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Convex Surfaces with Planar Polar Sets and Point-Source Shadow-Boundaries

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Generalizing certain characteristic properties of ellipsoids, we describe all convex hypersurfaces in $\mathbb{R}^n$, possibly unbounded, whose polar sets and point-source shadow-boundaries satisfy planarity conditions.

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1. Introduction and Preliminaries

Solid ellipsoids in $\mathbb{R}^n$ can be characterized as convex bodies whose polar sets and point-source shadow-boundaries satisfy certain planarity conditions (see the references below). Our goal is to generalize some of these characteristic properties to the case of convex solids with quadratic boundaries.

In what follows, by a *convex solid* we mean an n-dimensional closed convex set in $\mathbb{R}^n$ distinct from the entire space (*convex bodies* are compact convex solids). As usual, $\operatorname{bd} K$ and $\operatorname{int} K$ denote, respectively, the boundary and interior of a convex solid K. We recall that a *convex hypersurface* is the boundary of a convex solid in $\mathbb{R}^n$. This definition includes a hyperplane (as the boundary of a closed halfspace) and a pair of parallel hyperplanes (as the boundary of a closed slab between these hyperplanes).

In a standard way, a *quadric surface* (or a second degree surface) in $\mathbb{R}^n$ is the locus of points $x = (\xi_1, \ldots, \xi_n)$ which satisfy a quadratic equation

$$F(x) \equiv \sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k + 2\sum_{i=1}^{n} b_i\xi_i + c = 0, \tag{1}$$

where not all scalars a_{ik} are zero. We say that a quadric surface $Q \subset \mathbb{R}^n$ given by (1) is a *quadric hypersurface* provided its complement $\mathbb{R}^n \setminus Q$ contains

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at least two components. The latter happens if and only if either $F(x)$ is a complete square of a nonzero linear function, which describes a hyperplane, or both open sets $\{x \in \mathbb{R}^n : F(x) < 0\}$ and $\{x \in \mathbb{R}^n : F(x) > 0\}$ are nonempty.

There are different ways to define convex quadrics (see [19] for a discussion). The most general way is given in [15]: a convex hypersurface $S \subset \mathbb{R}^n$ is a *convex quadric* provided there is a quadric hypersurface $Q \subset \mathbb{R}^n$ and a component U of $\mathbb{R}^n \setminus Q$ such that U is a convex set and $S = \operatorname{bd} U$.

We will need some notation and terminology. A convex solid $K \subset \mathbb{R}^n$ is called *strictly convex* if $\operatorname{bd} K$ contains no line segment, and is called *regular* if every point $x \in \operatorname{bd} K$ belongs to precisely one support hyperplane of K. The recession cone of the convex solid K is given by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n \mid x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \geq 0\}.$$

It is known that $\operatorname{rec} K$ is a closed convex cone with apex o (the origin of $\mathbb{R}^n$), and $\operatorname{rec} K \neq \{o\}$ if and only if K is unbounded (see, e. g., [22] for general properties of convex sets). The subspace $\operatorname{lin} K = \operatorname{rec} K \cap (-\operatorname{rec} K)$ is called the *lineality space* of K. Given a subspace $L \subset \mathbb{R}^n$ complementary to $\operatorname{lin} K$, the solid K can be expressed as the direct sum $K = (K \cap L) \oplus \operatorname{lin} K$, where $K \cap L$ is a line-free closed convex set.

Given distinct points $u, v \in \mathbb{R}^n$, we write $[u, v]$ and (u, v) for the segment and the open segment with endpoints u and v; furthermore, $[u, v\rangle$ and $\langle u, v\rangle$ denote, respectively, the closed halfline and the line determined by u and v.

As shown in [15], a convex hypersurface $S \subset \mathbb{R}^n$ is a convex quadric if and only if there are Cartesian coordinates $\xi_1, \dots, \xi_n$ such that S is the locus of points $x = (\xi_1, \dots, \xi_n)$ given by one of the conditions

$$\begin{array}{ll}
a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, & 1 \leq k \leq n, \\
a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1,\ \xi_1 \geq 0, & 2 \leq k \leq n, \\
a_1\xi_1^2 = 0, & \\
a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0,\ \xi_1 \geq 0, & 2 \leq k \leq n, \\
a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, & 2 \leq k \leq n,
\end{array}$$

where all scalars a_i involved are positive. Several characterizations of convex quadrics are obtained in [13]–[21]. A recurrent geometric description of convex quadrics in $\mathbb{R}^n$ can be given as follows.

(a) Convex quadrics in $\mathbb{R}^2$ are ellipses, branches of hyperbolas, parabolas, convex cones, lines, and pairs of parallel lines.

(b) Convex quadrics in $\mathbb{R}^n$, $n \geqslant 3$, are ellipsoids, sheets of elliptic hyperboloids of two sheets, sheets of elliptic cones, elliptic paraboloids, and cylinders based on convex quadrics in $\mathbb{R}^{n-1}$.

2. Polar Sets

Let $l \subset \mathbb{R}^n$ be a line through a point p, expressed as $l = p + tc$, where $c \neq o$ and $t \in \mathbb{R}$. Given on $l \setminus \{p\}$ distinct points

$$u = p + t_1 c, \qquad v = p + t_2 c, \qquad p^* = p + t^* c,$$

the cross-ratio $(u, v; p, p^*)$ is defined by

$$(u, v; p, p^*) = \frac{t_1/t_2}{(t_1 - t^*)/(t_2 - t^*)}.$$

Obviously,

$$|(u, v; p, p^*)| = (\|u - p\|/\|v - p\|)/(\|u - p^*\|/\|v - p^*\|).$$

A point $p^* \in l$ is called *harmonic conjugate* of p with respect to u and v provided $(u, v; p, p^*) = -1$, which gives $t^* = 2t_1t_2/(t_1 + t_2)$. Clearly, harmonic conjugate p^* is a unique real point on l if $(u + v)/2 \neq p$.

