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Minimal Representation in a Quotient Space over a Lattice of Unbounded Closed Convex Sets

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In this paper we present conditions under which elements of a quotient space (Minkowski–Rådström–Hörmander space) over a lattice of unbounded closed convex sets with common recession cone have inclusion-minimal representation. We also discuss the uniqueness-up-to-translation of minimal representation.

1. Introduction

The law of cancellation holds true in the semigroup $\mathcal{B}(X)$ of nonempty bounded closed convex sets in real Hausdorff topological vector space X, where the addition $\dot{+}$ is defined by $A\dot{+}B = \overline{A+B} = \overline{\{a+b | a \in A, b \in B\}}$. Minkowski studied the semigroup of compact convex subsets of Euclidean spaces, therefore the algebraic or vector addition $A+B$ is also called the Minkowski addition. In infinite dimensional topological vector spaces we need to consider $A\dot{+}B$ instead of $A+B$. Rådström [14] applied the Cancellation law in order to embed the convex cone $\mathcal{B}(X)$, where X is a normed vector space, into a normed vector space. Hörmander [10] generalized the Rådström's result to the locally convex spaces and Urbański [18] to the arbitrary topological vector spaces.

Since Cancellation law does not hold in the semigroup $\mathcal{C}(X)$ of nonempty closed convex sets, this semigroup cannot be embedded into a vector space. However, Robinson [15] proved Cancellation law for the family $\mathcal{C}_V(\mathbb{R}^n)$ of nonempty closed convex subsets of $\mathbb{R}^n$ sharing common recession cone V. He also embedded $\mathcal{C}_V(\mathbb{R}^n)$ into a topological vector space (but not into a normed space). Bielawski and Tabor [2] restricted the family $\mathcal{C}_V(X)$, where X is a normed vec-

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tor space to such closed convex sets A that the Hausdorff distance $d_H(A, V)$ is finite. It enabled them to embed restricted $\mathcal{C}_V(X)$ into a normed vector space, generalizing in this way Rådström's result.

Let S be an *abstract convex cone with* 0 i.e. a commutative semigroup with zero in which a mapping $(\lambda, s) \longrightarrow \lambda s$ of $\mathbb{R}_+ \times S$ into S is defined such that

$$\begin{aligned} 1s &= s, \\ \lambda(\mu s) &= (\lambda\mu)s, \\ \lambda(s+t) &= \lambda s + \lambda t, \\ (\lambda+\mu)s &= \lambda s + \mu s \end{aligned}$$

for all $s, t \in S$ and $\lambda, \mu \in \mathbb{R}_+$.
Moreover assume that in this cone the *cancelation law* i.e. for all $a, b, c \in S$

$$a + c = b + c \text{ if and only if } b = c$$

is satisfied. Then in the Cartesian product $S \times S$ we can introduce an equivalence relation $\sim$ as follows

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c\,.$$

Let $\widetilde{S}$ denote the quotient set of this relation and let

$$[a, b] = [(a, b)]_\sim = \{(u, v) \in S \times S | a + v = b + u\}\,.$$

Then the quotient space $\widetilde{S}$ equipped with

- *addition*
$$[a, b] + [c, d] = [a + c, b + d]$$
- *scalar multiplication*
$$\mu[a, b] = \begin{cases} [\mu a, \mu b] & \text{if } \mu \geq 0 \\ [(-\mu)b, (-\mu)a] & \text{if } \mu < 0 \end{cases}$$

becomes a real vector space which is called the *Minkowski–Rådström–Hörmander space over the cone S*, shortly the *MRH space over S*.

