Journal of Convex Analysis
Volume 24 (2017), No. 2, 679–684

A Note on the Paper "Proximal Point Methods for Quasiconvex and Convex Functions with Bregman Distances on Hadamard Manifolds"

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Received: August 7, 2014
Revised manuscript received: February 1, 2016
Accepted: May 27, 2016

In this article, we provide an erratum to the result presented in [7, Proposition 3.4]. More specifically, we prove that the function associated with the exponential map in this proposition is not convex on Hadamard manifolds with negative sectional curvature, and present an alternative proof for the result in the case of null sectional curvature.

1. Introduction

In [7, Proposition 3.4], Quiroz and Oliveira claimed that the function described in Theorem 2.1 (see below) is an affine linear function, where M is a Hadamard manifold. Furthermore, the authors used this to generalize the proximal point method using Bregman distances to solve convex and quasiconvex optimization problems on noncompact Hadamard manifolds. In [1, Proposition 2.9], Colao et al. generalized the result presented in [7] and established existence results for the problem of mixed variational inequalities [1, Theorem 3.5] and fixed points of set-valued mappings [1, Theorem 3.10]. Moreover, in [1, Theorem 4.9] the authors gave sufficient conditions for the resolvent to have a full domain. More recently, in [4, Theorem 3.2], Zhou-Huang used this to establish the existence of a weak minimum.

In [9], Udriste considered the Poincaré plane, which is a Hadamard manifold with negative curvature equal to -1. Further, in [9, Theorem 2.2, p. 299], Udriste showed that affine functions are constant. This is enough to disprove Quiroz and Oliveira's Proposition 3.4 [7]. One of our goals is to prove that this result holds in Hadamard manifolds with zero curvature. We must determine if the considered function is convex in some Hadamard manifolds. We have

ISSN 0944-6532 / $ 2.50

answered this question, and demonstrated that the function is not convex in the case of strictly negative curvature.

This paper is organized as follows. In Section 2, we present a detailed proof that the function in Theorem 2.1 is an affine linear function in a Hadamard manifold with null sectional curvature. In Section 3, we prove that the same is not true in a Hadamard manifold with negative sectional curvature. In Section 4, we construct an example that illustrates the theorem proved in Section 3. Finally, in Section 5, we discuss the main results of our paper and present some perspectives for future work.

2. Hadamard Manifold with Null Sectional Curvature

Theorem 2.1. *Let M be a Hadamard manifold and $y \in M$. Then M has null sectional curvature if and only if for every nonzero $u \in T_yM$ the function $g : M \to \mathbb{R}$ defined by $g(x) = \langle \exp_y^{-1} x, u\rangle_y$ is an affine linear function.*

Proof. As the sectional curvature on $(M, \langle\, ,\, \rangle)$ is identically zero, it follows that the exponential map $\exp_y : T_yM \to M$ is a global isometry. In fact, from [3, Theorem 3.1, pp. 149] we have that $\exp_y : T_yM \to M$ is a diffeomorphism. Choose on T_yM the pullback metric to make it into a local Riemannian isometry. Then [8, Proposition 21, pp. 143] implies that $\exp_y$ is distance preserving, whence we conclude that it is a global isometry. Now, let $\gamma : [0, 1] \to M$ be a geodesic segment with $\gamma(0) = p_0$ and $\gamma(1) = p_1$. Since isometry preserves geodesics, we obtain that

$$\gamma(t) = \exp_y((1 - t) \exp_y^{-1}(p_0) + t \exp_y^{-1}(p_1)).$$

Therefore $g(\gamma(t)) = \langle (1-t) \exp_y^{-1}(p_0)+t \exp_y^{-1}(p_1), u\rangle_y$, implying $(g\circ\gamma)''(t) = 0$, i.e., g is an affine linear function.