Let $S \subset \mathbb{R}^n$ be a hypersurface and $p \in \mathbb{R}^n \setminus S$ be a point such that any line through p which is not supporting S meets S at most twice. (For instance, convex and quadric hypersurfaces have this property.) Denote by $H_p(S)$ the set (possibly, empty) of all harmonic conjugates of p with respect to the pairs of distinct points at which the lines through p meet S. The set $H_p(S)$ is called the *polar set* of p with respect to S.

It is known that for a quadric hypersurface $Q \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n \setminus Q$, the polar set $H_p(Q)$ lies in a hyperplane (for the sake of completeness, we give the proof of this statement in Lemma 2.3 below). In particular, this is true when Q is an ellipsoid. In this regard, there are various results characterizing solid ellipsoids among all convex bodies (see, e. g., Arocha, Montejano, and Morales [1], Gruber and Odor [4], Kelly and Straus [6], and Kojima [7]).

Theorem 2.1 provides a characteristic property of convex solids with quadratic boundary among all convex solids in $\mathbb{R}^n$. For $n = 2$ our method of proof is based on a local version of Kojima [7] argument (developed in Lemmas 2.4–2.7); for $n \geq 3$ it uses the fact that a convex hypersurface $S = \operatorname{bd} K$ is a convex quadric if and only there is a point $x \in \operatorname{int} K$ such that every section of $\operatorname{bd} K$ by a 2-dimensional plane through x is a convex quadric curve (see [13]).

Given a convex solid $K \subset \mathbb{R}^n$, the set $I(K)$ will denote the family, possibly empty, of all points $p \in \mathbb{R}^n \setminus K$ which satisfy the condition $K \not\subset p + \operatorname{rec} K$. For instance, if C is an n-dimensional closed convex cone with apex a, then $I(C) = \mathbb{R}^n \setminus (C \cup (2a - C)$. It is easy to see that (i) the set $I(K)$ is open, (ii) $I(K) = \varnothing$ if and only if K is a halfspace or a slab, (iii) $I(K)$ contains a certain neighborhood of K if K is strictly convex, (iv) $I(K) = \mathbb{R}^n \setminus K$ if K is a convex body.

Theorem 2.1. *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a convex solid which is neither a halfspace nor a slab, and let $S = \operatorname{bd} K$. The following conditions are equivalent.*

1) *For every point $p \in I(K)$, the polar set $H_p(S)$ lies in a hyperplane.*
2) *There is an open neighborhood U of K such that for every point p in $U \cap I(K)$, the polar set $H_p(S)$ lies in a hyperplane.*
3) *S is a convex quadric.*

Remark 2.2. It is easy to see that for a given a convex solid $K \subset \mathbb{R}^n$, the polar set $H_p(K)$ of a variable point $p \in I(K)$ continuously depends on p. Consequently, conditions *1*) and *2*) of Theorem 2.1 can be slightly refined by choosing a dense subset of points p in $I(K)$. For instance, that can be the set of sharp points of K, considered in the next section.

Proof of Theorem 2.1. *3*) $\Rightarrow$ *1*). By the definition of S, there is a quadric hypersurface $Q \subset \mathbb{R}^n$ such that $\operatorname{int} K$ is a component of $\mathbb{R}^n \setminus Q$ and $S = \operatorname{bd} K \subset Q$. Choose any point $p \in I(K)$. From the description of convex quadrics (see the introduction) it follows that $p \notin Q$. Furthermore, if a line l through p meets K along a nontrivial segment $[u, v]$, then $Q \cap l = \{u, v\}$. This argument implies that $H_p(S) \subset H_p(Q)$. Finally, Lemma 2.3 below shows that $H_p(Q)$, and whence $H_p(S)$, lies in a hyperplane.

Lemma 2.3. *For any quadric hypersurface $Q \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n \setminus Q$, the polar set $H_p(Q)$ lies in a hyperplane.*

Proof. Suppose that Q is given by (1) and the set $H_p(Q)$ is nonempty. Choose a line $l = p + te$ meeting Q at distinct points u and v such that $(u + v)/2 \neq p$ (otherwise $H_p(Q) = \varnothing$). With $p = (p_1, \dots, p_n)$ and $c = (c_1, \dots, c_n)$, every point $x = (\xi_1, \dots, \xi_n) \in l$ has coordinates $\xi_i = p_i + tc_i$, $1 \leq i \leq n$. Substituting these values of ξ_i into (1), we obtain a quadratic equation in t:

$$A(c)\, t^2 + 2B(c, p)\, t + C(p) = 0, \tag{2}$$

where

$$A(c) = \sum_{i,k=1}^{n} a_{ik} c_i c_k, \qquad B(c, p) = \tfrac{1}{2} \sum_{i=1}^{n} \tfrac{\partial F(p)}{\partial \xi_i}\, c_i, \qquad C(p) = F(p). \tag{3}$$

One has $A(c)B(c, p) \neq 0$ because the equation (2) has distinct solutions t_1 and t_2 (corresponding to u and v) such that $t_1 + t_2 \neq 0$. If p^* denotes the harmonic conjugate p^* of p with respect to u and v, then the value t^* in the expression $p^* = p + t^* e$ equals

$$t^* = \frac{2t_1 t_2}{t_1 + t_2} = -\frac{2F(p)/A(c)}{2B(c, p)/A(c)} = -\frac{F(p)}{B(c, p)}.$$

Thus

$$p^* = p - \frac{F(p)}{B(c, p)}\, c, \quad \text{or} \quad B(c, p)(p^* - p) + F(p)\, c = o. \tag{4}$$

