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Clarke and Limiting Subdifferentials of Integral Functionals

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In this article some comparisons (equality, strong density or complete characterizations) are made between the Clarke and limiting subdifferentials for the class of integral functionals defined on reflexive Lebesgue spaces L_p, without any locally Lipschitzian assumptions on these functionals. Let I_f be the integral functional associated with a normal integrand $f : \Omega \times E \to I\!R \cup \{\pm\infty\}$, where E is a finite dimensional space and Ω is endowed with an atomless σ-finite measure. At a given point x such that almost everywhere the Clarke subdifferential of f_ω at $x(\omega)$ coincide with the closed convex hull of the limiting subdifferential and such that the limiting subdifferential multifunction at x, $\omega \rightrightarrows \partial^L f_\omega(x(\omega))$, admits a $q = p(p-1)^{-1}$ integrable selection, we show that the limiting subdifferential of the integral functional is strongly dense in the Clarke subdifferential. This property allows us to give an exact computation of the Clarke subdifferential by a legitimate disintegration formula. As a consequence, when for almost every ω in Ω the function $f_\omega(.)$ is locally Lipschitz at $x(\omega)$ then the limiting subdifferential of I_f at x is strongly dense in the Clarke subdifferential of I_f at x. When the Lipschitz rate is q-integrable, there is the coincidence of the Clarke subdifferential with the limiting subdifferential of I_f at x and the disintegration formula is valid. Moreover such a disintegration formula is valid for the calculus of the Clarke normal (respectively tangent) cone at a given point to the set of p-integrable selections of a measurable multifunction.

Keywords: Nonsmooth analysis, Fréchet, Clarke, limiting subdifferentials, non locally Lipschitzian integral functionals

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1. Introduction

Being extensions of the convex subdifferential in the convex case [24], [21], [16], [3], the Clarke and limiting subdifferentials are very useful tools in nonconvex optimization theory, [5], [6], [19], [20], [22], [1]. The class of integral functionals is important for applications in many areas; it was first considered in the convex case by R. T. Rockafellar [25], [28], A. D. Ioffe and V. L. Levin [15]. When the measure is atomless, the computation of the Clarke subdifferential in the

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nonconvex and locally Lipschitzian cases was first proved using regularity assumptions by F. H. Clarke [5] for Lebesgue spaces, A. D. Ioffe proves an exact computation in case L_∞ [17], F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski gave a characterization in [6] for the Hilbert Lebesgue spaces and Lipschitz integrands independent of the random variable ω. A useful inclusion is presented in [9] without any Lipschitzian assumptions on the integral functional considered, and moreover an exact calculus in the Lebesgue spaces L_p, $p \neq \infty$ is proved in [11] for locally Lipschitzian integral functionals. In [4], [23], independently, N. H. Chieu and J. P. Penot prove that the limiting subdifferential coincide in L_1 with the global subdifferential of convex analysis without any Lipschitzian assumption. Given a locally Lipschitzian function f defined on a reflexive space X the Clarke and limiting subdifferentials are linked by the formula: $\partial^C f = \overline{\mathrm{co}}\partial^L f$. When this last formula is valid at some point x the function f is said to have the Clarke-limiting property at x. In this article, for integral functionals, it is proved that there is the strong density of the limiting subdifferential in the Clarke subdifferential at some point x of a reflexive Lebesgue space when a mild assumption holds and the integrand has the Clarke-limiting property at this point, Theorem 3.1. This allows us to obtain an exact calculus of the Clarke subdifferential by a disintegration formula. Clearly when the limiting subdifferential is strongly closed, there is the coincidence of the limiting and Clarke subdifferentials. As a consequence when for almost every ω the function f_ω is locally Lipschitzian at $x(\omega)$ then the limiting subdifferential of I_f at x is empty or strongly dense in the Clarke subdifferential. The major novelty is the lack of Lipschitzian assumptions on the integral functional (as in [9]), this permits, for example, to reach the case of a decomposable constraint. In this case, given a measurable multifunction it is proved, at a given point, that the Clarke tangent (respectively normal) cone to the set of p integrable selections is *exactly* the set of p integrable (respectively q integrable) selections of the tangent (respectively normal) cone to this measurable multifunction. This is a very surprising property which is not valid for the contingent cone (see [7, Chapter VII]). After the preliminaries the statements of the main results are in Section 3 and their proofs are given in the last section.

2. Preliminaries

For r in the set of real numbers $I\!R$, we will write $|r| = \max(r, -r)$. The set of extended real numbers is denoted by $\overline{I\!R} = I\!R \cup \{\pm\infty\}$. Let a normed space $(X, \|.\|)$. Given an extended real-valued function f defined on X, the domain of f is the set of points where f is finite, $x \xrightarrow{f} x_0$ means: $x \to x_0$ and $f(x) \to f(x_0)$. If X^* is the topological dual of X, we denote by $\langle .,. \rangle$ the duality pairing between X^* and X. Given a subset Y of X $\mathrm{co}\, Y$ denotes the convex hull of the set Y and the indicator function associated with Y is the function defined by $\iota_Y(x) = 0$ if $x \in Y$, $+\infty$ if not. Recall that given a subset Y of X, a vector topology τ on X, we denote by $\overline{Y}^{\tau}$ the closure of Y for the topology τ, and by $\overline{Y}^{\operatorname{seq}\tau} = \{\tau - \lim_n y_n, (y_n)_n \in Y^{\mathbb{N}}\}$ its sequential τ-closure; when τ is the norm topology we denote simply by $\overline{Y}^{\|\cdot\|}$. For a positive real number r, if f is an extended real-valued function defined on X, define the differential quotient of f at a point x_0 of its domain by

$$q(x_0, x, r) = r^{-1}(f(x_0 + rx) - f(x_0))$$

The contingent directional subderivate of f at x_0 is the function $f^{'}(x_0; .)$ defined a $x \in X$ by the formula:

$$f^{'}(x_0; x) = \liminf_{(x^{'}, r) \to (x, 0_+)} q(x_0, x^{'}, r)$$

and the contingent subdifferential of f at x_0 is the set

$$\partial f(x_0) = \{x^* \in X^* : \forall x \in X, \langle x^*, x\rangle \leq f^{'}(x_0; x)\}\,.$$

The function f is said to be Fréchet subdifferentiable at a point $x_0 \in X$ of its domain ([19, Definition 1.83] with $\epsilon = 0$) if there exists a linear continuous function $x^* \in X^*$ verifying:

$$\liminf_{x \to x_0} \frac{f(x) - f(x_0) - \langle x^*, x - x_0\rangle}{\|x - x_0\|} \geq 0\,.$$

In this case x^* is a Fréchet subderivative of f at x_0, $\partial^F f(x_0)$ denotes the set of Fréchet subderivatives of f at x_0. Classically the Fréchet subdifferential is norm closed and convex [22, Proposition 4.10]. Notice that when E is finite dimensional the Fréchet and contingent subdifferentials coincide [22, Proposition 4.6].