Conversely, assume that g is an affine linear function. We have by [5, Main Theorem] that there are a complete Riemannian manifold N_1 and an isometry $F : N_1 \times \mathbb{R} \to M$. If $F(x, a) = p$ and $F(n, t) = y$ then

$$g(F(x, a)) = \langle \exp_{(n,t)}^{-1}(x, a), (v, 0)\rangle_{(n,t)}$$

is an affine linear function $\forall\, (v, 0) \in T_{(n,t)}N_1 \times \mathbb{R}$. Since $\exp_{(n,t)}(v, w) = (\exp_n(v), \exp_t(w))$ for any $(v, w) \in T_{(n,t)}N_1 \times \mathbb{R}$, it follows that

$$\overline{g}(x) = g(F(x, a)) = \langle \exp_n^{-1}(x), v\rangle_n$$

is an affine linear function on N_1. Repeating the argument we conclude that the manifold M is isometric to $\mathbb{R}^k$, where $k = \dim M$. □

Remark 2.2. In the proof of the theorem above we used that the composition of an affine linear function with an isometry is still an affine linear function. This is an easily verifiable fact.

3. Hadamard Manifold with Negative Sectional Curvature

Theorem 3.1. *Let M be a Hadamard manifold with negative sectional curvature, $y \in M$ and $u \in T_yM$ nonzero. Then the function $g : M \to \mathbb{R}$, defined by $g(x) = \langle \exp_y^{-1} x, u\rangle_y$, is not convex.*

Proof. We initially consider $q = \exp_y(-u)$, which is on the geodesic $\gamma(t) = \exp_y(tu)$. We can assume, without loss of generality that $u \in T_yM$ is unitary. Consider a normalized geodesic $\alpha :]-\varepsilon, \varepsilon[\to M$, with $\alpha(0) = q$ and $\langle \alpha'(0), \gamma'(-1)\rangle_q = 0$, i.e., $\alpha(s)$ is orthogonal to $\gamma(t)$ in q.

Assume by contradiction that the function $g(x) = \langle \exp_y^{-1} x, u\rangle_y$ is convex, then, for all $s \in]-\varepsilon, \varepsilon[$ we get

$$-1 = g(\alpha(0)) = g\left(\alpha\left(\frac{1}{2}\cdot s + \frac{1}{2}(-s)\right)\right) \le \frac{1}{2}g(\alpha(s)) + \frac{1}{2}g(\alpha(-s)). \tag{1}$$

However, $g(\alpha(s)) = |\exp_y^{-1}(\alpha(s))| \cos(\beta) = \operatorname{dist}(y, \alpha(s)) \cos\beta$, where β is the angle between the vectors u and $\exp_y^{-1}(\alpha(s))$, and $g(\alpha(-s)) = \operatorname{dist}(y, \alpha(-s)) \cos\theta$, where θ is the angle between the vectors u and $\exp_y^{-1}(\alpha(-s))$. Moreover, we can choose s small enough so that the angles β and θ are larger than $\frac{\pi}{2}$. Substituting the equalities obtained in (1) results in

$$-2 \le \operatorname{dist}(y, \alpha(s)) \cos\beta + \operatorname{dist}(y, \alpha(-s)) \cos\theta.$$

Therefore,

$$\operatorname{dist}(y, \alpha(s)) \cos(\pi - \beta) + \operatorname{dist}(y, \alpha(-s)) \cos(\pi - \theta) \le 2. \tag{2}$$

Consider the geodesic triangle whose vertices are y, q and $\alpha(s)$. The unique geodesic connecting the points y and $\alpha(s)$ is $\exp_y(t(\exp_y^{-1}(\alpha(s))))$. Because the angle of the geodesic triangle, in the vertex y, is equal to the angle between the vectors $-u$ and w, its measure is $\pi - \beta$. By construction, the angle in the vertex q measures $\frac{\pi}{2}$. See Figure 3.1.

