j=0}^{j_1} \in \mathrm{ASK}(M)^{j_1+1}$ with $\mathrm{Hom}((M, \bigoplus_{j=0}^{j_1} \varpi_k^j K_j), (O, \varpi_k^{j_1} O) \subset \iota_{M^{\mathbb{D}}, j_1}^{-1}(\varpi_k^{-N} U)$ such that the addition $\prod_{j=0}^{j_1} \varpi_k^j K_j \hookrightarrow M$ is a homeomorphism onto the closed image. By Corollary 3.27, there is an $M_1 \in \mathrm{AS}\mathscr{C}(M)$ such that the addition $(\prod_{j=0}^{j_1} K_j) \times M_1 \to M$ is an isomorphism. By Corollary 2.17, the natural map $\varphi\colon M^{\mathbb{D}} \to (\prod_{j=0}^{j_1} (K_j)^{\mathbb{D}}) \times M_1^{\mathbb{D}}$ is an isomorphism. Let $\psi\colon M_1^{\mathbb{D}} \hookrightarrow M^{\mathbb{D}}$ denote the composite of the zero-extension $M_1^{\mathbb{D}} \hookrightarrow (\prod_{j=0}^{j_1} (K_j)^{\mathbb{D}}) \times M_1^{\mathbb{D}}$ and $\varphi^{-1}$. By the induction hypothesis, we have $\varphi^{-1}((\prod_{j=0}^{j_1} \mathrm{Hom}((K_j, \varpi_k^j K_j), (O, \varpi_k^{j_1} O))) \times \mathrm{Hom}(M_1, O)) \subset \iota_{M^{\mathbb{D}}, j_1}^{-1}(\varpi_k^{-N} U)$. In particular, $U_{j_1+1} := \psi^{-1}(\iota_{M^{\mathbb{D}}, j_1+1}^{-1}(\varpi_k^{-N} U)) \in$