In the case of $S = \mathcal{B}(X)$ or $S = \mathcal{K}(X)$, where the symbol $\mathcal{K}(X)$ denotes the family of all nonempty compact convex subsets of a real Hausdorff topological vector space X, the appropriate MRH spaces were investigated by many authors ([3, 10, 13, 14, 16, 18]) and in this case the MRH space is denoted by $\widetilde{X}$. There also arose the concept of minimal pair of compact convex sets and minimal pair of closed bounded and convex sets. We say that a pair $(A, B) \in \mathcal{B}^2(X)$(resp. $\in \mathcal{K}^2(X)$) is *minimal* if for any pair $(C, D) \in \mathcal{B}^2(X)$(resp. $\in \mathcal{K}^2(X)$) such that $C \subset A, D \subset B$ and $A + D = B + C$ we have $A = C$ and $B = D$. The concept of minimal pairs is by Minkowski duality closely related to the problem of minimal representation of DCH functions (differences of homogenous convex functions) which is a variant of the problem of minimal representation of DC functions (see [1, 19, 9]).

In [11] it was proved that *for any pair* $(A, B) \in \mathcal{K}^2(X)$ *there exists a minimal pair* $(C, D) \in [A, B]$ *such that* $C \subset A$ *and* $D \subset B$. Moreover it can be proved [7] that *if* X *is a reflexive Banach space then for any pair* $(A, B) \in \mathcal{B}^2(X)$ *there exists a minimal pair* $(C, D) \in [A, B]$ *such that* $C \subset A$ *and* $D \subset B$. Also it is known that if $X = c_0$ or $X = \ell^\infty$ then there exists a pair $(A, B) \in \mathcal{B}^2(X)$ such that there is no minimal pair in the quotient class $[A, B]$ [7]. Hence the following question arose: does in any nonreflexive Banach space there exist a pair $(A, B) \in \mathcal{B}^2(X)$ such that there is no minimal pair in $[A, B]$? We still don't know the answer, so it is an open problem.

Parallel to the issue of existence there also arose a problem of uniqueness of minimal pairs. This problem was solved by Grzybowski [4] and Sholtes [17] independently. It was shown that *if* $(A, B), (C, D) \in \mathcal{K}^2(\mathbb{R}^2)$ *are two equivalent minimal pairs then there exists a vector* $x \in \mathbb{R}^2$ *such that* $A = C + x, B = D + x$. It was also shown [4] that *if* $\dim(X) > 2$ *then there exist at least two equivalent minimal pairs* $(A, B), (C, D) \in \mathcal{K}^2(X))$ *such that* $A = C + x$ *and* $B = D + x$ *for no* $x \in X$.

In this paper we introduce the notion of minimal pairs of unbounded closed convex sets with common recession cone V. We prove in Theorem 2.5 under some conditions the existence of such minimal pairs. In order to prove it we investigate in Proposition 2.2 the translation of intersection property $(\bigcap_{t\in T}(A \dot{+} A_t) = A \dot{+} \bigcap_{t\in T} A_t)$ for chains of sets. Examples 2.3 and 2.4 show that the conditions cannot be weakened.

We also study the problem of uniqueness for minimal pairs of closed convex and unbounded sets. We affirm the uniqueness of equivalent minimal pairs up to translation in two-dimensional space. In Theorem 3.1 we give the formula for a minimal pair equivalent to a given pair $(A, B) \in \mathcal{C}_V^2(\mathbb{R}^2)$. The uniqueness up to translation follows from the formula. The approach from [4] appeared to be fruitful. In our case the criterion of minimality (Theorem 3.2) is simpler than the one for compact convex sets. We also find the criterion of minimality for a pair of nonbounded polygonal sets.

In the case of $\mathbb{R}^3$ we give Example 4.1 of three minimal equivalent pairs that are not connected by translation. There is some affinity with the example in [4]. This example is generalized to any pointed cone V in any finite-dimensional vector space (Theorem 4.2). We also provide pictures (projections) of these pairs of (polyhedral) sets. Moreover we give a continuum of equivalent minimal pairs that are not connected by translation.

