Journal of Convex Analysis
Volume 24 (2017), No. 3, 707–762

Reinventing Weak Barrelledness

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In memory of Professor Manuel Valdivia.

Received: August 22, 2015
Revised manuscript received: February 18, 2016
Accepted: March 10, 2016

With scores of novel theorems/examples we establish a hierarchy of 16 barrelled-type properties, consolidating decades of work by dozens of authors, defining/motivating/displaying optimal results. We prove our display answers all 4.29 billion interrelational questions at a glance. We solve the countable enlargement problems for separable weak barrelledness, uniformly relax certain Valdivia hypotheses to a characterization, and show that the existence of measurable cardinals depends on which of two hypotheses is optimal for Dierolf's dense subspace theorem.

Keywords: ℓ^∞-barrelled, dual locally complete, primitive, optimal results, measurable cardinals, countable enlargements

2010 Mathematics Subject Classification: Primary 46A08, 46A03; Secondary 03E55, 03E65

1. Introduction

This work, both innovative and retrospective, complements career-spanning joint and separate efforts. Our prequels cite results printed here with proofs for the first time. The completed paper transmutes obligation into joy. New results include and inspire solutions to old problems (see Corollary 5.8 and [45]) and illustrate the value of establishing Fig. 1.3, below, as a contextually rich way to comprehend the past and fashion the future of weak barrelledness.

Barrelledness (the Uniform Boundedness Principle) and its important weaker

*Supported by Spanish Ministerio de Economía y Competitividad. Secretaría General de Ciencia y Tecnología e Innovación ESP2013-41078-R.

ISSN 0944-6532 / $ 2.50

properties occupy a central place in the study of locally convex spaces. Some of the history is recounted in [56, pp. 230-233].

Our story began in 1971, when Saxon-Levin [46] and Valdivia [64] published the same result: Valdivia used *dual locally complete* (dlc), Saxon-Levin used a seemingly different weak barrelledness property. A quarter-century later, we realized [51] the crucial notions are equivalent (see Theorem 2.6(4), below). We saw [55, Theorems 5, 14] that *dlc* was the optimal support for the Tweddle-Yeomans Criterion, and continues to be so in the current expanded context (Theorems 5.1, 5.2). Theorem 3.15 shows *dlc* optimally supports dense subspace inheritance of most properties. Theorem 5.13 shows exactly when Dierolf's dense subspace theorem ([3], [26, 4.6.6], [31, Corollary 1]) is supported at the most general level of primitive spaces. By contrast, the splitting theorem [31, Theorem 2] extends no farther than Mackey $\aleph_0$-barrelled spaces [56].

To give meaning and proof for such claims of optimality, one desires a simple yet inclusive frame of reference, a manageably comprehensive list of weak barrelledness properties whose interrelationships are known generally, and under metrizability, separability or the Mackey topology. In this section we present our best candidate, Fig. 1.3, with preliminary Figs. 1.1 and 1.2 and requisite definitions.

Section 2 verifies the arrows in Fig. 1.3 with basic characterizations, novel use of pointwise eventually zero sequences (prompting solution [45] of Robertson's quarter-century-old problem), and a short proof that two of the properties are inherited by countable-codimensional subspaces. The same holds for the other twelve eligible properties via a brief argument in Section 4, aided by principles deduced in Section 3 about properties inherited from/by dense subspaces.

Section 4 studies additional stability of properties under quotients, products, etc., especially three-space-type stability.

Section 5 solves positively all of the separable weak barrelledness countable enlargement problems, assuming CH, and determines the optimal hypothesis for Dierolf's dense subspace theorem (i) if measurable cardinals exist and (ii) if they do not.

Section 6, with dozens of examples and results, determines which subsets of Fig. 1.3 are the precise subset of properties enjoyed by some locally convex space: They are exactly those subsets that are not immediately disallowed by the arrows of Fig. 1.3, with one obvious exception. We relate Kalton's *G-barrelled* to Fig. 1.3.

Section 7 explains why it is trebly convenient to omit *semi-Baire-like* from Fig. 1.3, and finds the optimal (weakest) hypothesis on a space $E \not\supset \varphi$ that will ensure E is semi-Baire-like.

The concluding comments of Section 8 remark a quite recent and felicitous marriage between weak barrelledness and (topological) P-spaces.

$$
\begin{array}{ccc}
\text{barrelled; } (p_\alpha)\text{b} & & \\
\Downarrow & & \\
\aleph_0\text{-barrelled; } (p_n)\text{b} & \Longrightarrow & \ell^\infty\text{-barrelled; } (f_n)\text{b} \\
\texttt{1) separable} & & \\
\Downarrow & & \Downarrow \\
\text{C-barrelled; } (p_n)\,0 & \Longrightarrow & c_0\text{-barrelled; } (f_n)\,0 \\
\Downarrow & & \Downarrow \\
\text{property (L); } p|_{A_n} & \Longrightarrow & \text{dlc; } f|_{A_n} \\
\texttt{2) Mackey} & & \\
\texttt{3) metrizable} & & \\
\Downarrow & & \Downarrow \\
\text{inductive; } p|_{E_n} & \Longrightarrow & \text{primitive; } f|_{E_n} \\
\texttt{4) Mackey} & &
\end{array}
$$

Figure 1.1.

Our initial Fig. 1.1 (cf. [38, 55]) fostered relatively optimal results and indicates our answer [52] to the Ferrando-López Pellicer question [7], to wit: the notions in box 2 are equivalent under the Mackey topology. Notions in each of the other three boxes similarly coincide under their respective assumptions, as was already known for boxes 1 and 3. The symbols after the semicolons recall symmetric *seminorm* vs. *linear form* characterizations/definitions for the left vs. right column [55]. The second and third rows constitute (the transpose of) Table 3 in [26, p. 507], the fourth row is part of Ruess' scheme [34, 35, 36].