Applying the law of cosines twice (see reference [8, p. 167]) to the triangle $\Delta(y, q, \alpha(s))$ we obtain

$$\operatorname{dist}^2(y, \alpha(s)) > \operatorname{dist}^2(q, \alpha(s)) + \operatorname{dist}^2(y, q) - 2\ \operatorname{dist}(q, \alpha(s)) \operatorname{dist}(y, q) \cos\frac{\pi}{2};$$
$$\operatorname{dist}^2(q, \alpha(s)) > \operatorname{dist}^2(y, \alpha(s)) + \operatorname{dist}^2(y, q) -\ \operatorname{dist}(y, \alpha(s)) \operatorname{dist}(y, q) \cos(\pi - \beta).$$

Because $\operatorname{dist}(y, q) = 1$, the obtained inequalities imply that

$$1 < \operatorname{dist}(y, \alpha(s)) \cos(\pi - \beta).$$

Similarly, using the triangle $\Delta(y, q, \alpha(-s))$, we obtain

$$1 < \operatorname{dist}(y, \alpha(-s)) \cos(\pi - \theta).$$

Therefore, $2 < \operatorname{dist}(y, \alpha(s)) \cos(\pi - \beta) + \operatorname{dist}(y, \alpha(-s)) \cos(\pi - \theta)$ which contradict (2). Thus, g cannot be a convex function. □

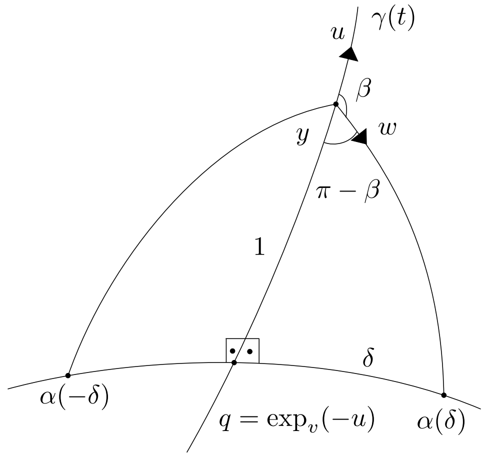

Figure 3.1.

4. The Particular Case $M = \mathbb{H}^2$

Let $\mathbb{L}^3$ denote the 3-dimensional Lorentz space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle p, q\rangle = p_1 q_1 + p_2 q_2 - p_3 q_3.$$

The 2-dimensional hyperbolic space

$$\mathbb{H}^2 = \{p \in \mathbb{L}^3;\ \langle p, p\rangle = -1,\ p_3 \geq 1\}$$

is a spacelike hypersurface in $\mathbb{L}^3$. That is, the metric induced by the inclusion $\iota : \mathbb{H}^2 \to \mathbb{L}^3$ is a Riemannian metric on $\mathbb{H}^2$. Further, the hyperbolic space has Gaussian curvature -1. A parameterization for the hyperbolic space is

$$\Psi(u, v) = (\sinh u\, \cos v,\ \sinh u\, \sin v,\ \cosh u)\,,$$

where $(u, v) \in [0, +\infty[\times[0, 2\pi[$. Given a point $p \in \mathbb{H}^2$, consider (u_0, v_0) such that $\Psi(u_0, v_0) = p$. Then, the curve $\gamma(u) = \Psi(u, v_0) : [0, +\infty[\to \mathbb{H}^2$ is the minimizing geodesic that satisfies $\gamma(0) = e_3$ and $\gamma(u_0) = p$. Therefore,

$$\operatorname{dist}(e_3, p) = \int_0^{u_0} |\gamma'(u)| du = \int_0^{u_0} \sqrt{\langle \gamma'(u), \gamma'(u)\rangle} du = u_0. \tag{3}$$

Now, consider the exponential map $\exp_{e_3} : T_{e_3}\mathbb{H}^2 \to \mathbb{H}^2$. Because $\gamma'(0) = (\cos v_0,\ \sin v_0,\ 0)$ unitary, we have by (3) that

$$\exp_{e_3}^{-1}(\Psi(u_0, v_0)) = \exp_{e_3}^{-1}(p) = u_0 \cdot \gamma'(0) = u_0 \cdot \Psi_u(0, v_0). \tag{4}$$