2. Existence of minimal pairs of convex sets

Let a Banach space X be a dual of a weakly compact generated normed space. In X every bounded sequence (x_n) has a weak-$*$ convergent subsequence. Let $V \subset X$ be a closed pointed convex cone. By Theorem 3.2 the cone $\mathcal{C}_V$ is naturally embedded into the quotient space $\widetilde{X}_V = \mathcal{C}_V \times \mathcal{C}_V/_\sim$. In this section we investigate the existence of inclusion-minimal representation of the quotient class $[A, B] = [(A, B)] \in \widetilde{X}_V$, where $A, B \in \mathcal{C}_V$. It should be noted that the existence of inclusion-minimal representation of the quotient class $[A, B] \in \mathcal{C}(X) \times \mathcal{B}(X)/_\sim$ was already proved for a reflexive Banach space X (Theorem 6.3 in [6]). However, since in the case of $[A, B] \in \widetilde{X}_V$ both sets A and B are unbounded, we need a new approach.

Lemma 2.1. *Let $A \in \mathcal{C}_V(X)$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements of the set A such that $||a_n|| \longrightarrow \infty$. Moreover, let the set $\{x \in V \mid \|x\| = 1\}$ be strictly separated from 0 by a hyperplane $H = \{x \in X | f(x) = 1\}$ which is closed in the weak-$*$ topology. Suppose that*

$$\frac{a_n}{||a_n||} \longrightarrow a_0$$

in the weak-$*$ topology, then $a_0 \neq 0$.

Proof. Since $A \in \mathcal{C}_V(X)$ then $A \subset V + B_r$ where B_r denotes a closed ball with center at zero and radius r. Thus $a_n = v_n + b_n$ where $v_n \in V$ and $b_n \in B_r$. Therefore

$$\frac{a_n}{||a_n||} = \frac{v_n}{||a_n||} + \frac{b_n}{||a_n||}.$$

But

$$\frac{b_n}{||a_n||} \longrightarrow 0$$

in the norm topology, hence

$$\frac{||v_n||}{||a_n||} \longrightarrow 1.$$

So we have

$$\begin{aligned} f(a_0) &= \lim_{n\to\infty} f\left(\frac{a_n}{||a_n||}\right) = \lim_{n\to\infty} f\left(\frac{a_n}{||v_n||}\right) = \lim_{n\to\infty} f\left(\frac{v_n}{||v_n||} + \frac{b_n}{||v_n||}\right) \\ &= \lim_{n\to\infty} f\left(\frac{v_n}{||v_n||}\right) + \lim_{n\to\infty} f\left(\frac{b_n}{||v_n||}\right) = \lim_{n\to\infty} f\left(\frac{v_n}{||v_n||}\right) \geqslant 1. \end{aligned}$$

Therefore $a_0 \neq 0$. $\square$

Proposition 2.2. *Let X be dual to some weakly compact generated normed space and $V \subset X$ be a closed pointed convex cone (i.e. $V \cap (-V) = \{0\}$).*

Moreover, let $\{x \in V \mid \|x\| = 1\}$ be strictly separated from 0 by a hyperplane H which is closed in the weak-$*$ topology. If the sets $A_t \in \mathcal{C}_V(X)$, $t \in T$, are weak-$*$ closed and $\{A_t\}$ is a chain with no least element, then for any weak-$*$ closed set $A \in \mathcal{C}_V(X)$ we have

$$\bigcap_{t\in T}(A\dot{+}A_t) = A\dot{+}\bigcap_{t\in T} A_t\,.$$

Proof. The inclusion $\supset$ is obvious. Let $z \in \bigcap_{t\in T} \overline{A + A_t}$. Then for every $\varepsilon > 0, t \in T$ there exist $a_t \in A, b_t \in A_t, c_t \in X$ such that $\|c_t\| \leqslant \varepsilon$ and $z = a_t + b_t + c_t$.