Let "space" mean "Hausdorff locally convex topological vector space over the scalar field $\mathbb{K}$ of reals or complexes". A *barrel* in a space E is an absorbing absolutely convex closed set; an $\aleph_0$-*barrel* is a barrel U that is the intersection of a sequence $\{U_n\}_n$ of absolutely convex closed neighborhoods of the origin; if each member of E is in all but finitely many of the U_n, then U is also a C-*barrel*. An increasing sequence $\sigma = \{A_n\}_n$ of absolutely convex sets in a space $(E, \mathcal{T})$ is a $\langle$*closed*$\rangle$ *absorbing sequence* if $\bigcup_n A_n$ is absorbing $\langle$and each A_n is closed$\rangle$; $\mathcal{T}_\sigma$ denotes the finest locally convex topology on E that coincides with $\mathcal{T}$ on each A_n. When $\sigma = \{A\}$, we will denote $\mathcal{T}_\sigma$ as $\mathcal{T}_A$.

A space E is *semi-Baire-like* (sBL) if some member of each closed absorbing sequence is an absorbing set (and thus a barrel) in E.

The following fourteen definitions are either the same as or are readily equivalent to the ones cited. Some of the terms differ from the original.
A space E is:

- *barrelled* if every barrel is a 0-neighborhood.
- $\langle\aleph_0$-$\rangle$ $(C$-$)$*barrelled* [12, 26] if every $\langle\aleph_0$-$\rangle$ $(C$-$)$barrel is a 0-neighborhood;
- S_σ [58] if there is an absorbing sequence of proper closed subspaces;

- *quasi-Baire* (QB) [47] if it is barrelled and non-S_σ;
- *Baire-like* (BL) [39] if it is barrelled and sBL;
- ℓ^∞*-barrelled* [24] if every $\sigma(E', E)$-bounded sequence is equicontinuous;
- c_0*-barrelled* [69] if every $\sigma(E', E)$-null sequence is equicontinuous;
- *dual locally complete* [66] if $(E', \sigma(E', E))$ is locally complete (cf. [26, 51]);
- *inductive* [9] if it is the inductive limit of every absorbing sequence of subspaces;
- *primitive* [57] if a linear form is continuous whenever it has continuous restrictions to each member of some absorbing sequence of subspaces.

A space $(E, \mathcal{T})$ has:

- *property (L)* [34] if $\mathcal{T} = \mathcal{T}_\sigma$ for each absorbing sequence σ;
- *property (LC)* [34] if $\mathcal{T}$ is compatible with each $\mathcal{T}_\sigma$;
- *property (C)* [24] if bounded sequences in $(E', \sigma(E', E))$ have cluster points;
- *property (S)* [24] if $(E', \sigma(E', E))$ is sequentially complete.

If the spaces E and F coincide as vector spaces and the topology on F is finer than that on E, then we say that F *dominates* E. Clearly, E is non-S_σ if it is dominated by a non-S_σ space. Also, *non-S_σ $\Rightarrow$ inductive* since one could insist defining steps be closed (cf. Theorem 2.7).

In the early 1970's, before Ferrando and Sánchez Ruiz [9] isolated inductivity and it became a core constituent of weak barrelledness, De Wilde, Houet, Saxon, Valdivia, et al. gave versions of the fact that *barrelled $\Rightarrow$ inductive*. Clearly,

$$\begin{array}{ccccc} \text{BL} & \Rightarrow & \text{QB} & \Rightarrow & \text{barrelled} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{sBL} & \Rightarrow & \text{non-}S_\sigma & \Rightarrow & \text{inductive} \end{array}$$

Note that every $C_c(X)$ and $C_p(X)$ space has a dense subspace that is dominated by a Banach space, thus has a dense non-S_σ subspace, thus is, itself, non-S_σ, hence inductive, hence primitive. *BL*, *QB* and *non-S_σ* form the *left periphery* of weak barrelledness: *BL* and *QB* naturally attach to the upper left of Fig. 1.1, *non-S_σ* to the lower left. The *right periphery*, consisting of properties (C) and (S), attaches to the right-center of Fig. 1.1, and our general view became that of Fig. 1.2 below:

The right periphery, if not the left, is clearly part of weak barrelledness. Still, the left periphery supports a metrizable dichotomy, i.e., the large boxes 3 and 5 in Fig. 1.3. And, as we just noted, *non-S_σ* establishes the weaker properties for $C(X)$ spaces.

Any space property P such that *Baire-like $\Rightarrow$ P $\Rightarrow$ primitive*, including all those studied here, is implicated in the separable quotient problem, since *A Banach space has a properly separable quotient if and only if it has a dense subspace which does not have property P* [1, 45, 53, 58].

$$
\begin{array}{ccccccccc}
\text{BL} & \Rightarrow & \text{QB} & \Rightarrow & \text{barrelled} & & & & \\
 & & & & \Downarrow & & & & \\
 & & & & \aleph_0\text{-barrelled} & \Rightarrow & \ell^\infty\text{-barrelled} & \Rightarrow & \text{property (C)} \\
 & & & & \Downarrow & & \Downarrow & & \\
\Downarrow & & \Downarrow & & \text{C-barrelled} & \Rightarrow & c_0\text{-barrelled} & & \Downarrow \\
 & & & & \Downarrow & & \Downarrow & & \\
 & & & & \text{property (L)} & \Rightarrow & \text{dlc} & \Leftarrow & \text{property (S)} \\
 & & & & \Downarrow & & \Downarrow & & \\
\text{sBL} & \Rightarrow & \text{non-}S_\sigma & \Rightarrow & \text{inductive} & \Rightarrow & \text{primitive} & &
\end{array}
$$

Figure 1.2.

Fig. 1.2 enlarged Fig. 1.1 to a total of fourteen named properties, excluding *sBL*. Fig. 1.3 interpolates two additional properties for a grand total of sixteen. In both cases, *sBL* may be inserted at the lower left, but is *not* one of the sixteen properties we formally associate with weak barrelledness. Indeed, *sBL* is expunged from Fig. 1.3, as justified in Section 7.