$\mathscr{O}(M_1^{\mathbb{D}})$ contains $\varpi_k M_1^{\mathbb{D}}$. Take a $(K'_{j_1+1}, h) \in \mathrm{K}(M_1) \times \mathbb{N}$ with $\mathrm{Hom}((M_1, K'_{j_1+1}), (O, \varpi_k^h O)) \subset U_{j_1+1}$. If $K'_{j_1+1} = \{0\}$ or $h = 0$, then we have $\mathrm{Hom}((M_1, K'_{j_1+1}), (O, \varpi_k^h O)) = \mathrm{Hom}(M_1, O)$, and hence $K_{j_1+1} := \{0\}$ satisfies $\psi(M_1^{\mathbb{D}}) = \psi(\mathrm{Hom}((M_1, K_{j_1+1}), (O, \varpi_k^{j_1+1} O))) \subset \iota_{M^{\mathbb{D}}, j_1+1}^{-1}(\varpi_k^{-N} U)$. Suppose $K'_{j_1+1} \neq \{0\}$ and $h > 0$. Assume $K'_{j_1+1} \subset \varpi_k M_1$. By $M \in \mathrm{ob}(\mathrm{LC}(O))$, $\Pi_M$ is a homeomorphism onto the closed image. Therefore $\Pi_M|_{\varpi_k^{-1}(K'_{j_1+1})} \colon \varpi_k^{-1}(K'_{j_1+1}) \to K'_{j_1+1}$ is an isomorphism. It implies $\Pi_{M_1}^{-1}(K'_{j_1+1}) \in \mathrm{K}(M_1)$. Replacing $(K'_{j_1+1}, h)$ by $(\Pi_{M_1}^{-1}(K'_{j_1+1}), h-1)$ in a repetitive way, it is reduced to the case where that $K'_{j_1+1}$ is not contained in $\varpi_k M_1$ and $h > 0$. Now we have $\mathrm{Hom}((M_1, K'_{j_1+1}), (O, \varpi_k^h O)) + \mathrm{Hom}(M_1, \varpi_k O) \subset U_{j_1+1}$. By $M_1 \in \mathscr{C}(M)$, we have $\varpi_k^j M_1 \in \mathscr{C}(M)$ for any $j \in \mathbb{N}$. Applying Lemma 4.29 in a repetitive way, we obtain a $(K_{j_1+1,j})_{j=0}^h \in \mathrm{K}(M_1)^{j_1+2}$ with $K'_{j_1+1} = \bigoplus_{j=0}^h \varpi_k^j K_{j_1+1,j}$, $K_{j_1+1,j} \in \mathrm{AS}(M_1)$ for any $j \in \mathbb{N} \cap [0,h]$, and $\bigoplus_{j=j_0}^{j_1+1} \varpi_k^j K_{j_1+1,j} \subset \varpi_k^{j_0} K_{j_1+1,j_0} + \varpi_k^{j_0+1} M_1$ for any $j_0 \in \mathbb{N} \cap [0,h]$. Put $K_{j_1+1} := \bigoplus_{j=0}^{h-1} K_{j_1+1,j} \in \mathrm{K}(M_1)$. We have $K'_{j_1+1} \subset (\bigoplus_{j=0}^h \varpi_k^j K_{j_1+1,j}) + \varpi_k^h M_1 \subset K_{j_1+1} + \varpi_k^h M_1$, and hence $\mathrm{Hom}((M_1, K_{j_1+1}), (O, \varpi_k^h O)) \subset \mathrm{Hom}((M_1, K'_{j_1+1}), (O, \varpi_k^h O)) \subset U_{j_1+1}$. By Corollary 3.27, there is a $K_{j_1+1}^{\perp} \in \mathrm{AS}\mathscr{C}(M_1)$ such that the addition $K_{j_1+1} \times K_{j_1+1}^{\perp} \to M_1$ is an isomorphism. Let $\mu \in \mathrm{Hom}((M_1, K_{j_1+1}), (O, \varpi_k O))$. We denote by $\mu_1 \in M_1^{\mathbb{D}}$ (resp. $\mu_2 \in M_1^{\mathbb{D}}$) the composite of $\mu$ and the idempotent $M_1 \twoheadrightarrow K_{j_1+1} \hookrightarrow M_1$ (resp. $M_1 \twoheadrightarrow K_{j_1+1}^{\perp} \hookrightarrow M_1$) induced by the addition $K_{j_1+1} \times K_{j_1+1}^{\perp} \to M_1$. Then we have $\mu_1 \in \mathrm{Hom}((M_1, M_1), (O, \varpi_k O))$ and $\mu_2 \in \mathrm{Hom}((M_1, K_{j_1+1}), (O, \{0\}))$, and hence $\mu = \mu_1 + \mu_2 \in U_{j_1+1}$. It implies $\mathrm{Hom}((M_1, K_{j_1+1}), (O, \varpi_k O)) \subset U_{j_1+1}$. By the induction hypothesis, we obtain $\mathrm{Hom}((M, \bigoplus_{j=0}^{j_1+1} \varpi_k^j K_j), (O, \varpi_k^{j_1+1} O)) \subset \iota_{M^{\mathbb{D}}, j_1+1}^{-1}(\varpi_k^{-N} U)$. By induction on $j_1 \in \mathbb{N}$, we get a sequence $(K_j)_{j \in \mathbb{N}} \in \mathrm{ASK}(M)^{\mathbb{N}}$ with $\mathrm{Hom}((M, \bigoplus_{j=0}^{j_1} \varpi_k^j K_j), (O, \varpi_k^{j_1} O)) \subset \iota_{M^{\mathbb{D}}, j_1}^{-1}(\varpi_k^{-N} U)$ for any $j_1 \in \mathbb{N}$ such that the addition $\prod_{j=0}^{j_1} \varpi_k^j K_j \hookrightarrow M$ is a homeomorphism onto the closed image. Since $M$ is complete, the addition $\bigoplus_{j \in \mathbb{N}} \varpi_k^j K_j \to M$ extends to an $A \in \mathrm{Hom}(\prod_{j \in \mathbb{N}} \varpi_k^j K_j, M)$ by Lemma 4.27. Put $K := A(\prod_{j \in \mathbb{N}} \varpi_k^j K_j) \in \mathrm{K}(M) \subset M$. We obtain $\mathrm{Hom}((M_k, K), (k, \varpi_k O)) \subset \iota(\varpi_k^{-N} U)$. It implies $\iota(U) = \Pi_{(M_k)^{\mathbb{D}_k}}^N(\iota(\varpi_k^{-N} U)) \in \mathscr{O}((M_k)^{\mathbb{D}_k})$. Thus $\iota$ is open. $\square$

The following is an immediate consequence of Proposition 2.8 and Theorem 4.25:

**Corollary 4.30.** *Let $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{f}}(O))$ with $M^{\mathbb{D}} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{f}}(O))$. For any $\ell \in M^{\mathbb{D}\mathbb{D}}$, there is a $K \in \mathrm{K}(M)$ such that* $\mathrm{Hom}((M, K), (O, \varpi_k^j O)) \subset \ell^{-1}(\varpi_k^j O)$ *for any $j \in \mathbb{N}$.*

Finally, we construct a $\mathbb{D}$-reflexive full subcategory of $\mathrm{Mod}_{\mathrm{l}}(O)$ containing separable discrete objects and separable compact objects.