An $\overline{\mathbb{R}}$-valued function f is said calm at a point $x \in \operatorname{dom} f$ if there exists a neighbourhood V of x such that for all $y \in V$

$$f(y) - f(x) \geq -c\|y - x\|\,.$$

The function f is said Lipschitzian on a subset Y of X if there exists a positive constant c, such that for all x, y in Y:

$$|f(y) - f(y^{'})| \leq c\|y - y^{'}\|\;;$$

the function f is said to be Lipschitzian around a point x of X (or locally Lipschitzian at x) if f is Lipschitz on a neighbourhood of x, in this case, the Lipschitz rate at $x \in X$, [28, 9.1], is defined by:

$$\operatorname{lip} f(x) = \limsup_{\substack{x^{'}, x^{''} \to x \\ x^{'} \neq x^{''}}} \frac{|f(x^{'}) - f(x^{''})|}{\|x^{'} - x^{''}\|}$$

The function f is said to be locally Lipschitzian when f is Lipschitzian around any point of X. The Clarke generalized directional subderivate of a function f defined on X with values in $\overline{I\!R}$, at a point x_0 of its domain is defined by [5, Theorem 2.9.1], see also [27], [26], [6], [22],

$$f^C(x_0; y) = \sup_{U \in \mathcal{N}(y)} \limsup_{\substack{r \to 0_+ \\ (x,\alpha) \xrightarrow{f} (x_0, f(x_0))}} \inf_{u \in U} r^{-1}(f(x+ru) - \alpha)$$

where $\mathcal{N}(x)$ is the family of neighbourhoods of x and $(x,\alpha) \xrightarrow{f} (x_0, f(x_0))$ means:

$$(x,\alpha) \in \operatorname{epi} f \quad \text{and} \quad (x,\alpha) \to (x_0, f(x_0)).$$

When $f = \iota_Y$, where Y is a closed subset of X, the relation

$$f^C(x_0; y) = \iota_{T^C_{x_0} Y}(y)$$

allows to define the Clarke tangent cone $T^C_{x_0} Y$ to the set Y at the point $x_0 \in Y$. The Clarke tangent cone to a set Y at a point $x \in Y$ is the set of elements z such that for every sequence $(r_n)_n$ of positive real numbers converging to zero, for every sequence $(x_n)_n$ of elements of Y converging to x, there exists a sequence $(z_n)_n$ converging to z verifying for every integer n: $x_n + r_n z_n \in Y$, see [5, Theorem 2.4.5].

When f is locally Lipschitzian at x_0, [5], [26], [27], [22], the Clarke directional subderivate at x_0 in the direction y takes the following form:

$$f^C(x_0; y) = f^\circ(x_0; y) = \limsup_{(x,\, r) \to (x_0,\, 0_+)} r^{-1}(f(x+ry) - f(x)) \ .$$

The Clarke subdifferential $\partial^C f(x_0)$ of f at x_0, [5], [26], [27], is the subset of the topological dual X^* of X, given by:

$$\partial^C f(x_0) = \{x^* \in X^* : \forall x \in X,\ \langle x^*, x \rangle \leq f^C(x_0; x)\},$$

Notice that we always have:

$$\partial^F f(x_0) \subset \partial f(x_0) \subset \partial^C f(x_0).$$

Definition 2.1 ([19], [22]). Suppose $(X, \|.\|)$ Asplund, and let $*\sigma$ be the weak star topology $\sigma(X^*, X)$. The limiting subdifferential of f at $x_0 \in \operatorname{dom} f$ is:

$$\partial^L f(x_0) = \operatorname{seq} - * \sigma \limsup_{x \xrightarrow{f} x_0} \partial^F f(x)$$

where for a multifunction M,

$$\operatorname{seq} - * \sigma \limsup_{x \longrightarrow x_0} M(x) = \{*\sigma - \lim_n y_n,\ y_n \in M(x_n),\ x_n \to x_0\},$$

Remark that in the above definition when M is a constant multifunction, we obtain the sequential weak star closure of M denoted by $\overline{M}^{\mathrm{seq}\,*\sigma}$. Given a subset Y of X, the normal cones are defined at some point $x \in Y$ by the formulas:

$$N_x^C Y = \partial^C \iota_Y(x); \qquad N_x^L Y = \partial^L \iota_Y(x)\,.$$

Definition 2.2. We will say that an $\overline{\mathbb{R}}$-valued function f defined on X has the Clarke-limiting property at some point x of its domain whenever

$$\overline{\mathrm{co}}^{*\sigma} \partial^L f(x) = \partial^C f(x)\,.$$

If X is an Asplund space, f is locally Lipschitzian at x then f has the Clarke-limiting property at x, [19, Theorem 3.57]; if $f = \iota_Y$ is the indicator function of a closed subset Y then f has the Clarke-limiting property at every point of Y since by [19, Theorem 3.57]

$$\partial^C \iota_Y(x) = N_x^C Y = \overline{\mathrm{co}}^{*\sigma} N_x^L = \overline{\mathrm{co}}^{*\sigma} \partial^L \iota_Y(x)\,.$$

In the sequel E is a finite dimensional space with Borel tribe $\mathcal{B}(E)$; $(\Omega, \mathcal{S}, \mu)$ is a measure space with a σ-finite non-negative measure μ and endowed with a tribe $\mathcal{S}$ which is μ-complete. For a measurable subset $A \in \mathcal{S}$, we set $A^c = \{\omega \in \Omega,\ \omega \notin A\}$ and 1_A stands for the characteristic function of A, $1_A(w) = 1$ if $w \in A$, 0 if $\omega \notin A$.

We consider the space $L_0(\Omega, E, \mathcal{S})$ of classes of measurable functions (for μ-almost everywhere equality) defined on Ω and with values in E also denoted by $L_0(\Omega, E)$. Let $L_p(\Omega, E, \mathcal{S}, \mu)$, $1 \le p \le \infty$, be the Lebesgue space of classes of p-μ-integrable functions (μ essentially bounded functions if $p = \infty$) defined on Ω with values in E. These spaces are also denoted by $L_p(\Omega, E)$ when there is no ambiguity on the tribe and the measure, and $\|x\|_p$ is the usual norm of an element $x \in L_p(\Omega, E)$. Let $L_q(\Omega, E^*)$ be the dual space of $L_p(\Omega, E)$, $1 \le p < \infty$, $p^{-1} + q^{-1} = 1$. Given a multifunction $M : \Omega \rightrightarrows E^*$ we denote by $L_q(M)$ the set of classes of measurable selections x^* of M which are in $L_q(\Omega, E^*)$. A subset X of $L_0(\Omega, E)$ is said decomposable in sense of [14] whenever $X1_A + X1_{A^c} \subset X$, where for every measurable set A, $X1_A = \{x1_A, x \in X\}$. Let us recall the following definitions.

Definition 2.3 (see [18]). Let X be a subset of $L_p(\Omega, E)$ with $1 \le p < \infty$. X is said to be p-equi-integrable (equi-integrable if $p = 1$), if for every positive number ϵ, there exist a positive constant δ and a measurable set K of finite measure such that:

(1) $\sup_{x \in X} \|x1_A\|_p < \epsilon$ for every measurable set A satisfying $\mu(A) \le \delta$.

(2) $\sup_{x \in X} \|x1_{K^c}\|_p < \epsilon$.