We say that a space E is

- *mc_0-barrelled* if $\{a_n f_n\}_n$ is equicontinuous whenever $\{f_n\}_n$ is $\sigma(E', E)$-bounded and $\{a_n\}_n$ is a null scalar sequence;
- *φ-complemental* if, for each closed $\aleph_0$-codimensional subspace F of E, each algebraic complement of F is a topological complement isomorphic to φ, an $\aleph_0$-dimensional space with its strongest locally convex topology.

Transparently, "E is mc_0-barrelled" is just a shorter way of saying "every sequence in $(E', \sigma(E', E))$ which converges to the origin in the sense of Mackey is equicontinuous". The longer phrase is in several Valdivia hypotheses [68, pp. 39, 40]. The mnemonic term *mc_0-barrelled* evokes a sequence $\{a_n f_n\}_n$ with $\{f_n(x)\}_n$ in $m = \ell^\infty$ for each $x \in E$ and $\{a_n\}_n$ in c_0. Also, mc_0 might remind us of $\underline{\text{M}}$ackey $\underline{\text{c}}$onvergence to $\underline{0}$.

Copies of φ determine when an (LF)-space is BL, QB, or neither [47]. The familiar *barrelled* $\Rightarrow$ *φ-complemental* was recently generalized [19, Lemma 3.2]. The original Saxon-Levin version: [*Property (S)* $\wedge$ *Mackey*] $\Rightarrow$ *φ-complemental* [46]. Valdivia [68, (10), p. 41] relaxed the hypothesis to [*dlc* $\wedge$ *Mackey*]. Fig. 1.3 indicates optimal relaxations, to *mc_0-barrelled* and to *inductive.*

Fig. 1.2 has an awkward arrow that points to the left, instead of down or to the right, and omits the two historically useful notions defined above. Fig. 1.3 remedies the anomaly and omissions. Its two additions nudge *dlc* and *primitive* one step to the right, while raising *property (S)* one level higher. From now on, "property" or "properties" will refer to one or more of the sixteen listed in Fig. 1.3, and "row(s)" , "column(s)" , and "box(es)" will also refer to those in Fig. 1.3, unless clearly indicated otherwise.

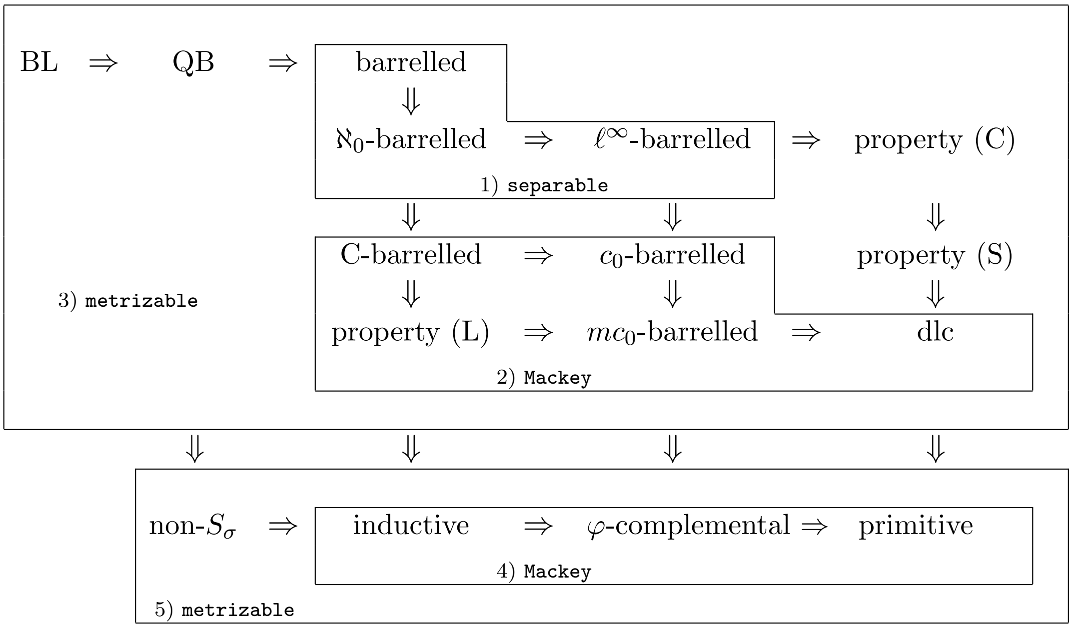

Figure 1.3.

There can be no arrow from c_0-*barrelled* to *property (S)*: Every Mackey dlc space without property (S) is c_0-barrelled, as indicated by box 2. (And, via Valdivia [65], has a *non*Mackey dense hyperplane; cf. [43].)

We will think of Fig. 1.3 as a 5×5 matrix with some vacant positions. Column 1 has only one entry, column 2, only two. Columns 3 and 4 enjoy an attractive *seminorm* vs. *linear form* symmetry (see (*i-iv*), below Theorem 2.7). Boxes 1-5 are known to be valid [26, 52, 53]. We will construct and cite examples to show the boxes cannot be drawn more liberally, and, among the sixteen, no property or combination of properties implies one not indicated by the arrows. The lone exception is the fact that, by definition, [*barrelled* $\wedge$ *non-*S_σ] $\Rightarrow$ *QB*.

2. Ten basic facts

For each space E, let $\mathfrak{p}(E)$ denote the set of all continuous seminorms on E. A sequence $\{\phi_n\}_n$ of scalar-valued functions on a space E is *pointwise eventually zero* if, for each $x \in E$, the evaluation $\phi_n(x)$ is zero for all but finitely many (*almost all*) $n \in \mathbb{N}$. By *countable*, we mean *finite or countably infinite.*

Theorem 2.1. *The following three statements about a space E are equivalent.*

(*1*) *E is an S_σ space.*

(*2*) *There is a closed $\aleph_0$-codimensional subspace M.*

(*3*) *Some linearly independent sequence $\{f_n\}_n$ in E' is pointwise eventually zero.*

Proof. (*1*) $\Rightarrow$ (*2*), (*3*). If $\{M_n\}_n$ is an absorbing sequence of proper closed subspaces, passing to a subsequence if necessary, we may assume that each M_n is a proper subspace of M_{n+1}, so that, by way of the Hahn-Banach theorem, some $f_n \in E'$ vanishes on M_n but not on all of M_{n+1}. Thus $M = \bigcap_n f_n^{\perp}$ is a closed $\aleph_0$-codimensional subspace of E, since $M \bigcap M_{n+1}$ is n-codimensional in M_{n+1}. It is also clear that $\{f_n\}_n$ is linearly independent and pointwise eventually zero.