**Definition 4.31.** An $M \in \mathrm{Mod}(O)$ is said to be *reductively $\sigma$-compact* if $M_{\overline{k}}$ is $\sigma$-compact. We denote by $\mathrm{LC}^{\sigma c}(O) \subset \mathrm{LC}(O)$ of reductively $\sigma$-compact locally convex $O$-modules, and put $\mathrm{LC}^{\sigma c\mathrm{f}}(O) := \mathrm{LC}^{\sigma c}(O) \cap \mathrm{LC}^{\mathrm{f}}(O)$.

We note that $\mathrm{LC}^{\sigma c\mathrm{f}}(O)$ contains separable objects in $\mathrm{ob}(\mathrm{LC}^{\mathrm{Ban}}(O)) \cup \mathrm{ob}(\mathrm{Mod}^{\mathrm{ch}}_{\mathrm{fl}}(O))$. Therefore the following is a partially extension of Corollary 4.24:

**Theorem 4.32.** *The full subcategory* $\mathrm{LC}^{\sigma c\mathrm{f}}(O) \subset \mathrm{Mod}_{\mathrm{l}}(O)$ *is $\mathbb{D}$-reflexive.*

In order to verify Theorem 4.32, it suffices to show the following two lemmata:

**Lemma 4.33.** *For any $M \in \mathrm{ob}(\mathrm{LC}^{\sigma c\mathrm{f}}(O))$, $M^{\mathbb{D}}$ lies in $\mathrm{ob}(\mathrm{LC}^{\sigma c\mathrm{f}}(O))$.*

**Lemma 4.34.** *For any $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{f}}(O))$ with $M^{\mathbb{D}} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{f}}(O))$, $\eta_M$ is an isomorphism.*

In order to show Lemma 4.33, we prepare the following lemma:

**Lemma 4.35.** *Every $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{cgh}}(O))$ is $(\mathrm{K}, O)$-admissible.*

We note that an $M \in \mathrm{ob}(\mathrm{Mod}(O))$ lies in $\mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(O))$ if and only if $M$ can be embedded into the direct product of a family in $\mathrm{ob}(\mathrm{LC}^{\mathrm{Ban}}(O))$ by Proposition 3.17 and Proposition 3.17 and [9] Theorem 3.4.10. Therefore Lemma 4.35 can be verified in a similar way to the proof of Lemma 4.3 using Proposition 2.24 and Proposition 2.25.

**Proof of Lemma 4.34.** By Theorem 2.22, Corollary 3.26, and Lemma 4.35, it suffices to verify the surjectivity of $\eta_M$. Let $m_0 \in M^{\mathbb{D}\mathbb{D}}$. By Corollary 4.30, there is a $K \in \mathrm{K}(M)$ with $\mathrm{Hom}((M, K), (O, \varpi_k^j O)) \subset m_0^{-1}(\varpi_k^j O)$ for any $j \in \mathbb{N}$. For any $\mu \in M^{\mathbb{D}}$ with $\mu|_K = 0$, we have $\mu \in \bigcap_{m \in K} \ker(\eta_M(m)) \subset \ker(\ell)$. Therefore $m_0$ induces an $O$-linear homomorphism $m_1 \colon K^{\mathbb{D}} \to O$ through the restriction map $M^{\mathbb{D}} \to K^{\mathbb{D}}$ by Theorem 3.21. We have $\varpi_k^j K^{\mathbb{D}} \subset m_1^{-1}(\varpi_k^j O)$ for any $j \in \mathbb{N}$ by the choice of $K$. It implies the continuity of $m_1$. By Proposition 4.23, there is a unique $m_2 \in K$ with $m_1 = \eta_K(m_2)$. In particular, we have $m_0(\mu) = \mu(m_2)$. Thus we obtain $m_0 = \eta_M(m_2)$. $\square$

**Proof of Lemma 4.33.** By $M \in \mathrm{ob}(\mathrm{LC}(O))$, we have $\varpi_k M \in \mathscr{C}(M)$ and hence $M_{\overline{k}} \in \mathrm{ob}(\mathrm{LC}^{\sigma c\mathrm{f}}(\overline{k}))$. Therefore we obtain $(M_{\overline{k}})^{\mathbb{D}_{\overline{k}}} \in \mathrm{ob}(\mathrm{LC}^{\sigma c\mathrm{f}}(\overline{k}))$ by Theorem 4.20. It ensures $(M^{\mathbb{D}})_{\overline{k}} \in \mathrm{ob}(\mathrm{LC}^{\sigma c\mathrm{f}}(\overline{k}))$ by Theorem 3.28, and hence $M^{\mathbb{D}} \in \mathrm{ob}(\mathrm{LC}^{\sigma c}(O))$. Take a countable $\mathscr{S} \subset \mathrm{K}(M_{\overline{k}})$ with $\bigcup_{\mathscr{K} \in \mathscr{S}} \mathscr{K} = M_{\overline{k}}$. For each $\mathscr{K} \in \mathscr{S}$, there is a $K \in \mathrm{K}(M)$ whose image in $M_{\overline{k}}$ coincides with $\mathscr{K}$ by Lemma 4.28. Take a complete representative $S \subset \mathrm{K}(M)$ of $\mathscr{S}$, and put $M_0 := \sum_{K \in S} K$. Then $M_0^{\mathbb{D}}$ is first countable by Lemma 2.30, while $M^{\mathbb{D}}$ is complete by Proposition 2.18 and Proposition 2.25. Let $\iota \in \mathrm{Hom}(M^{\mathbb{D}}, M_0^{\mathbb{D}})$