Let $d = (d_1, \dots, d_n)$, where

$$d_i = \tfrac{1}{2}\tfrac{\partial F(p)}{\partial \xi_i} = \sum_{k=1}^{n} a_{ik} p_k + b_i, \quad 1 \le i \le n.$$

One has $d \ne o$ because of $B(c, p) \ne 0$. Expressing $B(c, p)$ as the dot product

$$B(c, p) = d_1 c_1 + \dots + d_n c_n = d \cdot c,$$

we obtain

$$B(c, p)(p^* - p) = (d \cdot c)(p^* - p) = (d \cdot (p^* - p))c$$

because c and $(p^* - p)$ are collinear vectors. Thus (4) can be rewritten as

$$(d \cdot (p^* - p) + F(p))\, c = o,$$

which gives $d \cdot (p^* - p) + F(p) = 0$ because of $c \ne o$. With $p^* = (\xi_1^*, \dots, \xi_n^*)$, we rewrite the latter equality as

$$\sum_{i=1}^{n} d_i(\xi_i^* - p_i) + F(p) = 0. \tag{5}$$

Hence the coordinates $\xi_1^*, \dots, \xi_n^*$ of p^* satisfy the linear equation (5), which describes a hyperplane in $\mathbb{R}^n$. □

Since the implication *1)* ⇒ *2)* is obvious, it remains to show that *2)* ⇒ *3)*. We will proof this part by induction on n. The case $n = 2$ is divided into a sequence of Lemmas 2.4–2.7.

Lemma 2.4. *If a line-free convex solid $K \subset \mathbb{R}^2$ satisfies condition 2) of Theorem* 2.1 *and is not strictly convex, then K is a convex cone.*

Proof. Choose a maximal linear subset L of $\operatorname{bd} K$, which is either a closed segment $[x, z]$ of a halfline $[x, z\rangle$. Let $u \in \operatorname{rint} L$ and $v \in \operatorname{bd} K \setminus L$. Clearly, the segment $[u, v]$ meets $\operatorname{int} K$. Denote by l the line through u and v, and let $p \in l \cap (U \setminus K)$ such that $u \in (p, v)$. Obviously, $p \in I(K)$. Denote by p^* the harmonic conjugate of p with respect to u and v. Keeping p fixed and slightly varying u within $\operatorname{rint} L$ and v within $\operatorname{bd} K \setminus L$ such that the variable line $l = \langle u, v\rangle$ passes through p, we observe, due to condition *2)*, that $H_p(S)$ contains a small open segment (q_1, q_2), with $p^* \in (q_1, q_2)$. If v_1 and v_2 are the points of intersection of $\operatorname{bd} K \setminus L$ with the lines $\langle p, q_1\rangle$ and $\langle p, q_2\rangle$, then the linearity of $H_p(S)$ implies the inclusion $v \in (v_1, v_2) \subset \operatorname{bd} K \setminus L$. Consequently, $\operatorname{bd} K \setminus L$ belongs to a line. The latter is possible only if L and $\operatorname{bd} K \setminus L$ are halflines with common endpoint, that is, if K is a convex cone. □

Due to Lemma 2.4, we may further assume that the solid $K \subset \mathbb{R}^2$ is strictly convex. As above, let $S = \operatorname{bd} K$. Also, $J(K)$ will denote the set of points $p \in \mathbb{R}^2 \setminus K$ such that there are two lines through p both supporting K. Clearly, $J(K)$ is an open set in $\mathbb{R}^2$ and $J(K) \subset I(K)$.

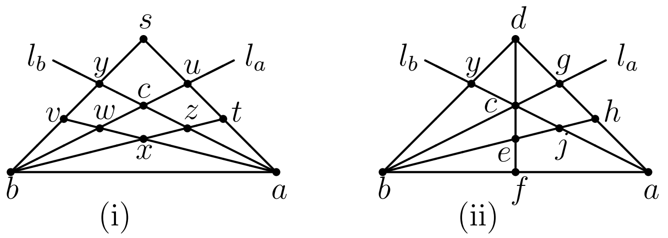

Figure 2.1: Illustration for Lemma 2.5

Lemma 2.5. *Let $K \subset \mathbb{R}^2$ be a strictly convex solid satisfying condition 2) of Theorem 2.1. If l_a is the line containing the polar set $H_a(S)$ of a point $a \in I(K)$, and l_b is the line containing the polar set $H_b(S)$ of a point $b \in l_a \cap J(K)$, then $a \in l_b$ and the polar set $H_c(S)$ of the intersection c of l_a and l_b lies on the line $\langle a, b\rangle$.*

Proof. First, we will prove the inclusion $a \in l_b$. Draw a line l through a which meets S and l_a at $\{s, t\}$ and u, respectively, such that $t \in (a, u) \subset (a, s)$. Rotating l further from b, we may assume that both open segments (b, s) and (b, t) meet $\operatorname{int} K$. Denote by v the point of intersection of S and (b, s). Let (a, v) meet l_a and S at w and x, respectively (see case (i) in Fig. 2.1). By the hypothesis, $(s, t; u, a) = (v, x; w, a)$. Put x' be the point of intersection of $[a, v]$ and $[b, t]$. Because the central projection with center b keeps the cross-ratio, one has $(s, t; u, a) = (v, x'; w, a)$. Therefore, $x = x' \in (b, t)$.

Denote the intersections of l_b with $[b, s]$ and $[b, t]$ by y and z, respectively. By the hypothesis, $(s, v; y, b) = (t, x; z, b)$. As above, this gives collinearity of y, z, and a. Hence $a \in \langle y, z\rangle = l_b$.