(*2*) $\Rightarrow$ (*1*). Given M, an absorbing sequence $\{M_n\}_n$ of proper closed subspaces with $M_1 = M$ and $\dim M_{n+1}/M_n = 1$ is obvious.

(*3*) $\Rightarrow$ (*2*). Given $\{f_n\}_n$ as in (*3*), the subspace $M := \bigcap_n f_n^{\perp}$ satisfies (*2*). $\square$

The seminorm translation of our [54, Lemma 3.1] constitutes

Theorem 2.2. *The following two statements about a space E are equivalent.*

(*1*) *E is inductive.*

(*2*) *Every pointwise eventually zero sequence in $\mathfrak{p}(E)$ is equicontinuous.*

Theorem 2.3. *The following three assertions about a space E are equivalent.*

(*1*) *E is φ-complemental.*

(*2*) *Every seminorm p which vanishes on some closed countable-codimensional subspace M is continuous.*

(*3*) *Every pointwise eventually zero sequence $\{f_n\}_n$ in E' is equicontinuous.*

Proof. Trivially, (*1*) $\Rightarrow$ (*2*). Now suppose (*2*) holds and $\{f_n\}_n$ is a pointwise eventually zero sequence in E'. Put $M_n = \bigcap_{i \geq n} f_i^{\perp}$ and note that $\{M_n\}_n$ covers E, so that M_1 is countable-codimensional in E, being finite-codimensional in each M_n. Writing $p(x) = \max_n |f_n(x)|$ for each $x \in E$ defines a seminorm p which is continuous by (*2*). Therefore $\{f_n\}_n$ is equicontinuous, and (*2*) $\Rightarrow$ (*3*).

(*3*) $\Rightarrow$ (*1*): Suppose (*3*) holds and M is a closed subspace with $\aleph_0$-dimensional algebraic complement N. Let V be an absolutely convex absorbing set in N. To complete the proof, we show that $M + V$ is a neighborhood of the origin in E. Standard procedure yields a biorthogonal sequence $\{x_n, f_n\}_n \subset E \times E'$ such that $\{x_n\}_n$ is a Hamel basis for N and each f_n vanishes on M. Rescaling and renaming $\{x_n, f_n\}_n$ if necessary, we may assume that $2^n x_n \in V$ for each n. By (*3*), $\{f_n\}_n$ is equicontinuous. Its polar in E is a 0-neighborhood contained in $M + V$ via the absolute convexity of V. $\square$

Rudimentarily, a linear form is continuous if and only if it vanishes on some closed finite-codimensional subspace. To characterize primitive spaces, replace "finite-codimensional" with "countable-codimensional".

Theorem 2.4. *The following three statements about a space E are equivalent.*

(*1*) *E is primitive.*

(*2*) *A linear form f is continuous if it vanishes on some closed countable-codimensional subspace F.*

(*3*) *Each pointwise eventually zero sequence $\{f_n\}_n$ in E' is $\sigma(E', E)$-summable.*

Proof. (*1*) $\Rightarrow$ (*2*). Given f and F as in (*2*), let $\{L_n\}_n$ be an increasing sequence of finite-dimensional subspaces whose union is an algebraic complement of F in E. Since each $f|_{F+L_n}$ is continuous, so is f, by virtue of (*1*).

(*2*) $\Rightarrow$ (*3*). Set $E_n = \bigcap_{k \geq n} f_k^{\perp}$. Now E is covered by $\{E_n\}_n$ and $\dim E_n/E_1 < n$, so $\dim E/E_1 \leq \aleph_0$. Since the pointwise sum of the series $\sum_n f_n$ vanishes on E_1, it is continuous by (*2*).

(*3*) $\Rightarrow$ (*1*). Suppose $\{H_n\}_n$ is an absorbing sequence of subspaces of E and $f \in E^*$ is given such that each $f|_{H_n}$ is continuous. Some f_n in E' agrees with f on H_n. Clearly, $\{f_n - f_{n+1}\}_n$ is pointwise eventually zero, and by (*3*), the pointwise sum $\sum_n (f_n - f_{n+1}) = f_1 - f$ is continuous; hence so is f. □

Characterizations of property (L), old and new (cf. [36, Proposition 2.2]):

Theorem 2.5. *The following six assertions about a space $(E, \mathcal{T})$ are equivalent.*

(*1*) *E has property (L).*

(*2*) *If $\{U_n\}_n$ is a sequence of absolutely convex closed 0-neighborhoods whose intersection $\bigcap_n U_n$ is absorbing, then $\bigcap_n nU_n$ is a neighborhood of 0.*

(*3*) *If $\{p_n\}_n$ is a pointwise bounded sequence in $\mathfrak{p}(E)$, then $\{n^{-1}p_n\}_n$ is equicontinuous.*

(*4*) *If $\{p_n\}_n$ is a pointwise bounded sequence in $\mathfrak{p}(E)$ and $\{a_n\}_n$ is a null scalar sequence, then $\{|a_n| p_n\}_n$ is equicontinuous.*

(*5*) *If $\{p_n\}_n$ is a pointwise bounded sequence in $\mathfrak{p}(E)$ and $\{\lambda_n\}_n$ is an absolutely summable scalar sequence, then the pointwise sum p of the series $\sum_n |\lambda_n| p_n$ is continuous.*

(*6*) *$\mathcal{T} = \mathcal{T}_A$ for each $\mathcal{T}$-barrel A.*

Proof. We have [(*1*) $\Leftrightarrow$ (*5*) $\Leftrightarrow$ (*6*)] from [51, 2.2] and its proof; [(*2*) $\Leftrightarrow$ (*3*)] via gauges; [(*4*) $\Rightarrow$ (*3*)] is trivial. So it suffices to show that [(*5*) $\Rightarrow$ (*4*)] and [(*3*) $\Rightarrow$ (*5*)].