denote the restriction map. We verify $M^{\mathbb{D}} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{f}}(O))$ by showing that $\iota$ is an isomorphism. We have $M = M_0 + \varpi_k M$ by the choice of $S$, and $M = M_0 + \varpi_k^j M$ by induction for any $j \in \mathbb{N}$. It implies that $M_0$ is dense in $M$ by the boundedness of $M$. By the density of $M_0 \subset M$ and the completeness of $O$, $\iota$ is bijective. Let $U \in \mathscr{O}(M^{\mathbb{D}})$. We show $\iota(U) \in \mathscr{O}(M_0^{\mathbb{D}})$. Take a $(K_0, n) \in \mathrm{K}(M) \times \mathbb{N}$ with $\mathrm{Hom}((M, K_0), (O, \varpi_k^n O))$. We construct an $F_n \in 2_{\mathrm{f}}^S$ with $K_0 \subset \sum_{K \in F_n} K + \varpi_k^n M$ in an inductive way. Put $F_0 := \emptyset$. By $M \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(O))$, we have $M_{\overline{k}} \in \mathrm{ob}(\mathrm{LC}^{\mathrm{h}}(\overline{k}))$ and $\mathrm{K}(M_{\overline{k}}) \subset \mathscr{C}(M_{\overline{k}})$. It implies that the image $\overline{K}_0$ of $K_0$ in $M_{\overline{k}}$ is compact. Therefore there is an $\mathscr{F}_1 \in 2_{\mathrm{f}}^{\mathscr{S}}$ with $\overline{K}_0 \subset \sum_{\mathscr{K} \in \mathscr{F}_1} \mathscr{K}$ by Lemma 2.30. Let $F_1 \in 2_{\mathrm{f}}^S$ the preimage of $\mathscr{F}_1$. Then we have $K_0 \subset \sum_{K \in F_1} K + \varpi_k M$. Suppose that we have constructed an $F_j \in 2_{\mathrm{f}}^S$ with $K_0 \subset \sum_{K \in F_j} K + \varpi_k^j M$ for a $j \in \mathbb{N} \cap [1, n-1]$. By $K_0 \in \mathrm{K}(M)$ and $F_j \subset \mathrm{K}(M)$, we have $K_0 + \sum_{K \in F_j} K \in \mathrm{K}(M)$ by the continuity of the addition. By $M \in \mathrm{ob}(\mathrm{LC}(O))$, we have $\varpi_k^j M \in \mathscr{C}(M)$ and $\Pi_M^j \colon M \to \varpi_k^j M$ is an isomorphism. Therefore we obtain $\varpi_k^j M \cap (K_0 + \sum_{K \in F_j} K) \in \mathrm{K}(\varpi_k^j M)$ and $(\Pi_M^j)^{-1}(K_0 + \sum_{K \in F_j} K) \in \mathrm{K}(M)$. By Lemma 2.30, there is an $F \in 2_{\mathrm{f}}^S$ with $(\Pi_M^j)^{-1}(K_0 + \sum_{K \in F_j} K) \subset \sum_{K \in F} K + \varpi_k M$. Put $F_{j+1} := F_j \cup F \in 2_{\mathrm{f}}^S$. We have $K_0 \subset \sum_{K \in F_j} K + (\varpi_k^j M \cap (K_0 + \sum_{K \in F_j} K)) = \sum_{K \in F_j} K + \varpi_k^j (\Pi_M^j)^{-1}(K_0 + \sum_{K \in F_j} K) \subset \sum_{K \in F_j} K + \varpi_k^j (\sum_{K \in F} K + \varpi_k M) \subset \sum_{K \in F_{j+1}} K + \varpi_k^{j+1} M$. By induction on $j$, we obtain an $F_n \in 2_{\mathrm{f}}^S$ with $K_0 \subset \sum_{K \in F_n} K + \varpi_k^n M$. It ensures $\mathrm{Hom}((M, \sum_{K \in F_n} K), (O, \varpi_k^n O)) \subset \mathrm{Hom}((M, K_0), (O, \varpi_k^n O)) \subset \iota(U)$. Thus $\iota$ is open. $\square$

This completes the proof of Theorem 4.32. We obtain a $\mathbb{D}$-reflexive full subcategory.

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