Case 1. The set $\{a_t\}_{t\in T}$ is not bounded. There exists a sequence $(a_{t_n})_{n\in\mathbb{N}}$ such that $\|a_{t_n}\|$ tends to infinity and (A_{t_n}) is decreasing. Since the sequence $\frac{a_{t_n}}{\|a_{t_n}\|}$ is bounded, some of its subsequence $\frac{a_{t_{n_k}}}{\|a_{t_{n_k}}\|}$ is convergent in the weak-$*$ topology to $a_0 \in X$. Now by Lemma 2.1 we get $a_0 \neq 0$ and, since

$$\frac{z}{\|a_{t_n}\|} \longrightarrow 0 \quad \text{and} \quad \frac{c_{t_n}}{\|a_{t_n}\|} \longrightarrow 0,$$

we obtain $\frac{b_{t_{n_k}}}{\|b_{t_{n_k}}\|} \longrightarrow -a_0$. Hence $a_0, -a_0 \in V$. Contradiction.

Case 2. The set $\{a_t\}_{t\in T}$ is bounded. The chain A_t is a decreasing net (or MS–sequence). Then there exist subnets $\{a_\sigma\}$ and $\{c_\sigma\}$ of the nets $\{a_t\}$ and $\{c_t\}$, respectively, such that $a_\sigma \longrightarrow a_0 \in A$ and $c_\sigma \longrightarrow c_0$, $||c_0|| \leqslant \varepsilon$. Therefore $b_\sigma \longrightarrow b_0 = z - a_0 - c_0$, and since the sets A_t are weak-$*$ closed, we have $b_0 \in \bigcap_{t\in T} A_t$. Hence

$$z = a_0 + b_0 + c_0 \in A + \bigcap_{t\in T} A_t + B_\varepsilon.$$

Since ε was arbitrary, we obtain

$$z \in A\dot{+}\bigcap_{t\in T} A_t\,. \qquad \square$$

Example 2.3. Let ℓ_1 denote the space of all absolutely summable sequences of real numbers with the norm $||a|| = \sum_j |a_j|$, and let B denote its closed unit ball. Let

$$B_n = \{a \in B : a_1 \leqslant a_2 \leqslant ... \leqslant a_n \text{ and } a_j \geqslant 0 \ \text{ for all } j \in \mathbb{N}\},$$

$S = \{a \in B : ||a|| = 1 \text{ and } a_j \geqslant 0 \ \text{ for all } j \in \mathbb{N}\}$, $A_n = S \cap B_n$. Then

$$0 \in \bigcap_{n\in\mathbb{N}}(B\dot{+}A_n) \neq B\dot{+}\bigcap_{n\in\mathbb{N}} A_n = \emptyset\,.$$

Example 2.3 shows that Proposition 2.2 does not hold if the sets A_n are not weak-$*$ closed.

The following example shows that the assumption in Proposition 2.2 about cone V, i.e. that $V \cap B$ is strictly weak-$*$ separated from 0, is essential even in Hilbert space ℓ_2.

Example 2.4. Let $X = \ell_2$ and let

$$V = \left\{ a = (a_0, ...) \in \ell_2 \mid a_0 \geqslant \sum_{j=1}^{\infty} \frac{|a_j|}{j} \right\},$$

$$A = \left\{ a \in \ell_2 \mid a_0 \geqslant \sum_{j=1}^{\infty} \frac{|a_j|}{j} - 2, a_0 \geqslant \sum_{j=1}^{\infty} \frac{a_j}{j} \right\},$$

$$A_n = \left\{ a \in \ell_2 \mid a_0 \geqslant \sum_{j=1}^{\infty} \frac{|a_j|}{j} - 1, a_0 \geqslant \sum_{j=1}^{n-1} \frac{|a_j|}{j} \right\},$$

where $n \geqslant 2$. Denote $p(a) = \sum_{j=1}^{\infty} \frac{|a_j|}{j}$. The function p is sublinear (subadditive and positively homogenous). By Hölder inequality

$$p(a) \leqslant \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \cdot \|a\|_2.$$

Then p is continuous in 0. Hence it is continuous, and V is a pointed closed convex cone in ℓ_2.