(*5*) $\Rightarrow$ (*4*). Let $\{p_n\}_n$ and $\{a_n\}_n$ be given as in (*4*). Choose a sequence $n_1 < n_2 < \cdots$ of natural numbers such that, for all $k \in \mathbb{N}$,

$$|a_i| \leq 2^{-k} \text{ for all } i > n_k.$$

If we define each q_k so that $q_k(x) = \max\{p_i(x) : n_k < i \leq n_{k+1}\}$ for each $x \in E$, then $\{q_k\}_k$ is a pointwise bounded sequence of continuous seminorms. By

(*5*), the pointwise sum q of $\sum_k 2^{-k} q_k$ is continuous. By construction, q majorizes $|a_i|\, p_i$ for $i > n_1$, therefore $\{|a_i|\, p_i\}_{i>n_1}$ is equicontinuous, and thus so is $\{|a_i|\, p_i\}_{i\geq 1}$.

(*3*) $\Rightarrow$ (*5*). Let $\{p_n\}_n$ and $\{\lambda_n\}_n$ be given as in (*5*). Choose $n_1 < n_2 < \cdots$ such that $k2^k \sum_{j>n_k} |\lambda_j| \leq 1$ for each k, and define $q_k = \max\{p_j : n_k < j \leq n_{k+1}\}$ as before. Since $\left\{\left(k2^k \sum_{j>n_k} |\lambda_j|\right) \cdot q_k\right\}_k$ is pointwise bounded, (*3*) ensures equicontinuity of $\left\{\left(2^k \sum_{j>n_k} |\lambda_j|\right) \cdot q_k\right\}_k$, so that the pointwise sum of the series

$$\sum_k 2^{-k} \left(2^k \sum_{n_k<j\leq n_{k+1}} |\lambda_j|\right) \cdot q_k = \sum_k \sum_{n_k<j\leq n_{k+1}} |\lambda_j|\, q_k$$

is a continuous seminorm which majorizes

$$\sum_k \sum_{n_k<j\leq n_{k+1}} |\lambda_j|\, p_j = \sum_{j>n_1} |\lambda_j|\, p_j.$$

The pointwise sum ρ of the latter series, then, is continuous, which yields continuity of the finite sum $p = \rho + \sum_{1\leq i\leq n_1} |\lambda_i|\, p_i$. $\square$

Selected *dlc* characterizations from [51, 2.3, 2.10] and [34] constitute

Theorem 2.6. *The following six statements about a space E are equivalent.*

(*1*) *E is dlc.*

(*2*) *If $\{f_n\}_n$ is a $\sigma(E', E)$-bounded sequence and $\{\lambda_n\}_n$ is an absolutely summable scalar sequence, then the series $\sum_n \lambda_n f_n$ is $\sigma(E', E)$-convergent.*

(*3*) *If $\{A_n\}_n$ is an absorbing sequence in a countable-codimensional subspace F and each A_n is closed in E, then F is closed.*

(*4*) *If an absolutely convex set A is closed in E with countable-codimensional span F, then F is closed.*

(*5*) *A linear form is continuous if it vanishes on some set A given as in* (4).

(*6*) (Ruess [34]; cf. [28]) *E has property (LC).*

The vertical implication arrows in column 5 are now obvious. For row 5, we prove

Theorem 2.7. *Non-S_σ $\Rightarrow$ inductive $\Rightarrow$ φ-complemental $\Rightarrow$ primitive.*

Proof. *Non-S_σ $\Rightarrow$ inductive.* Let $\{p_n\}_n$ be a pointwise eventually zero sequence in $\mathfrak{p}(E)$. The increasing sequence $\left\{\bigcap_{k\geq n} p_k^{-1}(0)\right\}_n$ of closed subspaces covers E. If E is non-S_σ, then $\bigcap_{k\geq m} p_k^{-1}(0) = E$ for some m, which means that $\{p_n\}_n$ has at most m distinct members, and hence must be equicontinuous.

Inductive $\Rightarrow$ φ-complemental. If $\{f_n\}_n$ is pointwise eventually zero in E', so is $\{|f_n|\}_n$ in $\mathfrak{p}(E)$.

φ-complemental $\Rightarrow$ primitive. If E is φ-complemental and $\{f_n\}_n$ is pointwise eventually zero in E', so is $\{2^n f_n\}_n$, and the latter is equicontinuous. Hence $\sum_n f_n = \sum_n 2^{-n} (2^n f_n)$ is $\sigma(E', E)$-convergent. □

We see that a space E is $\aleph_0$-barrelled, is C-barrelled, has property (L), or is inductive, respectively, if and only if each sequence of continuous seminorms is equicontinuous when it (i) is pointwise bounded, (ii) is pointwise null, (iii) is the product of a pointwise bounded sequence with a null scalar sequence, or (iv) is pointwise eventually zero. Apparently, these four conditions on E are progressively weaker; e.g., if $\{p_n\}_n$ satisfies (iv), then $\{p_n\}_n = \{n^{-1}(np_n)\}_n$ also satisfies (iii). The vertical implication arrows of column 3 are therefore equally apparent: $\aleph_0$-barrelled $\Rightarrow$ C-barrelled $\Rightarrow$ property (L) $\Rightarrow$ inductive. To similarly command column 4 and its two new entries, we simply replace "seminorms" with "linear forms" in $(i$-$iv)$. Indeed, the vertical implication arrows in columns 3 and 4 are clearly valid.