For the second statement, draw a line through c, which meets S at $\{d, e\}$ and the line $\langle a, b\rangle$ at f (see case (ii) in Fig. 2.1). Let (a, d) meet l_a and S at g and h, respectively. Suppose that (b, h) meets l_b and S at j and k, respectively. By the hypothesis,

$$(a, g; h, d) = (a, c; j, y) = (h, k; j, b) = -1.$$

Hence the points d, c, k are collinear. Consequently, $e = k$. This argument gives the equality $(d, e; c, f) = (d, h; g, a)$. Thus $(d, e; c, f)$ is a harmonic range. In conclusion, the polar set $H_c(S)$ lies on the line $\langle a, b\rangle$. $\square$

Lemma 2.6. *A convex quadric curve $P \subset \mathbb{R}^2$ supported by the lines $\eta = \pm\xi$ at the points $(1, \pm 1)$ is given by*

$$\xi^2 + (a-1)\eta^2 - 2a\eta + a = 0, \quad a \geq 0, \ y \geq 0. \tag{6}$$

The curve P is the convex cone $\eta = |\xi|$ when $a = 0$, a branch of hyperbola when $0 < a < 1$, the parabola $\eta = (\xi^2 + 1)/2$ when $a = 1$, and an ellipse when $a > 1$.

Proof. Let $Q \subset \mathbb{R}^2$ be a quadric curve containing the convex quadric P. Suppose that Q is expressed by

$$f(\xi, \eta) \equiv a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 + 2b_1\xi + 2b_2\eta + c = 0. \tag{7}$$

It is well-known that the tangent line to Q at a regular point $u = (p, q) \in Q$ is given by the linear equation

$$(a_{11}p + a_{12}q + b_1)\xi + (a_{12}p + a_{22}q + b_2)\eta + (b_1p + b_2q + c) = 0.$$

The conditions on P at $(1, 1)$ and $(1, -1)$ result in similar conditions on Q. Consequently, the coefficients of $f(\xi, \eta)$ satisfy a certain system of linear equations, whose solution is

$$a_{12} = 0, \qquad a_{22} = c - a_{11}, \qquad b_1 = 0, \qquad b_2 = -c, \qquad a_{11}, c \in \mathbb{R}.$$

Therefore, (7) becomes

$$a_{11}\xi^2 + (c - a_{11})\eta^2 - 2c\eta + c = 0.$$

Observe that $a_{11} \neq 0$, since otherwise Q would be the line $\eta = 1$. With $a = c/a_{11}$, we obtain (6). According to the standard classification of quadric curves, the shape of Q, and whence the form of P, depend on the sign of the invariant

$$a_{12}^2 - a_{11}a_{22} = 4(1 - a).$$

Consequently, P has the stated shape for all $a \geq 0$. Since the convex curve P, bounded by the lines $\eta = \pm\xi$ should lie in the region $\eta \geq |\xi|$, the quadric curve Q cannot be a hyperbola whose branches are disjoint from this region. Consequently, $a \geq 0$. $\square$

Lemma 2.7. *If $K \subset \mathbb{R}^2$ is a strictly convex solid satisfying condition 2) of Theorem 2.1, then the curve $S = \operatorname{bd} K$ is a convex quadric.*

Proof. First, we are going to prove that any point $x \in S$ belongs an open piece of convex quadric $\gamma \subset S$. For this, choose a point $p \in U \cap J(K)$ satisfying the following properties: (i) the line $l = \langle p, x\rangle$ meets $\operatorname{int} K$ such that $K \cap l$ is a segment, (ii) if h_1 and h_2 are the halflines with endpoint p supporting K at z_1 and z_2, respectively, and if c is the point of intersection of l and the line $\langle z_1, z_2\rangle$, then the harmonic conjugate c^* of c with respect to z_1 and z_2 belongs to $U \cap J(K)$, as depicted in Fig. 2.2. By Lemma 2.5, the line $m = \langle p, c^*\rangle$ contains the polar set $H_c(S)$.

Denote by u the point at which the lines m and $\langle x, z_1\rangle$ meet. The line containing $H_u(S)$ meets $\langle p, z_1\rangle$ at a certain point v such that the line $\langle v, x\rangle$ supports K at x. Choosing suitable coordinates in $\mathbb{R}^2$, we may assume that $p = (0, 0)$, $z_1 = (-1, 1)$, and $z_2 = (1, 1)$. Lemma 2.6 implies the existence of the unique convex quadric C_x passing through x and supported by the lines $\eta = \pm\xi$ at

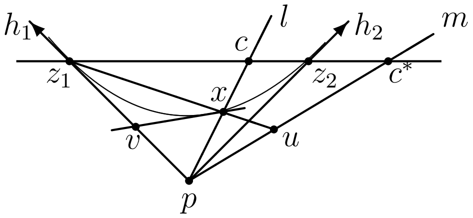

Figure 2.2: Illustration for Lemma 2.7

z_1 and z_2, respectively. Since m contains the polar set $H_c(C_x)$, the line $\langle v, x\rangle$ supports C_x at x.

By continuity, the above argument holds when p is fixed and x varies within a small arc γ of S. Therefore, γ is an envelope of the respective arcs of C_x, $x \in \gamma$. Since the family of convex quadric curves C_x is nested (as it follows from Lemma 2.6), the latter is possible only if γ itself is an arc of convex quadric curve.