Similarily, A and A_n are closed convex subsets of ℓ_2. Since ℓ_2 is reflexive, V, A and A_n are weakly closed and weak-$*$ closed. Since $V \subset A \subset V - 2e_0$ and $V \subset A_n \subset V - e_0$, where $e_0 = (1, 0, ...)$, the recession cone of A and of each A_n is V. Concerning the Hausdorff metric we have $d_H(A, V) \leqslant 2$ and $d_H(A_n, V) \leqslant 1$. Therefore, $A, A_n \in \mathcal{C}_V(\ell_2)$.

Notice, that $\bigcap_{n=2}^{\infty} A_n = V$, and $A \dot{+} \bigcap_{n=2}^{\infty} A_n = A \dot{+} V = A$. Let us observe that $-e_0 \notin A$, but $-ne_n - e_0 \in A$ and $ne_n \in A_n$, where e_n is the n-th canonical basis vector in ℓ_2. Therefore, $-e_0 = (-ne_n - e_0) + ne_n \in \bigcap_{n=2}^{\infty}(A \dot{+} A_n)$ and $-e_0 \notin A \dot{+} \bigcap_{n=2}^{\infty} A_n$.

We have just showed that Proposition 2.2 does not hold true any longer if we omit the assumption that the set $\{a \in V \mid \|a\| = 1\}$ is strictly separated from 0 by some hyperplane H which is closed in the weak-$*$ topology.

In the product $\mathcal{C}_V^2(X) = \mathcal{C}_V(X) \times \mathcal{C}_V(X)$ we introduce an equivalence relation $\sim$ as follows:

$$(A, B) \sim (C, D) \text{ iff } A \dot{+} D = B \dot{+} C.$$

By $\widetilde{X}_V$ we denote the quotient set $\mathcal{C}_V^2(X)/_\sim$. Let $[A,B] \in \widetilde{X}_V$. We introduce a partial order $\leqslant$ within the quotient class $[A,B]$ as follows:

$$(A,B) \leqslant (C,D) \text{ if and only if } A \subset C \text{ and } B \subset D.$$

Notice that in no quotient class $[A,B] \in \widetilde{X}_V$ there exists a minimal pair. By *minimal pair* $(C,D) \in [A,B]$ we understand a minimal element of $[A,B] \cap (\mathcal{C}_0(X) \times \mathcal{C}_V(X))$ with respect to $\leqslant$, where $\mathcal{C}_0(X) = \{A \in \mathcal{C}_V(X) \mid 0 \in A\}$.

Theorem 2.5. *Let X be a reflexive Banach space and $V \subset X$ be a closed pointed convex cone. Moreover, let the set $\{x \in V \mid \|x\| = 1\}$ be strictly separated from 0 by a hyperplane H which is closed in the norm topology. Then in every equivalence class $[A,B] \in \widetilde{X}_V$ there exists a minimal pair.*

Proof. Let $(A_t, B_t) \in [A,B] \cap (\mathcal{C}_0(X) \times \mathcal{C}_V(X))$ be a chain. By Proposition 2.2 we get

$$B \dot{+} \bigcap_t A_t = \bigcap_t (B \dot{+} A_t) = \bigcap_t (A \dot{+} B_t) = A \dot{+} \bigcap_t B_t.$$

Hence the element $(A_0, B_0) = (\bigcap_t A_t, \bigcap_t B_t) \in [A,B]$ is a lower bound of the chain $\{(A_t, B_t)\}_{t \in T}$. Thus by Kuratowski–Zorn Lemma there exists a minimal element in $[A,B]$, and the proof is complete. $\square$

In fact any element of $[A,B] \cap (\mathcal{C}_0(X) \times \mathcal{C}_V(X))$ is bounded from below by some minimal element.