It is easy to see that $\gamma(t) = \cosh t\, e_3 + \sinh t\, e_1$ is the geodesic such that $\gamma(0) = e_3$ and $\gamma'(0) = e_1 \in T_{e_3}\mathbb{H}^2$. Let $q = \gamma(-1) = \cosh 1\, e_3 - \sinh 1\, e_1$. It is not difficult to see that $e_2 \in T_q\mathbb{H}^2$ and $\langle e_2, \gamma'(-1)\rangle = 0$. Now consider the geodesic $\beta : \mathbb{R} \to \mathbb{H}^2$, defined by $\beta(s) = \cosh s\, q + \sinh s\, e_2$.

Thus, we prove that the function $g : \mathbb{H}^2 \to \mathbb{R}$, defined as $g(p) = \langle \exp_{e_3}^{-1}(p), e_1\rangle$, is not convex. For this, we analyze $g(\beta(s))$. If $\Psi(u_0(s), v_0(s)) = \beta(s)$, it follows that

$$\begin{aligned}(\sinh u_0(s) \cos v_0(s),\ \sinh u_0(s) \sin v_0(s),\ \cosh u_0(s))&\\ &= \cosh s\, q + \sinh s\, e_2\\ &= \cosh s\,(\cosh 1\, e_3 - \sinh 1\, e_1) + \sinh s\, e_2\\ &= (-\sinh 1 \cosh s,\ \sinh s,\ \cosh 1 \cosh s).\end{aligned}$$

Therefore, using (4) we get

$$\begin{aligned}g(\beta(s)) &= \langle \exp_{e_3}^{-1}(\beta(s)), e_1\rangle = \langle u_0(s)\cdot \Psi_u(0, v_0(s)), e_1\rangle\\ &= -\frac{u_0(s)\cdot \sinh 1 \cdot \cosh s}{\sinh u_0(s)} = -\tanh 1 \frac{u_0(s)\cdot \cosh 1\cdot \cosh s}{\sinh u_0(s)}\\ &= -\tanh 1 \frac{u_0(s)\cdot \cosh u_0(s)}{\sinh u_0(s)} = -\tanh 1 \cdot u_0(s)\cdot \tanh\, u_0(s).\end{aligned}$$

A consequence of the equality $\cosh u_0(s) = \cosh 1 \cosh s$ is that, $u_0(s)$ is an even function and

$$\lim_{s\to\pm\infty} u_0(s) = +\infty.$$

Thus, $g(\beta(s))$ is also an even function and

$$\lim_{s\to\pm\infty} g(\beta(s)) = -\infty. \tag{5}$$

Now, assume by contradiction that g is convex. Then, for all s,

$$-1 = g(\beta(0)) = g\left(\beta\left(\frac{1}{2}s + \frac{1}{2}(-s)\right)\right) \leq \frac{1}{2}g(\beta(s)) + \frac{1}{2}g(\beta(-s)) = g(\beta(s))$$

which contradict (5).

5. Conclusions

In Theorem 2.1, we proved that the function in question is affine on Hadamard manifolds with zero curvature. In Theorem 3.1, we established that the function is not convex in Hadamard manifolds with negative curvature. This function has a role in proximal regularization. Now, the open problem is to ensure the existence of a convex function that has sufficient properties to prove the convergence of the sequence generated by the proximal point method, when, for example, applied to the equilibrium problem.

During the submission process of this paper we became aware that Kristály-Li-Lopez-Nicolae [6] have made available on arXiv a work in which similar results to ours are established, but using different techniques. More specifically, to ensure that the manifold M is a Euclidean space they use Choquet theorem [9, Theorem 6.5], while in our paper we apply a theorem proved by Innami [5, Main Theorem]. We emphasize that a draft of our work has been made available in the optimization-online repository [2].

Acknowledgements. The authors would like to express their gratitude to an unknown referee for valuable suggestions on the improvement of the paper.

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