By the above argument, S may be expressed as a countable union of overlapping pieces of strictly convex quadric curves. Since any quadric curve is uniquely determined by five points, with no four on a line (see, e. g., [20, Theorem 4]), it follows that S itself is a strictly convex quadric curve. $\square$

To finalize the proof of the statement *2)* $\Rightarrow$ *3)* of Theorem 2.1, it remains to consider the case $n \geq 3$. Choose a point $q \in \operatorname{int} K$ and a two-dimensional plane $N \subset \mathbb{R}^n$ through q. We state that the relative boundary of the plane set $K \cap N$ is a convex quadric. Indeed, this is true if $K \cap N$ is a closed halfplane of a slab of N. Suppose that $K \cap N$ is neither a halfplane nor a slab of N. Then $K \cap N$ is line-free. Clearly,

$$(U \cap N) \cap I(K \cap N) \subset U \cap N \cap I(K).$$

If $p \in (U \cap N) \cap I(K \cap N)$, then, by condition *2)*, there is a hyperplane H containing $H_p(K)$. The inclusion

$$H_p(K \cap N) = H_p(K) \cap N \subset H \cap N$$

shows that the polar set $H_p(K \cap N)$ lies on the line $H \cap N$. Thus $K \cap N$ satisfies condition *2)* of Theorem 2.1 with respect to $U \cap N$ in N. A combination of Lemmas 2.4 and 2.7 shows that the curve $\operatorname{bd} K \cap N = \operatorname{rbd}(K \cap N)$ is a convex quadric (indeed, it is either a convex cone, which is a degenerate convex quadric, or a strictly convex quadric).

Since the above argument holds for every 2-dimensional plane N through q, Theorem 3 from [13] shows that $\operatorname{bd} K$ is a convex quadric. $\square$

3. Point-Source Shadow-Boundaries

If $K \subset \mathbb{R}^n$ is a convex solid, and p is a point in $\mathbb{R}^n \setminus K$, then the *shadow-boundary* of K with respect to p, denoted $S_p(K)$, is the set (possibly, empty) of points in $\mathrm{bd}\, K$ at which the halflines with endpoint p support K. (Various types of shadow-boundaries of convex solids are studied in [10, 12].) As shown in [17, Lemma 3.1], $S_p(K) = \varnothing$ if and only if K is a slab or $K \subset p + \mathrm{int}\,(\mathrm{rec}\, K)$.

Given a convex set $M \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n \setminus M$, denote by $C_p(M)$ the convex cone with apex p generated by M; equivalently, $C_p(M)$ is the union of all halflines $[p, x\rangle = \{p + \lambda(x - p) : \lambda \geq 0\}$, where $x \in M$. In these terms, for a convex solid $K \subset \mathbb{R}^n$, one has

$$S_p(K) = \mathrm{bd}\, K \cap \mathrm{bd}\, C_p(K). \tag{8}$$

It is well known that the shadow boundary of a solid ellipsoid in $\mathbb{R}^n$ lies in a hyperplane. In this regard, there are various results characterizing solid ellipsoids among all convex bodies. For instance, Kubota [8, 9] showed that a regular convex body $K \subset \mathbb{R}^3$ is a solid ellipsoid provided every shadow-boundary $S_p(K)$, $p \in \mathbb{R}^3 \setminus K$, lies in a plane. Burton [2] refined this result, by proving that a convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is a solid ellipsoid if and only if it satisfies the following condition:

(A) for every point $p \in \mathbb{R}^n \setminus K$, there is a hyperplane $H \subset \mathbb{R}^n$ satisfying the inclusion

$$S_p(K) \subset H. \tag{9}$$

In fact, Burton [2] showed that the points p in condition (A) can be chosen at a distance less than a given scalar $\delta > 0$ from K. Following Burton's ideas, a characterization of all convex solids $K \subset \mathbb{R}^n$ satisfying condition (A) is described in [17].

Marchaud [11] (for $n = 3$) and Gruber [3] (for all $n \geq 3$) proved, with some additional restrictions on the placement of p, a slightly stronger statement: a convex body $K \subset \mathbb{R}^n$ is a solid ellipsoid if and only if it satisfies the condition

(B) for every point $p \in \mathbb{R}^n \setminus K$, there is a hyperplane H such that

$$C_p(H \cap K) = C_p(K). \tag{10}$$

From the proof of Theorem 3.4 (see below) it follows that condition (B), applied to convex solids, characterizes solid ellipsoids, paraboloids, and cylinders based on their low-dimensional versions. Restricting p to a small neighborhood of K also allows to include solid hyperbolids and cylinders based on their low-dimensional versions, but not solid elliptic cones. To involve elliptic cones into consideration, we will use some new conditions on a convex solid $K \subset \mathbb{R}^n$.

A nonempty shadow-boundary $S_p(K)$ of a convex solid $K \subset \mathbb{R}^n$ is called *sharp* provided every halfline with endpoint p which supports K has precisely one common point with K; the respective point p is also called *sharp* (see, e. g., [10]). Ivanov [5] proved that the set of non-sharp points in $\mathbb{R}^n \setminus K$ has σ-finite $(n-1)$-dimensional Hausdorff measure. This fact immediately implies that the set of all sharp points of a convex solid $K \subset \mathbb{R}^n$ is dense in $I(K)$. We recall that $I(K)$ denotes the set of all points $p \in \mathbb{R}^n \setminus K$, possibly empty, which satisfy the condition $K \not\subset p + \operatorname{rec} K$. In these terms, the following conditions on a convex solid $K \subset \mathbb{R}^n$ will be used:

(C) for every *sharp* point $p \in I(K)$, there is a hyperplane $H \subset \mathbb{R}^n \setminus \{p\}$ satisfying the inclusion (9).

(D) for any point $p \in I(K)$, there is a hyperplane $H \subset \mathbb{R}^n \setminus \{p\}$ such that every halfline h with apex p which supports K along a *bounded* set meets H at a point from $S_p(K)$.

We will say that the convex solid K satisfies any of condition (A)–(D) with respect to a set $U \subset \mathbb{R}^n$, provided this condition holds for all points $p \in U \cap I(K)$.