Every finite dimensional vector space X is reflexive and for every pointed closed convex cone $V \subset X$ the set $\{x \in V \mid \|x\| = 1\}$ is strictly separated from 0 by some hyperplane. Therefore, the following theorem is an obvious corollary from the last theorem.

Corollary 2.6. *Let X be a finite dimensional vector space and $V \subset X$ be a closed pointed convex cone. Then in every equivalence class $[A,B] \in \widetilde{X}_V$ there exists a minimal pair.*

3. Uniqueness of minimal pairs of convex sets in a two-dimensional vector space

Two dimensional vector space can be identified with $\mathbb{R}^2$. The recession cone is an angle with vertex 0 of measure less then π, possibly 0. Assume that the x-axis is bisecting the recession cone V and that the negative part of the x-axis is contained in V. We also assume that the measure of the angle V is equal to $\pi - 2\vartheta$, where $0 < \vartheta \leqslant \frac{\pi}{2}$. Our approach is similar to the one that was applied first in [4] and [5] to compact convex subsets of $\mathbb{R}^2$.

Let $\mathcal{A}$ be the abstract cone of nondecreasing real functions f on the open interval $(-\vartheta, \vartheta)$, such that $f(0) = 0$ and $f(t) = \frac{f(t^+)+f(t^-)}{2}$ where $f(t^+) =$

$\lim_{s\to t^+} f(s), f(t^-) = \lim_{s\to t^-} f(s)$, for $t \in (-\vartheta, \vartheta)$. Let us notice that every function $f \in \mathcal{A}$ has bounded variation on any closed subinterval of $(-\vartheta, \vartheta)$.

Define the ordering in $\mathcal{A}$ the following way. For the functions $f, g \in \mathcal{A}$ we have $f \leqslant g$ if and only if $g - f \in \mathcal{A}$. Then the infimum $\inf(f, g)$ is the greatest function in $\mathcal{A}$ such that $f - \inf(f, g), g - \inf(f, g) \in \mathcal{A}$.

For $A \in \mathcal{C}_V(\mathbb{R}^2), u \in \mathbb{R}^2$ we have the face $A(u) = \{a \in A | \langle u, a\rangle = \max_{b\in A}\langle u, b\rangle\}$, where $\langle x, y\rangle$ is the inner product. Obviously, $A(u)$ is either a segment or singleton. Let the boundary function $h_A : (-\vartheta, \vartheta) \longrightarrow \partial A$, where $h_A(t)$ is the center of $A(\cos t, \sin t)$. We also define the arc length function $f_A : (-\vartheta, \vartheta) \longrightarrow \mathbb{R}$, where $f_A(t)$ is the length of the arc contained in ∂A joining $h_A(0)$ and $h_A(t)$ for $t \geqslant 0$ and minus the length of the arc joining $h_A(0)$ and $h_A(t)$ for $t < 0$. The function f_A belongs to $\mathcal{A}$. The mapping $\mathcal{C}_V \ni A \mapsto f_A \in \mathcal{A}$ is a homomorphism of abstract convex cones i.e. $f_{A+B} = f_A + f_B$ and $f_{\alpha A} = \alpha f_A$ for all $A, B \in \mathcal{C}_V, \alpha \geqslant 0$.

For a nondecreasing function $f \in \mathcal{A}$ we define the function $h_f : (-\vartheta, \vartheta) \longrightarrow \mathbb{R}^2$ by $h_f(t) = \int_0^t(-\sin s, \cos s)df(s)$ for $t \geqslant 0$, where the latter is the Stieltjes integral, and $h_f(t) = -\int_t^0(-\sin s, \cos s)df(s)$ for $t < 0$.

We denote $A_f = \overline{\operatorname{conv}(h_f(-\vartheta, \vartheta)) + V}$. Then $A_{f_A} = A - h_A(0)$ and $f_{A_f} = f$.