Definition 2.4. A subset X of $L_1(\Omega, \mathbb{R})$ is said uniformly integrable when it is bounded and equi-integrable.

Given an extended real valued function v defined on Ω, the upper integral I_v of v or $\int_\Omega^* v d\mu$ is defined by:

$$I_v = \int_\Omega^* v d\mu = \inf\left\{\int_\Omega u d\mu, u \in L_1(\Omega, I\!R), v \leq u, \mu - a.e\right\}$$

We set $v^+ = \sup(0, v)$, $v^- = (-v)^+$. Given an integrand $f : \Omega \times E \to \overline{I\!R}$, for $(\omega, e) \in \Omega \times E$ denote $f_\omega(e) = f(\omega, e)$. If f is measurable and for every $\omega \in \Omega$ f_ω is lower semicontinuous, f is said to be normal. For such an integrand we consider the differential quotient integrand q defined on $\Omega \times E^2 \times I\!R$ by

$$q(\omega, e_0, e, r) = r^{-1}(f_\omega(e_0 + re) - f_\omega(e_0)),$$

and the Lipschitz rate integrand of f as the integrand $\operatorname{lip} f$ defined by $\operatorname{lip} f(\omega, e) = \operatorname{lip} f_\omega(e)$. For a measurable E-valued function x, the symbol $f(x)$ denotes the map $\omega \mapsto f_\omega(x(\omega))$. The integrand f is said to have the Clarke-limiting property along some function x if for every $\omega \in \Omega$ f_ω has the Clarke-limiting property at $x(\omega)$. The integral functional I_f associated with f is the functional defined at every point x of $L_0(\Omega, E)$ by:

$$I_f(x) = I_{f(x)} = \int_\Omega^* f(x) d\mu\,.$$

In the sequel $1 < p < \infty$, $p^{-1} + q^{-1} = 1$; we will use the following condition on f:

($\mathcal{H}_p$) *There exist a positive constant k and an element a of $L_1(\Omega, I\!R)$, such that for almost every $\omega \in \Omega$, for every $e \in E$,*

$$f(\omega, e) \geq -k\|e\|^p - a(\omega)\,.$$

Remark that for a normal integrand f, when $I_f \neq \infty$, $\mathcal{H}_p$ characterizes both the strong semicontinuity and the properness of I_f on $L_p(\Omega, E)$ ([2]).

3. Main results

In the remaining of the paper, by I_f we will denote the restriction to $L_p(\Omega, E)$ of the integral functional associated to a normal integrand f. The main results are the following

Theorem 3.1. *Suppose the measure μ is atomless and the space $L_p(\Omega, E)$ is separable. Let f be a normal integrand satisfying the assumption ($\mathcal{H}_p$). Then at every p-integrable element x of the domain of I_f we have the following inclusion:*

$$\overline{L_q(\partial^L f(x))}^{\operatorname{seq} *\sigma} \subset \partial^L I_f(x)\,.$$

If in addition $L_q(\partial^L f(x)) \neq \emptyset$ then

$$L_q(\overline{\operatorname{co}}\, \partial^L f(x)) \subset \overline{\partial^L I_f(x)}^{\|\cdot\|}\,,$$

and if moreover the integrand f has the Clarke-limiting property at x:

$$\partial^C I_f(x) = L_q(\partial^C f(x)) = \overline{\partial^L I_f(x)}^{\|\cdot\|} .$$

Corollary 3.2. *Suppose the measure μ is atomless and the space $L_p(\Omega, E)$ is separable. Let f be a normal integrand satisfying the assumption $(\mathcal{H}_p)$. Then at every p-integrable element x of the domain of I_f such that* $\operatorname{lip} f(x)$ *is q-integrable we have the coincidence of the Clarke and limiting subdifferentials.*

$$\partial^C I_f(x) = L_q(\partial^C f(x)) = \partial^L I_f(x) .$$

Theorem 3.3. *Suppose the measure μ is atomless and the space $L_p(\Omega, E)$ is separable. Let f be a normal integrand satisfying the assumption $(\mathcal{H}_p)$. Then at every p-integrable element x of the domain of I_f such that* $\operatorname{lip} f(x)$ *is almost everywhere finite we have the following density relation:*

$$\partial^C I_f(x) = \overline{\partial^L I_f(x)}^{\|\cdot\|} .$$

In addition whenever $\partial^C I_f(x) \neq \emptyset$ then:

$$\partial^C I_f(x) = L_q(\partial^C f(x)) = \overline{\partial^L I_f(x)}^{\|\cdot\|} .$$

Theorem 3.4. *Suppose the measure μ is atomless and the space $L_p(\Omega, E)$ is separable. Let $M : \Omega \rightrightarrows E$ be a measurable multifunction with nonempty closed values. Then for every $x \in L_p(M)$,*

$$\overline{N_x^L L_p(M)}^{\|\cdot\|} = N_x^C L_p(M) = L_q(N_x^C M) .$$

The next corollary is nothing else than a striking example of an interchange between the symbols integral and upper epi-limit:

Corollary 3.5. *With all the assumptions of Theorem* 3.1 *or Corollary* 3.2 *(or Theorem* 3.4 *with $f(\omega, e) = \iota_{M(\omega)}(e)$) we get for every $y \in L_p(\Omega, E)$,*

$$I_f^C(x; y) = I_{f^C(x,.)}(y) .$$

Corollary 3.6. *With the assumptions of Theorem* 3.4*, we have:*

$$T_x^C L_p(M) = L_p(T_x^C M) .$$

These results are proved in the next section.

4. Proofs

In this section we suppose that E is an Euclidean space of dimension d and $1 < p < \infty$; $*\sigma$ denotes the weak star topology $\sigma(L_q(\Omega, E^*), L_p(\Omega, E)$. Let us recall first the following result (proved in [11]), it is an idea due to M. Valadier (when E is a separable Banach space and $q = 1$).

Proposition 4.1 ([11, Proposition 3.5]). *Let $M : \Omega \rightrightarrows E^*$, be a measurable multifunction. Suppose that the measure μ is atomless. Then, when $L_q(M) \neq \emptyset$,*

$$\overline{L_q(M)}^{*\sigma} = L_q(\overline{\text{co}}M)\,.$$

We will need the following sequential version of the preceding result:

Theorem 4.2. *Suppose $1 < p < \infty$, the measure μ is atomless and that $L_p(\Omega, E)$ is separable. Let $M : \Omega \rightrightarrows E^*$ be a multifunction with measurable graph, then whenever $L_q(M) \neq \emptyset$:*

$$\overline{\overline{L_q(M)}^{\text{seq}\,*\sigma}}^{\|\cdot\|_q} = L_q(\overline{\text{co}}M)\,.$$

Proof of Theorem 4.2. Let us prove before the following certainly known lemmas:

Lemma 4.3. *If $M : \Omega \rightrightarrows E^*$ is a multifunction with measurable graph and non empty values, then for any measurable selection x^* of $\text{co}\,M$ there exist a finite number of measurable selections $(x_i)_{1\leq i\leq d+1}$ of M and measurable non negative functions $(\lambda_i)_i$ such $\sum_i \lambda_i = 1$, with $x^* = \sum_i \lambda_i x_i^*$.*