We prove the implications in row 4:

Theorem 2.8. *Property (L) $\Rightarrow$ mc_0-barrelled $\Rightarrow$ dlc.*

Proof. *Property (L) $\Rightarrow$ mc_0-barrelled.* Suppose $\{f_n\}_n \subset E'$ and $\{a_n\}_n$ are as in the definition of mc_0-barrelled. If E has property (L), then $\{|a_n|\,|f_n|\}_n$ is equicontinuous, and thus so is $\{a_n f_n\}_n$.

mc_0-barrelled $\Rightarrow$ dlc. Let $\{f_n\}_n$ be a $\sigma(E', E)$-bounded sequence, with $\{\lambda_n\}_n \in \ell^1$. Choose a sequence $n_1 < n_2 < \cdots$ of natural numbers such that, for all $k \in \mathbb{N}$,

$$\sum\nolimits_{n_k < i \leq n_{k+1}} |\lambda_i| \leq k^{-1} 2^{-k}.$$

Now $\left\{k 2^k \sum_{n_k < i \leq n_{k+1}} \lambda_i f_i\right\}_k$ is $\sigma(E', E)$-bounded; if E is mc_0-barrelled, then $\left\{2^k \sum_{n_k < i \leq n_{k+1}} \lambda_i f_i : k \in \mathbb{N}\right\}$ is equicontinuous, and therefore

$$\sum\nolimits_{i > n_1} \lambda_i f_i = \sum\nolimits_{k \geq 1} 2^{-k} \left(2^k \sum\nolimits_{n_k < i \leq n_{k+1}} \lambda_i f_i\right)$$

is $\sigma(E', E)$-convergent. Thus so is $\sum_i \lambda_i f_i = \sum_{i \leq n_1} \lambda_i f_i + \sum_{i > n_1} \lambda_i f_i$. □

We conclude that all the implication arrows in Fig. 1.3 are valid.

Theorems 2.4 and 2.6 symmetrically shorten proofs of the final two basic facts.

Theorem 2.9. *Every countable-codimensional subspace F of a primitive space E is primitive.*

Proof. Let G be a closed countable-codimensional subspace of F and suppose some f in F^* vanishes on G. Choose h in E^* which agrees with f on F and vanishes on the closure $\overline{G}$ in E. Since E is primitive, $h \in E'$; thus $f = h|_F \in F'$. □

Theorem 2.10 (Valdivia [67]). *Every countable-codimensional subspace F of a dlc space E is dlc.*

Proof. Let A be an absolutely convex closed subset of F with $\dim F/\operatorname{sp} A \leq \aleph_0$ and suppose some f in F^* vanishes on A. Choose h in E^* which agrees with f on F and vanishes on the closure $\overline{A}$ in E. Since E is dlc, $h \in E'$; thus $f \in F'$. □

3. Density and the exceptional fifth column

Throughout this section, assume F is a dense subspace of a space E. A subspace G is *between F and E* if $F \subset G \subset E$; both F and E qualify.

At a glance, Fig. 1.3 improves Valdivia's (8-10) in [68, pp. 40, 41]. His previous six results (2-7) take F to be either barrelled, or Mackey and dlc, or, most generally, mc_0-barrelled. We uniformly relax the hypotheses to the assumption that every subspace between F and E be dlc. But precisely when does every subspace between F and E possess a given property?

Theorem 3.1. *If F enjoys a given property not in the fifth column, so does E.*

Proof. The non-S_σ case is obvious. For the other eleven: The closure in E of a 0-neighborhood in F is a 0-neighborhood in E. □

If F is mc_0-barrelled, every subspace between F and E is dlc (in fact, is mc_0-barrelled), so the latter condition does, indeed, relax the former. Note that any proper (LB)-space with a dense step [48] shows that sBL is not generally passed from F to E.

The fifth-column exceptions to Theorem 3.1 are plentiful, constrained by little more than Theorem 3.1 itself and boxes 2 and 4.

Theorem 3.2. *Some non-primitive spaces contain dense primitive hyperplanes (necessarily non-Mackey and S_σ by box 4 and Theorem 3.1). In fact, any primitive S_σ space F furnished with a suitable compatible topology is a dense hyperplane in some non-primitive Mackey space (E,τ). Compatibility ensures (F,τ) is primitive.*

Proof. By Theorems 2.1 and 2.4, there is a linearly independent pointwise eventually zero sequence $\{f_n\}_n$ in F', and it is $\sigma(F',F)$-summable to some $f = \sum_n f_n$ in F'. We assume that f is not in the span of $\{f_n\}_n$; if, initially, f is a linear combination of the first m of the f_n's, then we replace $\{f_n\}_n$ with $\{f_n\}_{n>m+1}$ and note that $\sum_{n>m+1} f_n$ is a non-trivial linear combination of $f_1, \ldots, f_{m+1}$ and therefore not in the span of $\{f_n\}_{n>m+1}$. Let H be a hyperplane of F' which contains $\{f_n\}_n$ but not f. Then H is necessarily a $\sigma(F',F)$-dense hyperplane of F', and separates points of F. Now let E be a vector space that contains F as a 1-codimensional subspace, and fix $x \in E \setminus F$. Let $H^- = \{g \in E^* : g(x) = 0 \text{ and } g|_F \in H\}$ and define $f^- \in E^*$ so that $f^-(x) = 1$ and $f^-|_F = f$. We define E' to be the span of H^- and f^-. It is easily seen that E' separates points of E, and E with the Mackey topology $\mu(E, E')$ contains F as a dense hyperplane, since 0 is the only member of E' that vanishes on F. It is also clear that E induces a compatible topology on F, since the set of restrictions to F of members of E' coincides with the original F'. Thus, by duality invariance, F is a primitive hyperplane of E. If f_n^- denotes the extension of f_n that vanishes at x, then by definition $\{f_n^-\}_n \subset E'$, and the sequence is pointwise eventually zero, but the pointwise sum $\sum_n f_n^-$ is not continuous, since it agrees with f^- on the dense subspace F but not at x. Therefore E is not primitive. □

So any proper strict (LB)-space F with a suitably weaker compatible topology is a dense hyperplane of some non-primitive Mackey space E, and F has the duality invariant property (C): None of the fifth-column properties are generally transmitted from F to E.