Lemma 3.1. *For a convex solid $K \subset \mathbb{R}^n$ distinct from a halfspace or a slab, one has* (A) $\Rightarrow$ (C) $\Leftrightarrow$ (D) *with respect to any open set $U \subset \mathbb{R}^n$. Furthermore,* (B) $\Leftrightarrow$ (D) *provided for any point $p \in U \cap I(K)$ all boundary halflines of the cone $C_p(K)$ meets K along bounded sets.* □

Proof. The implication (A) $\Rightarrow$ (C) is obvious, while the equivalence of conditions (C) and (D) follows from the following arguments: (i) the set of sharp points is dense in $I(K)$, (ii) the shadow-boundary $S_p(K)$ continuously depends on a variable sharp point $p \in I(K)$ (this is not true for all points in $I(K)$).

If every boundary halfline of the cone $C_p(K)$ meets K along a bounded set, then the cone $C_p(K)$ is closed, and the equivalence of conditions (B) and (D) becomes obvious. □

Remark 3.2. Any of the condition (C) and (D) does not imply that the hyperplane H meets every boundary halfline of the cone $C_p(K)$, as it follows from condition (B). For instance, if

$$K = \{(x, y) : x \geq 0, y \geq 0, x + y \geq 1\} \subset \mathbb{R}^2 \quad \text{and} \quad p = (2, -1),$$

then

$$C_p(K) = \{p\} \cup \{(x, y) : x + y \geq 1 \text{ and } y > -1\}.$$

Consequently,

$$S_p(K) = \{(x, y) : x \geq 0, y \geq 0, x + y = 1\},$$

and any line $H \subset \mathbb{R}^2$ given by $y = kx$, $k > 0$, satisfies condition (C) and does not meet the boundary halfline $\{(x, -1) : x \geq 1\}$ of $C_p(K)$.

Remark 3.3. Both conditions (C) and (D) are weaker than condition 1) of Theorem 2.1. Indeed, given a halfline h with apex p which supports K along a bounded set, there is a small scalar $\varphi > 0$ such that for any halfline h' with apex p which meets $\operatorname{int} K$ and forms with h an angle less than φ, the set $K \cap h'$ is a segment. Hence h' meets the polar set $H_p(K)$. Consequently, if $H_p(K)$ lies in a hyperplane H, then h meets H at a limit point of $H_p(K)$, which belongs to $S_p(K)$.

Theorem 3.4. *Given a convex solid $K \subset \mathbb{R}^n$, $n \geq 3$, which is neither a halfspace nor a slab, the following statements are equivalent.*

1) *K satisfies any of the equivalent conditions* (C) *and* (D).

2) *There is an open neighborhood U of K such that K satisfies any of the equivalent conditions* (C) *and* (D) *with respect to U.*

3) *K has one of the shapes below:*
   - (a) $\operatorname{bd} K$ *is a convex quadric,*
   - (b) $\dim(\operatorname{lin} K) = n-2$ *and K is a direct sum of* $\operatorname{lin} K$ *and a* 2*-dimensional line-free convex solid,*
   - (c) $\dim(\operatorname{lin} K) = n-3$ *and K is a direct sum of* $\operatorname{lin} K$ *and a* 3*-dimensional line-free closed convex cone.*

We observe that the above shapes (a)–(c) are not mutually exclusive: a cylinder based on a solid 2-dimensional convex quadric is a particular case of (b), and a cylinder based on a convex solid 3-dimensional elliptic cone is a particular case of (c).

The following lemma gives an important step in the proof of Theorem 3.4.

Lemma 3.5. *If a line-free convex solid $K \subset \mathbb{R}^3$ satisfies condition 2) of Theorem* 3.4 *and is not strictly convex, then K is a convex cone.*

Proof. Since K is not strictly convex, there is a plane $L \subset \mathbb{R}^3$ supporting K such that the exposed face $F = K \cap L$ is not a singleton. Denote by V the open halfspace of $\mathbb{R}^3$ bounded by L and disjoint from K. Assuming, for contradiction, that K is not a convex cone, we divide the argument into two cases.

Case 1. $\dim F = 1$. Then F is either a line segment or a halfline. Denote by l the line containing F.

1a) Suppose first that F is a halfline with endpoint x. Since K is not a cone, we can choose a plane N through l which meets $\operatorname{int} K$ such that the section $K \cap N$ is not a plane cone. Then the curve $\Gamma = (\operatorname{bd} K \cap N) \setminus F$ is not a halfline. Consequently, there is a point $y \in U \cap (l \setminus K)$ sufficiently close to x such that a certain open halfline h with endpoint y does not lie in l and supports $K \cap N$ along a bounded set. Denote by V_1 and V_2 be the opposite open halfspaces of $\mathbb{R}^3$ determined by N.

By a continuity argument, there are two distinct open halflines h_1 and h_2 with common endpoint y, such that both halflines are sufficiently close to h, lie in V_1 and V_2, respectively, and support K along bounded sets. Again by continuity, we can choose a point $p \in U \cap (L \setminus l)$ sufficiently close to y such that the following condition is satisfied: there are open halflines h_1' and h_2' with common endpoint p both supporting K along bounded sets, with $K \cap h_1' \subset V_1$ and $K \cap h_2' \subset V_2$. Clearly, all three sets F, $K \cap h_1'$ and $K \cap h_2'$ lie in $S_p(K)$. According to condition *2*), there is a plane $H \subset \mathbb{R}^3 \setminus \{p\}$ which meets both sets $K \cap h_1'$ and $K \cap h_2'$. Furthermore, H meets all halflines $[p, x\rangle$, $x \in F$. Because every halfline $[p, z\rangle$, $z \in F$, meets K at a single point, the latter condition implies the inclusion $F \subset H$. Consequently, H cannot meet both sets $K \cap h_1'$ and $K \cap h_2'$, since they lie in the opposite halfspaces V_1 and V_2. The obtained contradiction shows that K is a cone with apex x.