Proof of Lemma 4.3. Let $F = [0,1]^{d+1}$, and consider the multifunctions and maps:

$$\Gamma(\omega) = \Big\{(\lambda, e^*) \in F \times (E^*)^{d+1} : \forall 1 \leq i \leq d+1, e_i^* \in M(\omega),$$
$$\sum_i \lambda_i = 1;\ x^*(\omega) = \sum_i \lambda_i e_i^*\Big\}$$

$\Lambda(\omega, \lambda) = M(\omega)^{d+1}$, $u(\omega, \lambda, e^*) = 1 - \sum_i \lambda_i$, $v(\omega, \lambda, e^*) = x^*(\omega) - \sum_i \lambda_i e_i^*$. It suffices to prove that Γ admits a measurable selection. The multifunction Λ is with measurable graph and the maps u and v are measurable, therefore $\text{graph}(\Gamma) = \text{graph}\,\Lambda \cap u^{-1}(0) \cap v^{-1}(0)$ is measurable. From the Caratheodory theorem, Γ is with nonempty values, due to [3, Theorem III 22] there exists a measurable selection.

Lemma 4.4. *If B^* is the unit ball of E^*, then when $L_q(M) \neq \emptyset$,*

$$L_q(\text{co}\,M) \subset \overline{\bigcup_{\substack{\beta\in L_q(\Omega,I\!R)\\ M\cap\beta B^*\neq 0}} L_q(\text{co}(M \cap \beta B^*))}^{\|\cdot\|_q}\,.$$

Proof of Lemma 4.4. Let β_0 be an element of $L_q(\Omega, I\!R)$ with positive values and $x_0^* \in L_q(M)$. Given $x^* \in L_q(\text{co}\,M)$, from Lemma 4.3 there exist a finite family of measurable selections $(x_i^*)_{1\leq i\leq dimE+1}$ of M and measurable non negative functions $(\lambda_i)_i$ such that $\sum_i \lambda_i = 1$, with $x^* = \sum_i \lambda_i x_i^*$. Set $\Omega_n = \{\omega : \max_i \|x_i^*(\omega)\| \leq n\beta_0(\omega)\}$. Then up to a negligible set, $(\Omega_n)_n$ is an increasing covering of Ω by measurable sets. For each integer n define $x_n^* = \sum_i \lambda_i y_i^*$ with $y_i^* = (x_i^* 1_{\Omega_n} + x_0^* 1_{\Omega_n^c}) \in L_q(M \cap \beta_n B^*)$ where $\beta_n = n\beta_0 + \|x_0^*\|$. Then $x^* - x_n^* = (x^* - x_0^*) 1_{\Omega_n^c}$, this proves that the sequence (x_n^*) norm converges to x^* therefore

$$x^* \in \overline{\bigcup_n L_q(\mathrm{co}(M \cap \beta_n B^*))}^{\|\cdot\|_q} \subset \overline{\bigcup_{\substack{\beta \in L_q(\Omega, I\!R) \\ M \cap \beta B^* \neq 0}} L_q(\mathrm{co}(M \cap \beta B^*))}^{\|\cdot\|_q}.$$

The proof of Lemma 4.4 is complete.

End of the proof of Theorem 4.2. Since $L_q(\mathrm{co}\, M)$ is convex, and $*\sigma$ is also the weak topology of the reflexive space $L_q(\Omega, E^*)$, then $\overline{L_q(\mathrm{co}\, M)}^{\|\cdot\|_q}$ is $*\sigma$ closed, therefore from Proposition 4.1 and the above lemma:

$$L_q(\overline{\mathrm{co}} M) = \overline{L_q(\mathrm{co}\, M)}^{*\sigma} = \overline{L_q(\mathrm{co} M)}^{\|\cdot\|_q} \subset \overline{\bigcup_{\substack{\beta \in L_q(\Omega, I\!R) \\ M \cap \beta B^* \neq 0}} L_q(\mathrm{co}(M \cap \beta B^*))}^{\|\cdot\|_q}.$$

Because $L_p(\Omega, E)$ is separable, and for every $\beta \in L_q(\Omega, I\!R)$ such that $M \cap \beta B^* \neq 0$, $M \cap \beta B^*$ is bounded, we get from Proposition 4.1:

$$L_q(\mathrm{co}(M \cap \beta B^*)) \subset \overline{L_q(M \cap \beta B^*)}^{*\sigma} = \overline{L_q(M \cap \beta B^*)}^{\mathrm{seq}\, *\sigma} \subset \overline{L_q(M)}^{\mathrm{seq}\, *\sigma}.$$

Therefore:

$$\begin{aligned} L_q(\overline{\mathrm{co}} M) &\subset \overline{\bigcup_{\substack{\beta \in L_q(\Omega, I\!R) \\ M \cap \beta B^* \neq 0}} L_q(\mathrm{co}(M \cap \beta B^*))}^{\|\cdot\|_q} \\ &\subset \overline{\overline{L_q(M)}^{\mathrm{seq}\, *\sigma}}^{\|\cdot\|_q} \subset \overline{L_q(M)}^{*\sigma} = L_q(\overline{\mathrm{co}} M). \end{aligned}$$

This proves Theorem 4.2.

Definition 4.5. We say that a sequence $(f_k)_k$ of $\overline{I\!R}$-valued measurable integrands satisfies eventually a property $\mathcal{P}$ if there exists an integer l such the sequence $(f_k)_{k \geq l}$ satisfies the property $\mathcal{P}$.

Theorem 4.6. *When assumption $(\mathcal{H}_p)$ is satisfied the following property holds:*

$(\mathcal{H})$ *For every p-integrable element x of $\mathrm{dom}\, I_f$ with $L_q(\partial^F f(x)) \neq \emptyset$, there exists an increasing covering $(\Omega_n)_n$ of Ω by measurable sets of finite measure such that for every integer n, for every $\epsilon > 0$, there exists positive constants $\delta_{n,\epsilon}$ such that for every sequence $(r_k)_k$ of positive real numbers converging to zero, the sequence of integrands $(q(x, ., r_k))_k$ satisfies eventually on $\Omega_n \times E$*

$$q_\omega(x(\omega), e, r_k) \geq -\epsilon \|e\|^p - \delta_{n,\epsilon}.$$

Proof of Theorem 4.6. First let us prove the following result used in the proof of [8, Theorem 5.8].