The following theorem describes the construction of a minimal pair equivalent to the given one (see [4]):

Theorem 3.1. *Let* $(A, B) \in \mathcal{C}_V^2(\mathbb{R}^2)$. *Denote* $f_1 = f_A - \inf(f_A, f_B)$ *and* $f_2 = f_B - \inf(f_A, f_B)$. *Then the pair* $(A_{f_1}, A_{f_2} + h_B(0) - h_A(0))$ *is minimal and belongs to* $[A, B]$.

The following theorem, the Criterion of Minimality (compare Lemma 5.1 in [4]) follows from Theorem 3.1.

Corollary 3.2 (Criterion of Minimality). *Let* $(A, B) \in \mathcal{C}_V^2(\mathbb{R}^2)$. *The pair* (A, B) *is minimal if and only if* $0 \in A(\cos t, \sin t)$ *for some* $t \in (-\vartheta, \vartheta)$ *and* $\inf(f_A, f_B) = 0$.

The following theorem is a corollary to Corollary 3.2.

Corollary 3.3. *Let* $(A, B) \in \mathcal{C}_V^2(\mathbb{R}^2)$. *If the boundaries of* A *and* B *are unions of segments and/or rays then the pair* (A, B) *is minimal if and only if the sets have no parallel bounded sides and* 0 *belongs to one of* A*'s bounded sides.*

4. Lack of the uniqueness of minimal pairs of convex sets in a more-than-two-dimensional vector space

As we know from [4] in the family of compact convex subsets of $\mathbb{R}^3$ there exist two minimal pairs of sets not connected by translation. In fact if there

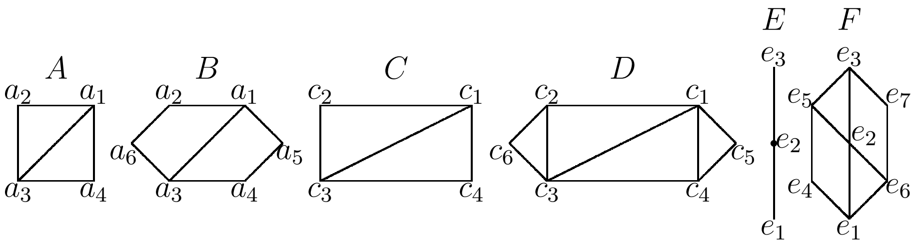

Figure 4.1: Orthogonal projections of sets from Example 4.1 on the plane Ox_1x_2

exist two of such pairs then there exist uncountably many of minimal pairs not connected by translation [8, 12]. These results hold true in all higher dimensional Hausdorff topological vector spaces.

In this section we also show that in $\mathbb{R}^3$ for every pointed convex cone V there exist two equivalent minimal pairs of sets in $\mathcal{C}_V$ that are not translates of each other.

Example 4.1. Let $V = \{x \in \mathbb{R}^3 | x_1 = x_2 = 0, x_3 \geqslant 0\}$. Let us denote the following points $a_1 = (1,1,0), a_2 = (-1,1,1), a_3 = (-1,-1,0), a_4 = (1,-1,1)$, $a_5 = (2,0,1), a_6 = (-2,0,1), c_1 = (2,1,0), c_2 = (-2,1,1), c_3 = (-2,-1,0)$, $c_4 = (2,-1,1), c_5 = (3,0,1), c_6 = (-3,0,1), e_1 = (0,-2,1), e_2 = (0,0,0)$, $e_3 = (0,2,1)$, $e_4 = (-1,-1,2)$, $e_5 = (-1,1,1), e_6 = (1,-1,1), e_7 = (1,1,2)$. Also, let $A = \operatorname{conv}\{a_1,a_2,a_3,a_4\} + V$, $B = \operatorname{conv}\{a_1,a_2,a_3,a_4,a_5,a_6\} + V$, $C = \operatorname{conv}\{c_1,c_2,c_3,c_4\}+V$, $D = \operatorname{conv}\{c_1,c_2,c_3,c_4,c_5,c_6\}+V$, $E = \operatorname{conv}\{e_1,e_2,e_3\}+V$, $F = \operatorname{conv}\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\} + V$.