1b) Suppose now that F is a segment, say $[x, z]$. We state the existence of a plane N through l which meets $\operatorname{int} K$ such that at least one of the components of the set $(\operatorname{bd} K \cap N) \setminus F$ is not a halfline. Indeed, assume for a moment that no such a plane exists. Then every section of K by a plane through l is an unbounded polygonal region with three sides: F, and two halflines with endpoints x and z, respectively. From this argument if follows that $\operatorname{bd} K$ is the union of two polygonal regions M_1 and M_2, of three sides each, with a common base $[x, z]$, and two open conic surfaces, say Ω_x and Ω_z, with apices x and z, respectively. Let h_x^i and h_z^i be the boundary halflines of M_i, with endpoints x and z, respectively, $i = 1, 2$. Choose a sharp point $p \in U \setminus l$ sufficiently close to x so that the shadow-boundary $S_p(K)$ consists of F and two open halflines $h_x \subset \Omega_x$ and $h_z \subset \Omega_z$ which have x and z as endpoints, respectively. Clearly, p can be chosen such that h_x and h_z become sufficiently close to h_x^1 and h_z^2, respectively; consequently, the polygonal line $h_x \cup F \cup h_z$ cannot lie in a plane. The latter contradicts condition *2*).

Hence there is a plane N through l, such that at least one of the components of the set $(\operatorname{bd} K \cap N) \setminus F$ is not a halfline. Assume that this component is the arc of $\operatorname{bd} K \setminus F$ with endpoint x. Repeating the argument from the case 1a), we obtain a contradiction. Thus F cannot be a segment. Summing up, K is a cone with apex x if $\dim F = 1$.

Case 2. $\dim F = 2$. Choose an extreme point x of F. First, we are going to prove that F is a cone with apex x. Indeed, assume the contrary. Then a certain line $l \subset L$ through x meets $\operatorname{rbd} F$ at another point z such that the open segment (x, z) lies in $\operatorname{rint} F$. Since K is line-free, there is a plane N through l which meets $\operatorname{int} K$ so that the set $K \cap N$ is bounded. Choose a point $p \in l \cap (U \setminus K)$ such that $z \in (p, x)$. Denote by h and h' the closed halflines with endpoint p both supporting $K \cap N$ such that $h \subset l$. Let q be the nearest to p point in $K \cap N \cap h'$.

Clearly, $N \setminus K \subset I(K)$. Since the set $I(K)$ is open and the set of sharp points of K is dense in $I(K)$, one can find a sequence of sharp points $p_1, p_2, \dots$ in $U \cap V \cap I(K)$ converging to p. By a continuity argument, there are sequences $x_i \in S_{p_i}(K)$ and $q_i \in S_{p_i}(K)$, $i \geq 1$, converging to x and q, respectively.

According to condition *2*), there is a plane $H_i \subset \mathbb{R}^3 \setminus \{p\}$ such that $S_{p_i}(K) \subset H_i$, $i \geq 1$. Furthermore, if $U_r(x)$ is a sufficiently small open ball of radius $r > 0$ centered at x, then the plane curves $\gamma_i = S_{p_i}(K) \cap U_r(x)$ tend to the plane curve $\gamma = \operatorname{rbd} F \cap U_r(x)$. The curve γ is not a line interval because its interior point x belongs to $\operatorname{ext} F$. Hence there is an index i_0 such that every curve γ_i, $i \geqslant i_0$, is not a line interval, implying that every plane H_i, $i \geqslant i_0$, is uniquely determined by γ_i. Consequently, the sequence $H_1, H_2, \dots$ tends to L. Therefore, there is an index $i_1 \geq i_0$ such that $q_i \in S_{p_i}(K) \setminus H_i$ for all $i \geq i_1$. The latter contradicts the inclusions $S_{p_i}(K) \subset H_i$, $i \geq i_1$. Hence F must be a convex cone with apex x.

Next, we are going to show that K also is a convex cone with apex x. For this, choose a boundary halfline m of F and denote by l the line containing m. Let N be a plane through l which meets $\operatorname{int} K$ such that the set $K \cap N$ is not a plane cone (otherwise K is a cone with apex x). Then the curve $\Gamma = (\operatorname{bd} K \cap N) \setminus m$ is not a halfline. Consequently, there is a point $y \in U \cap (l \setminus K)$ sufficiently close to x such that a certain open halfline h with endpoint y does not lie in l and supports $K \cap N$ along a bounded set. Denote by V_1 and V_2 be the opposite open halfspaces of $\mathbb{R}^3$ determined by N.

By a continuity argument, there are two distinct open halflines h_1 and h_2 with endpoint y, such that both are sufficiently close to h, lie in V_1 and V_2, respectively, and support K along bounded sets. Again by continuity, we can choose a sharp point $p \in (U \cap V) \setminus N$ sufficiently close to y such that the following condition is satisfied: there are open halflines h_1' and h_2' with common endpoint p both supporting K such that the sets $K \cap h_1' \subset V_1$ and $K \cap h_2' \subset V_2$ are singletons. Put $K \cap h_1' = \{z_1\}$ and $K \cap h_2' = \{z_2\}$. Obviously, $S_p(K)$ contains a curve γ which is close to m in the neighborhood of x. Since the set $m \cup \{z_1, z_2\}$ does not belong to a plane, p can be chosen so close to y that the set $\gamma \cup \{z_1, z_2\} \subset S_p(K)$ does not lie in any plane, contrary to condition *2*). Summing up, K is a cone with apex x. □

Proof of Theorem 3.4. *3*) $\Rightarrow$ *1*). (a) Suppose first that $\operatorname{bd} K$ is a convex quadric. By the proved in [17], for any point $p \in I(K)$ there is a hyperplane H satisfying the inclusion (9). Consequently, K satisfies condition (C).