Lemma 4.7. *If f is a normal integrand satisfying $(\mathcal{H}_p)$, then for every $x \in \operatorname{dom} I_f$, with $L_q(\partial^F f(x)) \neq \emptyset$ there exists an increasing covering $(\Omega_n)_n$ of Ω by measurable sets of finite measure such that for every integer n, $(\omega, e) \in \Omega_n \times E$,*

$$f(\omega, x(\omega)+e) - f(\omega, x(\omega)) \geq -n \max(\|e\|, \|e\|^p)\,.$$

Proof of Lemma 4.7. If $L_q(\partial^F f(x)) \neq \emptyset$ then for almost every $\omega \in \Omega$, f_ω is calm at $x(\omega)$. The integrands f and g with $g(\omega, s, e) = s\|e\|$, are normal. Thus the integrand $h(\omega, r, s, e) = f(\omega, x(\omega)+e) - f(\omega, x(\omega)) + g(\omega, s, e)$ is normal. Let B be the unit ball of E. The multifunction M with values $M(\omega, r, s) = x(\omega) + rB$ is measurable; thus the function $k(\omega, r, s) = \inf_{e \in M(\omega,r,s)} h(\omega, r, s, e)$ is measurable ([25, Theorem 2K]). The multifunction N defined by:

$$N(\omega) = \{(r,s) : 0 < r,\ 0 < s,\ k(\omega, r, s) \geq 0\}$$

has a measurable graph and nonempty values (since $f(\omega, .)$ is calm at $x(\omega)$), thus, with [3, III 22], any measurable selection $(r(.), s(.))$ of N satisfies: for every $\omega \in \Omega$,

$$\|e\| \leq r(\omega) \Rightarrow \quad f(\omega, x(\omega)+e) - f(\omega, x(\omega)) \geq -s(\omega)\|e\|\,.$$

Moreover, condition $(\mathcal{H}_p)$ yields for every $e \in E$:

$$f(\omega, x(\omega)+e) - f(\omega, x(\omega)) \geq -k\|x(\omega)+e\|^p - b(\omega)\,,$$

with $b(\omega) = a(\omega) + |f(\omega, x(\omega))|$. Thus, by the convexity of $\|.\|^p$,

$$f(\omega, x(\omega)+e) - f(\omega, x(\omega)) \geq -k2^{p-1}\|e\|^p - c(\omega)\,,$$

with $c(\omega) = b(\omega) + k2^{p-1}\|x(\omega)\|^p$. If $\|e\| \geq r(\omega)$ then $c(\omega) \leq c(\omega) r(\omega)^{-p}\|e\|^p$, hence for every $e \in E$,

$$f(\omega, x(\omega)+e) - f(\omega, x(\omega)) \geq -\max(s(\omega)\|e\|, t(\omega)\|e\|^p)\,,$$

where $t(\omega) = (k2^{p-1} + c(\omega) r(\omega)^{-p})$ is a positive, finite valued, measurable function. Let $(\Lambda_n)_n$ be an increasing covering of Ω by measurable sets of finite measure. Define $\Omega_n = \Lambda_n \cap \{\max(s(.), t(.)) \leq n\}$, then $(\Omega_n)_n$ satisfies the conclusion of Lemma 4.7. This achieves the proof of Lemma 4.7.

Theorem 4.6 is an immediate consequence of Lemma 4.7. Indeed fix ϵ and the integer n. For $(\omega, e) \in \Omega_n \times E$, for every sequence $(r_k)_k$ of positive real numbers converging to the origin we get:

$$\begin{aligned} f(\omega, x(\omega)+r_k e) - f((\omega, x(\omega)) \geq & -n r_k \max(\|e\|, r_k^{p-1}\|e\|^p) \\ = & -r_k \max(n\|e\|, n r_k^{p-1}\|e\|^p)\,, \end{aligned}$$

and the Young inequality $n\|e\| \leq p^{-1}\epsilon\|e\|^p + q^{-1}\epsilon^{-\frac{q}{p}}n^q$ gives for k large enough with $nr_k^{p-1} \leq p^{-1}\epsilon$:

$$\begin{aligned}\max(n\|e\|, nr_k^{p-1}\|e\|^p) &\leq \max(p^{-1}\epsilon\|e\|^p + q^{-1}\epsilon^{-\frac{q}{p}}n^q, nr_k^{p-1}\|e\|^p)\\ &\leq \epsilon\|e\|^p + q^{-1}\epsilon^{-\frac{q}{p}}n^q\,.\end{aligned}$$

This proves the conclusion of Theorem 4.6 with $\delta_{n,\epsilon} = q^{-1}\epsilon^{-\frac{q}{p}}n^q$.

Theorem 4.8. *Suppose the assumption $(\mathcal{H}_p)$ is satisfied. There exists some point $(x_0, x_0^*) \in L_p(\Omega, E) \times L_q(\Omega, E^*)$ with $x_0^* \in \partial^F I_f(x_0)$. Moreover for every $x \in \operatorname{dom} I_f$ and $x^* \in L_q(\partial^F f(x))$ there exists a sequence $(x_l, x_l^*)_l$ strongly convergent to (x, x^*) such that $x_l \xrightarrow{I_f} x$ and for every integer l, $x_l^* \in \partial^F I_f(x_l)$ with moreover $\|x_l^*\|_q \leq \|x^*\|_q + \|x_0^*\|_q$.*

Proof of Theorem 4.8. The first assertion is a consequence of [19, Corollary 2.29 (c)]. We will use Theorem 4.6 in order to prove the last assertion of Theorem 4.8. Let us first prove the following lemmas:

Lemma 4.9. *Let $x|A$ be the restriction to a set A of the function x. The restriction $I_{A,f}$ of the integral functional I_f to a measurable set A is defined on $L_p(A, E)$ by*

$$I_{A,f}(x) = \int_A^* f(x)d\mu\,.$$

Given elements x, y of $L_p(\Omega, E)$ and x^*, y^* of $L_q(\Omega, E^*)$ satisfying:

$$x^*|A \in \partial^F I_{A,f}(x|A) \quad \text{and} \quad y^*|A^c \in \partial^F I_{A^c,f}(y|A^c)\,,$$

then setting $z = x1_A + y1_{A^c}$, $z^* = x^*1_A + y^*1_{A^c}$, we have: $z^* \in \partial^F I_f(z)$.

Proof of Lemma 4.9. Given a sequence $(z_n)_n$ norm converging to z, the sequence $(z_n|A)_n$ norm converges to $x|A$ in $L_p(A, E)$ and the sequence $(z_n|A^c)_n$ norm converges to $y|A^c$ in $L_p(A^c, E)$. Since $x^*|A \in \partial^F I_{A,f}(x|A)$, and $y^*|A^c \in \partial^F I_{A,f}(y|A^c)$, for every $\epsilon > 0$ for n large enough:

$$I_{A,f}(z_n|A) - I_{A,f}(x|A) - \langle x^*|A, z_n|A - x|A\rangle \geq -\epsilon\|z_n|A - x|A\|_{L_p(A,E)}\,,$$
$$I_{A,f}(z_n|A^c) - I_{A,f}(y|A^c) - \langle y^*|A^c, z_n|A^c - y|A^c\rangle \geq -\epsilon\|z_n|A^c - y|A^c\|_{L_p(A^c,E)}\,,$$

That is:

$$\int_A^* (f(z_n) - f(x))d\mu - \langle x^*1_A, z_n - x\rangle \geq -\epsilon\|(z_n - x)1_A\|_{L_p(\Omega,E)}\,,$$
$$\int_{A^c}^* (f(z_n) - f(y))d\mu - \langle y^*1_{A^c}, z_n - y\rangle \geq -\epsilon\|(z_n - y)1_{A^c}\|_{L_p(\Omega,E)}\,.$$

But since $(z_n - z)1_A = (z_n - x)1_A$, $\|(z_n - x)1_A\|_{L_p(A,E)} \leq \|(z_n - z)\|_{L_p(\Omega,E)}$, since $(z_n - z)1_{A^c} = (z_n - y)1_{A^c}$, $\|(z_n - y)1_{A^c}\|_{L_p(A^c,E)} \leq \|(z_n - z)\|_{L_p(\Omega,E)}$, we obtain:

$$I_f(z_n) - I_f(z) - \langle z^*, z_n - z\rangle \geq -2\epsilon\|(z_n - z)\|_{L_p(\Omega,E)}.$$

This proves that $z^* \in \partial^F I_f(z)$. The proof of Lemma 4.9 is complete.