Theorem 3.3. *Some non-dlc Mackey spaces contain dense non-Mackey dlc hyperplanes. In fact, any non-BL dlc space F furnished with an appropriate compatible topology is a dense dlc hyperplane in some non-dlc Mackey space (E, τ). Clearly, (F, τ) cannot be Mackey* (by box 2 and Theorem 3.1).

Proof. There is a linearly independent $\sigma(F, F')$-bounded sequence $\{f_n\}_n$ in F'. Otherwise, F has its weak topology and is barrelled. The weak topology means F cannot contain a copy of φ, hence F is BL by [39, Theorem 2.1], a contradiction. The pointwise limit f of $\sum_k 2^{-k} f_k$ is in F', and, as before, we may assume that f is not in the span of $\{f_n\}_n$. Let H be any hyperplane of F' which contains $\{f_n\}_n$ but does not contain f. Then H is necessarily a $\sigma(F', F)$-dense hyperplane of F'. We define E, x, H^-, f^-, E', and f_n^- as before, and note that $\sum_n 2^{-n} f_n^-$ is not continuous on E. Thus E is not dlc and the conclusion holds. □

Note from the proof that merely requiring F to be non-barrelled under its weak topology would maximally relax the *non-BL* hypothesis on F.

Valdivia (see his proof of [68, (1), p. 38]) likely knew the purely algebraic

Lemma 3.4. *Let $\{A_n\}_n$ be an increasing sequence of convex sets in a vector space E with $0 \in A_1$. There exists an increasing sequence $\{B_n\}_n$ of balanced convex sets such that each $B_n \subset A_n$ and $\bigcup_n B_n$ absorbs the same points of E as does $\bigcup_n A_n$.*

Proof. We assume the scalar field $\mathbb{F} = \mathbb{C}$; the real case is easier. Simply let B_n be the balanced convex hull of

$$C_n := \frac{1}{2}\left[A_n \bigcap (-A_n) \bigcap (iA_n) \bigcap (-iA_n)\right].$$

If n is given and $x_1, \dots, x_k \in C_n$, and $\lambda_1, \dots, \lambda_k$ are scalars with $\sum_{j=1}^k |\lambda_j| \le 1$, then each $\pm 2x_j, \pm 2ix_j \in A_n$ and $1 - \left(\frac{1}{2}\sum_{j=1}^k |\mathrm{Re}(\lambda_j)| + \frac{1}{2}\sum_{j=1}^k |\mathrm{Im}(\lambda_j)|\right) =: b \ge 0$. Thus each point

$$\sum_{j=1}^k \lambda_j x_j = b \cdot 0 + \frac{1}{2}\sum_{j=1}^k |\mathrm{Re}(\lambda_j)|(\pm 2x_j) + \frac{1}{2}\sum_{j=1}^k |\mathrm{Im}(\lambda_j)|(\pm 2ix_j)$$

in B_n is also in the convex set A_n; i.e., $B_n \subset A_n$.

Now suppose x is a point absorbed by $\bigcup_n A_n$. Since $\{A_n\}_n$ is increasing, there exists $\epsilon > 0$ and $n \in \mathbb{N}$ such that $\pm\epsilon x, \pm\epsilon i x \in A_n$. Therefore $\frac{\epsilon}{2}x \in C_n$, so that $\lambda x \in B_n$ for all $|\lambda| \le \epsilon/2$; i.e., B_n absorbs x. □

Theorem 3.5. *The following three statements are equivalent.*

(i) *Every $\sigma(E', F)$-bounded set is $\sigma(E', E)$-bounded.*
(ii) *For each increasing sequence $\{A_n\}_n$ of convex sets covering F, the sequence $\{n\overline{A}_n\}_n$ covers E.*
(iii) *For each increasing sequence $\{B_n\}_n$ of balanced convex sets covering F, the sequence $\{n\overline{B}_n\}_n$ covers E.*

Proof. (The bipolar theorem)
Lemma 3.4 yields (i) ⇔ (iii) and (ii) ⇔ (iii). □

A locally convex space is *Baire-like* if it is not the union of an increasing sequence of balanced convex nowhere dense sets [39]. Locally convex Baire spaces are defined by omitting “balanced convex” ; they are characterized by omitting “convex” and retaining “balanced” [40]. If we omit “balanced” and retain “convex”, we characterize Baire-like spaces, by virtue of the Lemma. This means, for example, that Valdivia’s (4) [68, p. 39] simply says *If F is barrelled and E is Baire, then F is Baire-like.*

In our terminology, Valdivia’s (2, 3) [68, p. 39] say *The (equivalent) conditions* (i) *and* (ii) *of Theorem* 3.5 *hold if F is mc_0-barrelled.* Levin-Saxon [24, Lemma] used a non-comparable hypothesis. Both hypotheses relax to a characterization in Theorem 3.14. Valdivia’s (6) is a special case of his (3); see box 2. His (7): *If F is mc_0-barrelled and E is barrelled, then F is barrelled*, an obvious corollary to Theorem 3.15. His (5) cites Amemiya-Kōmura [1], our Corollary 3.21. Indeed, all of (2-7) is corollary to Theorems 3.14 and 3.15 in our ensuing study of the fifth column.

Theorem 3.6. *Every subspace between F and E enjoys property (C) if and only if E has property (C) and every $\sigma(E', F)$-bounded set is $\sigma(E', E)$-bounded.*

Proof. Suppose every subspace between F and E enjoys property (C). Then E has property (C), trivially. If B is a subset of E' that is bounded at the points of F but not at some point x in E, then there is a sequence $\{f_n\}_n \subset B$ with $|f_n(x)| > n$ for each $n \in \mathbb{N}$. Set $G = F + \mathrm{sp}\{x\}$ and $h_n = f_n/f_n(x)$. Since G

has property (C), there is a (unique) $\sigma(G', G)$-cluster point h of $\{h_n|_G\}_n$ which clearly vanishes on F and is 1 at x, contradicting density of F. Therefore the boundedness condition holds.