Assume that K has one of the shapes (b), (c). Choose any point $p \in I(K)$ and a plane L through p which is complementary to $\operatorname{lin} K$. Then $2 \leq \dim L \leq 3$ and $K = \operatorname{lin} K \oplus (K \cap L)$. Clearly, $S_p(K) = \operatorname{lin} K \oplus S_p(K \cap L)$. From the standard properties of 2-dimensional convex solids and 3-dimensional convex cones we conclude the existence of a line N (if $\dim L = 2$) or a 2-dimensional plane N (if $\dim L = 3$) such that $N \subset L$ and every halfline h with apex p which supports $K \cap L$ along a bounded set meets N at a point from $S_p(K \cap L)$. It is easy to see that the hyperplane $H = \text{lin}\, K \oplus N$ has a similar property with respect to K.

Since the implication *1)* $\Rightarrow$ *2)* is obvious, it remains to show that *2)* $\Rightarrow$ *3)*. This part of proof is organized by induction on $n \geq 3$.

Let $n = 3$ and $K \subset \mathbb{R}^3$ be a convex solid which satisfies condition *2)* of Theorem 3.4. If K contains a line, then K is a cylinder based on a 2-dimensional line-free closed convex set M; that is, K has shape (b). Suppose K is line-free. If K is not strictly convex, then Lemma 3.5 shows that K is a convex cone. Let K be strictly convex. Then one can choose a neighborhood U' of K such that $U' \subset U \cap I(K)$. Since every point in U' is sharp for K, condition *2)* implies the existence of a plane $H(p)$ containing $S_p(K)$. Therefore, according to Lemma 4.4 of [17], the boundary of K is a strictly convex quadric.

Let $n > 3$. We continue the proof of *2)* $\Rightarrow$ *3)* assuming that it holds for all $m \leq n-1$, where $n \geq 4$. Let K be a convex solid in $\mathbb{R}^n$ that satisfies condition *2)* of the theorem. Since K is neither a halfspace nor a slab, one has $\dim(\text{lin}\, K) \leq n-2$.

First, we consider the case when $\text{lin}\, K \neq \{o\}$. Let $L \subset \mathbb{R}^n$ be a subspace complementary to $\text{lin}\, K$. Then $K = \text{lin}\, K \oplus M$, where $M = K \cap L$ is a line-free closed convex set. By the above, $\dim L = n - \dim(\text{lin}\, K) \geqslant 2$. Choose a point $p \in L \cap U \cap I(K)$. It is easy to see that $S_p(K) = \text{lin}\, K \oplus S_p(M)$. By condition *2)*, there is a hyperplane H is such that every halfline h with apex p which supports K along a bounded set meets H at a point from $S_p(K)$. Clearly, $H = \text{lin}\, K \oplus G$, where the plane $G = H \cap L$ has dimension $\dim L - 1$ and has a similar property with respect to M. Since M is line-free, the inductive assumption implies that M is a two-dimensional line-free convex solid (if $\dim M = 2$), or M is a three-dimensional convex cone (if $\dim M = 3$), or $\text{rbd}\, M$ is a line-free convex quadric (if $\dim M \geqslant 3$). Therefore, $K = \text{lin}\, K \oplus M$ has one of the shapes (a)–(c).

Finally, it remains to consider the case when $\text{lin}\, K = \{o\}$ (so that K is line-free). Choose any point $q \in \text{int}\, K$ and a hyperplane N through q such that the set $K \cap N$ is bounded. Let p be a point in $U \cap (N \setminus K)$. Since $K \cap N$ is bounded, one has $I(K \cap N) = N \setminus K$. By condition *2)*, there is a hyperplane $H \subset \mathbb{R}^n$ such that every halfline h with apex p which supports K along a bounded set meets H at a point from $S_p(K)$. From the obvious equality $S_p(K \cap N) = N \cap S_p(K)$ is follows that the $(n-2)$-dimensional plane $H \cap N$ satisfies condition *2)* with respect to $K \cap N$. Since $3 \leq \dim(K \cap N) = n - 1$, the inductive assumption gives that $K \cap N$ is an $(n-1)$-dimensional solid ellipsoid. Considering all hyperplanes N through q which meet K along bounded sets, we observe that the ellipsoid sections $N \cap \text{bd}\, K$ cover the set $Q = \text{bd}\, K \setminus (q - \text{rec}\, K)$. By Theorem 2 from [14], Q is a piece of a convex quadric (see the picture below).

If $\dim(\text{rec}\, K) \leq n-1$, then Q is dense in $\text{bd}\, K$, implying that $\text{bd}\, K$ is a convex quadric. Suppose that $\dim(\text{rec}\, K) = n$ and choose a sequence of points $q_1, q_2, \dots \in \text{int}\, K \cap \text{int}\,(q - \text{rec}\, K)$ converging to a point $q_0 \in \text{bd}\, K \cap \text{int}\,(q -$

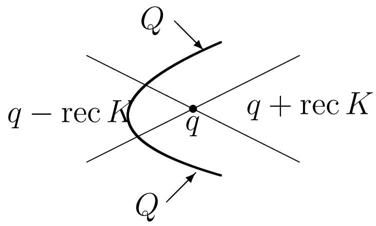

Figure 3.1:

$\operatorname{rec} K$). By the above argument, every set $Q_i = \operatorname{bd} K \setminus (q_i - \operatorname{rec} K)$ is a piece of convex quadric. Obviously,

$$Q_1 \subset Q_2 \subset \dots \quad \text{and} \quad Q_1 \cup Q_2 \cup \dots = \operatorname{bd} K \setminus \{q_0\},$$

implying that $\operatorname{bd} K$ is a convex quadric. □

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