As an immediate consequence we get:

Corollary 4.10. *The graph of the multifunction $x \rightrightarrows \partial^F I_f(x)$ is a decomposable set of $L_p(\Omega, E) \times L_q(\Omega, E^*)$. At any point x of the domain of I_f the Fréchet subdifferential $\partial^F I_f(x)$ is a closed convex decomposable set.*

The proof of the decomposability property of Corollary 4.10 is due to Lemma 4.9 and the following lemma:

Lemma 4.11. *Given an element x in $L_p(\Omega, E)$ and $x^* \in \partial^F I_f(x)$, then for every measurable set A,*

$$x^*|A \in \partial^F I_{A,f}(x|A)$$

Proof of Lemma 4.11. Let a measurable set A and a sequence $(z_n)_n$ norm converging to $x|A$ in $L_p(A, E)$. If $\overline{z}_n$ is the null extension of z_n to Ω, define $x_n = \overline{z}_n + x1_{A^c}$. Then $\|x - x_n\|_{L_p(\Omega,E)} = \|x|A - z_n\|_{L_p(A,E)}$ This proves that $(x_n)_n$ is norm converging to x in $L_p(\Omega, E)$. Since x^* is a Fréchet subderivative of I_f at x, for every ϵ and every integer n large enough we get:

$$I_f(x_n) - I_f(x) - \langle x^*, x_n - x\rangle \geq -\epsilon\|(x_n - x)\|_{L_p(\Omega,E)} = -\epsilon\|z_n - x|A\|_{L_p(A,E)}\,.$$

Or equivalently:

$$I_{A,f}(z_n) - I_{A,f}(x|A) - \langle x^*|A, z_n - x|A\rangle \geq -\epsilon\|z_n - x|A\|_{L_p(A,E)}\,,$$

Therefore $x^*|A \in \partial^F I_{A,f}(x|A)$. This achieves the proof of Lemma 4.11 and Corollary 4.10.

Now in order to end the proof of Theorem 4.8 let us introduce the following definitions and result:

Definition 4.12. Given a sequence $(f_n)_n$ of $\overline{I\!R}$-valued integrands defined on $\Omega \times E$, we will say that a sequence $(x_n)_n$ satisfies the lower compactness property (respect to $(f_n)_n$) if the sequence $(f_n^-(x_n))_n$ is eventually uniformly integrable in $L_1(\Omega, I\!R)$. Let $1 \leq p \leq \infty$. An integrand f is said to have the Fréchet-differential lower compactness property (denoted Fréchet-dlcp) at $x \in L_p(\Omega, E)$, if for every sequence $(r_n)_n$ of positive real numbers converging to 0, every bounded sequence $(x_n)_n$ in $L_p(\Omega, E)$ has the lower compactness property with respect to the sequence of differential quotients $(q(x, ., r_n))_n$.

Theorem 4.13 ([12], [13, Theorem 10.9]). *If $p \neq 1$ and the integrand f has the Fréchet-dlcp at $x \in \operatorname{dom} I_f$, then:*

$$L_q(\partial^F f(x)) \subseteq \partial^F I_f(x)\,.$$

End of the proof of Theorem 4.8. Due to Theorem 4.6 and [12], [13, Theorem 8.4], so that the restriction of f to each Ω_n has the Fréchet-dlcp at $x|\Omega_n$. Using Theorem 4.13 for every $x^* \in L_q(\partial^F f(x))$,

$$x^*|\Omega_n \in \partial^F I_{\Omega_n,f}(x|\Omega_n)\,.$$

Due to Lemma 4.11, we get

$$x_0^*|\Omega_n \in \partial^F I_{\Omega_n,f}(x_0|\Omega_n)\,.$$

Setting $x_n = x1_{\Omega_n}+x_0 1_{\Omega_n^c}$, $x_n^* = x^*1_{\Omega_n}+x_0^*1_{\Omega_n^c}$, with Lemma 4.9 we obtain that $x_n^* \in \partial^F I_f(x_n)$. Since x and x_0 are in $\operatorname{dom} I_f$ and $(\Omega_n)_n$ is a covering of Ω, the sequence $(x_n)_n$ graph converges to x and the sequence $(x_n^*)_n$ norm converges to x^*, and for each integer n we have: $\|x_n^*\|_q \le \|x^*\|_q + \|x_0^*\|_q$. The proof of Theorem 4.8 is complete.

Theorem 4.14. *Suppose f is a normal integrand satisfying the condition $(\mathcal{H}_p)$, then for every $x \in \operatorname{dom} I_f$, we have*

$$\overline{L_q(\partial^L f(x))}^{\mathrm{seq}\,*\sigma} \subset \partial^L I_f(x).$$

Proof of Theorem 4.14. When E is finite dimensional the contingent and Fréchet subdifferentials coincide. The result [11, Lemma 3.9] can be expressed in the following form:

Lemma 4.15 ([11, Lemma 3.9]). *If f is a normal integrand, then for every $x \in \operatorname{dom} I_f$, and every $x^* \in L_q(\partial^L f(x))$, there exist a sequence $x_k \xrightarrow{I_f} x$ and a sequence $x_k^* \to x^*$, such that for every integer k: $x_k^* \in L_q(\partial^F f(x_k))$ and $\|x_k^*\|_q \le \|x^*\|_q + 1$.*

Let $x^* \in \overline{L_q(\partial^L f(x))}^{\mathrm{seq}\,*\sigma}$. There exists a sequence $(x_n^*)_n$ of elements of $L_q(\partial^L f(x))$ $*\sigma$ converging to x^* therefore the sequence $(x_n^*)_n$ is $L_q(\Omega, E^*)$-norm bounded by some real number m. Using Lemma 4.15, for each integer n there exists a sequence $((x_k, x_{n,\,k}^*))_k$ strongly convergent to (x, x_n^*) such that $x_k \xrightarrow{I_f} x$ and for every integer k, $x_{n,\,k}^* \in L_q(\partial^F f(x_k))$ and $\|x_{n,\,k}^*\|_q \le \|x_n^*\|_q + 1 \le m+1$. Using Theorem 4.8, for each pair of integers n, k there exists a sequence $((x_{k,\,l}, x_{n,\,k,\,l}^*))_l$ strongly convergent to $(x_k, x_{n,\,k}^*)$ such that $x_{k,\,l} \xrightarrow{I_f} x_k$ and for every integer l, $x_{n,\,k,\,l}^* \in \partial^F I_f(x_{k,\,l})$ and moreover $\|x_{n,\,k,\,l}^*\|_q \le \|x_{n,\,k}^*\|_q + \|x_0^*\|_q$. Thus for every integers n, k, l

$$\|x_{n,\,k,\,l}^*\|_q \le \|x_{n,\,k}^*\|_q + \|x_0^*\|_q \le m + 1 + \|x_0^*\|_q\,.$$

The topology $*\sigma$ being metrisable on bounded sets, the graph convergence being semi-metrisable, by a diagonal argument there exists a sequence $((x_{k(i),\,l(i)}, x_{n(i),\,k(i),\,l(i)}^*))_i$ satisfying the properties: $x_{k(i),\,l(i)} \xrightarrow{I_f} x$, $((x_{n(i),\,k(i),\,l(i)}^*))_i$ $*\sigma$-converges to x^* and for every integer i, $x_{n(i),\,k(i),\,l(i)}^* \in \partial^F I_f(x_{k(i),\,l(i)})$. That is $x^* \in \partial^L I_f(x)$. The proof of Theorem 4.14 is complete.