In the following picture (Fig. 4.1) we present orthogonal projections of all mentioned sets on the plane Ox_1x_2.

In Fig. 4.1 solid lines are projections of all the edges that are not orthogonal to the plane plane Ox_1x_2. Visible polygons are projections of all bounded facets of the listed sets.

All sets A, B, C, D, E and F belong to $\mathcal{C}_V(\mathbb{R}^3)$. Moreover, $A, C, E \in \mathcal{C}_0(\mathbb{R}^3)$. It can be proved that the three pairs (A, B), (C, D) and (E, F) are inclusion-minimal. Notice that these three pairs are mutually equivalent, as we see in the picture below (Fig. 4.2).

It is easy to observe that these pairs are not connected by translation. Using these pairs we can easily construct continuum of equivalent minimal pairs not connected by translation. For example $((1-t)A+tE, (1-t)B+tF)$, $0 \leqslant t \leqslant 1$.

The example is also correct if we replace the ray V with any cone contained in the cone $\{x \in \mathbb{R}^3 | x_3 \geqslant 2\sqrt{x_1^2 + x_2^2}\}$. This leads us to the following theorem.

Theorem 4.2. *If $V \subset \mathbb{R}^n$, $n \geqslant 3$ is a pointed cone then in $\mathcal{C}_V^2$ there exist a continuum of equivalent minimal pairs not connected by translation.*

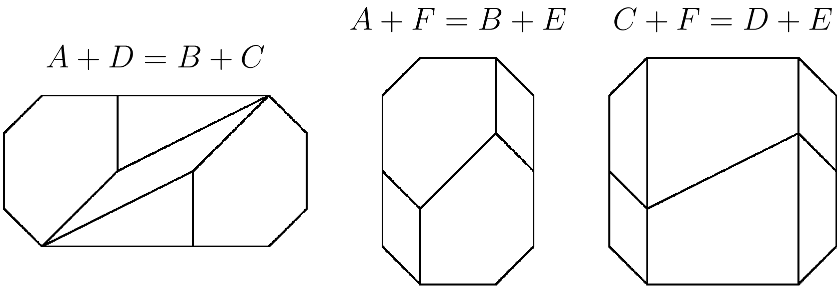

Figure 4.2: Orthogonal projections of sums of sets from Example 4.1.

Proof. For $n = 3$ Example 4.1 serves as the proof of our theorem in case of a cone V contained in the cone $\{x \in \mathbb{R}^3 | x_3 \geqslant 2\sqrt{x_1^2 + x_2^2}\}$. Since for any pointed cone V in $\mathbb{R}^3$ we can find a linear transformation which maps V onto a cone contained in the cone $\{x \in \mathbb{R}^3 | x_3 \geqslant 2\sqrt{x_1^2 + x_2^2}\}$, the theorem holds true for $n = 3$.

For $n > 3$ and any pointed cone $V \subset \mathbb{R}^n$ we can assume (using linear transformation if necessary) that $V \subset \{x \in \mathbb{R}^n | x_3 \geqslant 2\sqrt{x_1^2 + x_2^2}\}$. Let us define the sets A, B, C, D, E and F in the same way as in Example 4.1 taking the points a_i, c_i, e_i as elements of $\{x \in \mathbb{R}^n | x_4 = \ldots = x_n = 0\}$. Then the pairs $(A, B), (C, D)$ and (E, F) are equivalent minimal pairs in $\mathcal{C}_0(\mathbb{R}^n) \times \mathcal{C}_V(\mathbb{R}^n)$ that are mutually not connected by translation. The same is true for the pairs $((1-t)A + tE, (1-t)B + tF)$, $0 \leqslant t \leqslant 1$. $\square$

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