Using Theorem 4.2 and Theorem 4.14 we obtain the complete proof of the two first assertions of Theorem 3.1: if $L_q(\partial^L f(x)) \ne \emptyset$, then

$$L_q(\overline{\mathrm{co}}\partial^L f(x)) = \overline{\overline{L_q(\partial^L f(x))}^{\mathrm{seq}\,*\sigma}}^{\|\cdot\|_q} \subset \overline{\partial^L I_f(x)}^{\|\cdot\|_q}.$$

For the last assertion of Theorem 3.1, remark that since f has the Clarke-limiting property at x, for every $\omega \in \Omega$

$$\partial^C f_\omega(x_0(\omega)) = \overline{\mathrm{co}}\, \partial^L f_\omega(x_0(\omega)).$$

Moreover $L_q(\partial^L f(x))$ being nonempty from the first part of Theorem 3.1, and the main result of [9] we have the chain of inclusions

$$L_q(\partial^C f(x)) = L_q(\overline{\mathrm{co}}\, \partial^L f(x)) \subset \overline{\partial^L I_f(x)}^{\|\cdot\|_q} \subset \partial^C I_f(x) \subset L_q(\partial^C f(x))\,.$$

The proof of Theorem 3.1 is complete.

If the Lipschitz rate $\operatorname{lip} f(x)$ is q integrable, since we have $d(0, \partial^L f(x)) \leq \operatorname{lip} f(x)$, the proof of Corollary 3.2 is a consequence of the following two lemmas:

Lemma 4.16. *Let $M : \Omega \rightrightarrows E$ be a multifunction with nonempty values and with measurable graph. Then $L_q(M) \neq \emptyset$ if and only if the function $d(0, M(\omega)) = \inf_{e \in M(\omega)} \|e\|$ is q-integrable.*

Proof of Lemma 4.16. . The necessity part is obvious. For the converse, keeping a measurable ([3, Lemma III 39 and Application]) selection x of M such for all ω, $d(0, M(\omega)) = \|x(\omega)\|$, we deduce that $L_q(M) \neq \emptyset$.

Lemma 4.17. *When the Lipschitz rate* $\operatorname{lip} f(x)$ *is q integrable then $\partial^L I_f(x) = \partial^C I_f(x)$.*

Proof of Lemma 4.17. In this case due to Lemma 4.16, $L_q(\partial^L f(x)) \neq \emptyset$ and the $*\sigma$ topology is metrisable on this set. With Theorem 3.1 the following chain of inclusions achieves the proof of Lemma 4.17 and Corollary 3.2.

$$\begin{aligned} L_q(\overline{\mathrm{co}}\partial^L f(x)) &= \overline{L_q(\partial^L f(x))}^{\mathrm{seq}\,*\sigma} \subset \partial^L I_f(x)) \\ &\subset \partial^C I_f(x) \subset L_q(\partial^C f(x)) = L_q(\overline{\mathrm{co}}\partial^L f(x))\,. \end{aligned}$$

Proof of Theorem 3.3. Let us use the following definition and prove the following lemmas:

Definition 4.18. Let $f : \Omega \times E \to \overline{\mathbb{R}}$ be a normal integrand. The integral functional is said to have the graph property on $L_p(\Omega, E)$ at a p-integrable function x of $\operatorname{dom} I_f$ when for every graph-converging sequence $(x_n)_n \xrightarrow{I_f} x$ the sequence $((x_n, f(x_n))_n$ converges to $(x, f(x))$ in $L_p(\Omega, E) \times L_1(\Omega, \mathbb{R})$.

Lemma 4.19 (Graph convergence). *Let $1 < p < \infty$, f be a normal integrand, x be a p-integrable function in* $\operatorname{dom} I_f$. *When f satisfies the growth condition $(\mathcal{H}_p)$, then the integral functional I_f has the graph property at this point on $L_p(\Omega, E)$.*

Proof of Lemma 4.19. Let a sequence $(x_n)_n \xrightarrow{I_f} x$. Extracting a subsequence we may suppose that $(x_n)_n$ converges almost everywhere to x. Due to the growth condition $(\mathcal{H}_p)$, for every integer n, $f_n(x_n) \geq -k\|x_n\|^p - a$. Let $g_n = \inf(f_n(x_n) - f(x_0), 0)$. Since the sequence $(\|x_n\|^p)_n$ is uniformly integrable, the sequence $(g_n)_n$ is uniformly integrable in $L_1(\Omega, I\!R)$. Moreover f being normal, the sequence $(g_n)_n$ converges almost everywhere to 0. On uniformly integrable sets the a.e. convergence being stronger than or equal to the norm convergence in $L_1(\Omega, I\!R)$, [18] or [29, Theorem 16.6], the sequence $(-g_n)_n$ converges to zero in $L_1(\Omega, I\!R)$. But $-g_n = (f_n(x_n) - f(x))^-$ and

$$\int_\Omega (f_n(x_n) - f(x))^+ d\mu = \int_\Omega (f_n(x_n) - f(x)) d\mu + \int_\Omega (f_n(x_n) - f(x))^- d\mu\,,$$

therefore $\lim_n \int_\Omega (f_n(x_n) - f(x))^+ d\mu = 0$. This proves that

$$\begin{aligned}&\lim_n \int_\Omega |f_n(x_n) - f(x)| d\mu \\ = &\lim_n \int_\Omega (f_n(x_n) - f(x))^+ d\mu + \lim_n \int_\Omega (f_n(x_n) - f(x))^- d\mu = 0\,.\end{aligned}$$

Hence the sequence $\left((x_n, f(x_n))\right)_n$ converges to $(x, f(x))$ in $L_p(\Omega, E) \times L_1(\Omega, I\!R)$. This last property being valid for every sequence $(x_n)_n \xrightarrow{I_f} x$ such that $(x_n)_n$ converges almost everywhere, we deduce that I_f has the graph property at x. $\square$

Lemma 4.20. *Let f be a measurable integrand and x be a measurable function. The Lipschitz rate* $\operatorname{lip} f(x)$ *of f along x is $\mathcal{S}$-measurable.*

Proof of Lemma 4.20. The set $\Lambda(\omega) = \{(e, e') \in E^2 : 0 < \|e - e'\|\}$ is a Borel set thus the constant multifunction Λ is with graph $\mathcal{S} \otimes \mathcal{B}(E)$-measurable. For every integer n the multifunction

$$\Theta_n(\omega) = \left\{(e, e') \in E^2 : \max(\|x(\omega) - e\|, \|x(\omega) - e'\|) \leq \frac{1}{n}\right\}$$

is with graph $\mathcal{S} \otimes \mathcal{B}(E)$-measurable ([3, Theorem III 41]). Therefore the multifunction $\Gamma_n(\omega) = \Lambda(\omega) \cap \Theta_n(\omega)$ is with $\mathcal{S} \otimes \mathcal{B}(E)$-measurable graph (and nonempty values). Due to [3, Theorem III 39], the function

$$u_n(\omega) = \sup\left\{\frac{|f(\omega, e) - f(\omega, e')|}{\|e - e'\|}; (e, e') \in \Gamma_n(\omega)\right\}$$

is $\mathcal{S}$ measurable. Since the Lipschitz rate $\operatorname{lip} f(x)$ is the pointwise limit of the sequence $(u_n)_n$, it is $\mathcal{S}$ measurable. The proof of Lemma 4.20 is complete.

End of proof of Theorem 3.3. If $\partial^C I_f(x) = \emptyset$ then it coincides with $\partial^L I_f(x)$). Suppose $\partial^C I_f(x) \neq \emptyset$, then by [19, Theorem 3.57] we have $\partial^L I_f(x) \neq \emptyset$. From

Lemma 4.20, $\operatorname{lip} f(x)$ is $\mathcal{S}$ measurable and almost everywhere finite. Let $(\Omega_k)_k$ be an increasing covering of Ω by measurable sets of finite measure, define $\Lambda_k = \{\omega \in \Omega_k : \operatorname{lip} f_\omega(x(\omega)) \leq k\}$. Up to a negligible set $(\Lambda_k)_k$ is an increasing covering of Ω by measurable sets of finite measure. Then for every integer k, $\operatorname{lip} f(x)|\Omega_k \in L_q(\Omega_k, I\!R)$. Let $y^* \in L_q(\partial^C f(x))$ and $z^* \in \partial^L I_f(x)$. By definition there exist a sequence $(x_n)_n$ graph-convergent to x, a sequence $(z_n^*)_n$ weakly convergent to z^* in $L_q(\Omega, E^*)$ such that for every integer n, $(x_n, z_n^*) \in \operatorname{graph}(\partial^F I_f)$. Moreover $y^*|\Lambda_k \in L_q(\partial^C f(x|\Lambda_k))$, hence due to Corollary 3.2 for every integer k

$$y^*|\Lambda_k \in \partial^L I_{\Lambda_k, f}(x|\Lambda_k)\,.$$

Therefore for every integer k there exist a sequence $(x'_{n,k})_n$ such that the restrictions to Λ_k graph-converges to $x|\Lambda_k$ in $L_p(\Lambda_k, E)$, a sequence $(y^*_{n,k})_n$ such that the restrictions to Λ_k weakly converge to $y^*|\Lambda_k$ in $L_q(\Lambda_k, E^*)$, with for every integer k $(x'_{n,k}, y^*_{n,k})|\Lambda_k \in \operatorname{graph}(\partial^F I_{\Lambda_k,f})$. Define $v_{n,k} = x'_{n,k} 1_{\Lambda_k} + x_n 1_{\Lambda_k^c}$ and $v^*_{n,k} = y^*_{n,k} 1_{\Lambda_k} + z^*_n 1_{\Lambda_k^c}$. From Lemma 4.11,

$$v^*_{n,k}|\Lambda_k^c = z^*_n|\Lambda_k^c \in \partial^F I_{\Lambda_k^c, f}(x_n|\Lambda_k^c)\,,$$

moreover we have

$$v^*_{n,k}|\Lambda_k = y^*_{n,k}|\Lambda_k \in \partial^F I_{\Lambda_k, f}(x'_{n,k})\,.$$

Lemma 4.9 ensures that $v^*_{n,k} \in \partial^F I_f(v_{n,k})$. But due to Lemma 4.19 the sequence $(v_{n,k})_n$ graph-converges to x. Moreover the sequence $(v^*_{n,k})_n$ weakly converges to $v^*_k = y^* 1_{\Lambda_k} + z^* 1_{\Lambda_k^c}$. This proves that $v^*_k \in \partial^L I_f(x)$. Since $(v^*_k)_k$ strongly converges to y^* we deduce that $y^* \in \overline{\partial^L I_f(x)}^{\|\cdot\|}$ and prove that $L_q(\partial^C f(x)) \subset \overline{\partial^L I_f(x)}^{\|\cdot\|}$. Due to the main result of [9] we get the following chain of inclusions:

$$L_q(\partial^C f(x)) \subset \overline{\partial^L I_f(x)}^{\|\cdot\|} \subset \partial^C I_f(x) \subset L_q(\partial^C f(x))\,.$$

This completes the proof of Theorem 3.3.

Proof of Theorem 3.4. Let $M : \Omega \rightrightarrows E$ be a multifunction with nonempty closed values and with measurable graph. Then $\iota_{L_p(M)} = I_f$ where the integrand f is defined by $f(\omega, e) = \iota_{M(\omega)}(e)$. For an element x of $L_p(M)$ we have $\partial^L f(x) = N_x^L M$, since $0 \in N_x^L M$, then $L_q(\partial^L f(x)) \neq \emptyset$. As noted before the integrand f has the Clarke-limiting property; therefore with Theorem 3.1:

$$\begin{aligned}\overline{N_x^L(L_p(M))}^{\|\cdot\|_q} &= \overline{\partial^L \iota_{L_p(M)}(x)}^{\|\cdot\|_q} = \overline{\partial^L I_f(x)}^{\|\cdot\|_q}\\ &= \partial^C I_f(x) = N_x^C(L_p(M)) = L_q(N_x^C M)\,,\end{aligned}$$

since

$$N_x^C(L_p(M)) = \partial^C I_f(x) = L_q(\partial^C f(x)) = L_q(N_x^C M)\,.$$

This ends the proof of Theorem 3.4.

The proof of Corollary 3.5 is analog to that of the last part of Theorem 3.3 in [11].

The proof of Corollary 3.6 is a consequence of Corollary 3.5 and [22, Section 5.3, Exercise 1, p. 381], since for any set Y, and $x \in Y$,

$$\iota_Y^C(x,.) = \iota_{T_x^C Y}(.)\,.$$

Keeping $f(\omega, e) = \iota_{M(\omega)}(e)$, $f_\omega^C(x(\omega), e) = \iota_{T^C_{x(\omega)}M(\omega)}(e) = g(\omega, e)$, since $\iota_{L_p(M)} = I_f$, and $I_g = \iota_{L_p(T_x^C M)}$, we deduce the chain of equalities:

$$\iota_{T_x^C L_p(M)}(.) = \iota^C_{L_p(M)}(x,.) = I_f^C(x,.) = I_{f^C(x,.)} = I_g = \iota_{L_p(T_x^C M)}\,.$$

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