Conversely, suppose E has property (C) and the boundedness condition holds. If G is between F and E and A is a $\sigma(G', G)$-bounded set, then the set A^- of unique continuous linear extensions in E' is $\sigma(E', F)$-bounded, hence $\sigma(E', E)$-bounded, and has a $\sigma(E', E)$-cluster point f. Clearly, $f|_G$ is a $\sigma(G', G)$-cluster point of A; i.e., G has property (C). □

In Theorem 3.6 we cannot substitute F for E in "*E has property (C)*" (Examples 6.16, 6.19), but we can, and do, in the *property (S)*, *dlc*, and *primitive* analogs.

Theorem 3.7. *Assume property (S) is enjoyed either* (*a*) *by F, or* (*b*) *by E. The following two statements are equivalent.*

(1) *Every subspace between F and E enjoys property (S).*
(2) *Every $\sigma(E', F)$-null sequence is $\sigma(E', E)$-null.*

Proof. (*2*) ⇒ (*1*). Let G be a subspace between F and E and let $\{f_n\}_n$ be a $\sigma(E', G)$-Cauchy sequence in E'. We claim $\{f_n\}_n$ must be $\sigma(E', E)$-Cauchy; if not, then for some $x \in E \backslash G$ and subsequence $\{g_n\}_n$ of $\{f_n\}_n$ there is an $\epsilon > 0$ such that $|g_n(x) - g_{n+1}(x)| \geq \epsilon$ for all $n \in \mathbb{N}$, so that $\{g_n - g_{n+1}\}_n$ contradicts (*2*). Let f be the pointwise limit of $\{f_n\}_n$ in E^*. To show $f|_G$ is continuous, we show f is continuous; it clearly is in case (*b*). Case (*a*) assumes $f|_F$ is continuous. Let g be the unique member of E' which agrees with f on F. Since $\{g - f_n\}_n$ is $\sigma(E', E)$-null by (*2*), we have $g = f$, as desired.

¬(*2*) ⇒ ¬(*1*). Assume some $\sigma(E', F)$-null sequence $\{f_n\}_n$ is not $\sigma(E', E)$-null. Then for some $x \in E \backslash F$ and subsequence $\{h_n\}_n$ we have $|h_n(x)| \geq 1$ for all $n \in \mathbb{N}$. Set $g_n = h_n / h_n(x)$ and note that $G := F + \operatorname{sp}\{x\}$ is a subspace between F and E that does not have property (S), since the pointwise limit of $\{g_n|_G\}_n$ is 1 at x and vanishes on the dense F. □

As mentioned earlier, *c_0-barrelled $\not\Rightarrow$ property (S)*, a third-row anomaly that occurs when and only when the completion lacks property (S).

Theorem 3.8. *When F is c_0-barrelled, F has property (S) if and only if E has.*

Proof. Two simple observations suffice: (1) A space G is c_0-barrelled and has property (S) if and only if every $\sigma(G', G)$-Cauchy sequence is equicontinuous; and (2) A sequence $\{f_n\}_n \subset E'$ is equicontinuous and $\sigma(E', E)$-Cauchy if and only if $\{f_n|_F\}_n$ is equicontinous and $\sigma(F', F)$-Cauchy. □

Theorem 3.9. *If F is c_0-barrelled and E is C-barrelled, then so is F.*

Proof. Let $\{p_n\}_n$ be a pointwise null sequence in $\mathfrak{p}(F)$. If the unique sequence $\{p_n^-\}_n$ of extensions in $\mathfrak{p}(E)$ is not pointwise null, then there is a subsequence $\{q_n\}_n$ and a point $x \in E$ such that $q_n(x) \geq 1$ for all $n \in \mathbb{N}$. The Hahn-Banach theorem ensures a sequence $\{f_n\}_n$ in E' with $|f_n(x)| \geq 1$ and $\{f_n|_F\}_n$ pointwise null. By hypothesis, $\{f_n|_F\}_n$ is equicontinuous. Hence $\{f_n\}_n$ is equicontinuous and has a $\sigma(E', E)$-cluster point f. Necessarily, $|f(x)| \geq 1$ and $f(F) = \{0\}$, contradicting density. Therefore $\{p_n^-\}_n$ is pointwise null, and then equicontinuous by hypothesis. Thus $\{p_n\}_n$ is equicontinuous. $\square$

Corollary 3.10. *If the subspaces between F and E share a third-row property, then they share every third-row property that E enjoys.*

Proof. There are three cases. When the given shared property is

- *C-barrelled* then the subspaces also share *c_0-barrelled*, trivially, and also share *property (S)* when E does, by Theorem 3.8;
- *c_0-barrelled* then the subspaces also share [*property (S)*]⟨*C-barrelled*⟩ if E does, by [Theorem 3.8]⟨Theorem 3.9⟩;
- *property (S)* then the subspaces also share [*c_0-barrelled*]⟨*C-barrelled*⟩ if E does, by [Theorem 3.7]⟨the two previous half-cases⟩. $\square$

Theorem 3.11. *Assume either* (a) *F is primitive, or* (b) *E is primitive. The following two statements are equivalent.*

(*1*) *Every subspace between F and E is primitive.*
(*2*) *If a sequence in E' is eventually zero at the points of F, then it is eventually zero at the points of E.*

Proof. (*2*) $\Rightarrow$ (*1*). Let G be a subspace between F and E and let $\{f_n\}_n$ be a sequence in E' that is eventually zero at the points of G, hence at the points of E, by (*2*). Let f be the pointwise sum in E^* of the series $\sum_n f_n$. To show that $f|_G$ is continuous, we show that f is continuous; the result is clear in case (b). Case (a) assumes $f|_F$ is continuous. Let g be the unique member of E' which agrees with f on F. Since the sequence of partial sums is pointwise eventually constant, the sequence $\left\{g - \sum_{n \leq k} f_n\right\}_k$ is eventually zero at points of F; thus, by (*2*), is eventually zero at points of E, so that g and f agree on all of E, as desired.

$\neg$(*2*) $\Rightarrow \neg$(*1*). If $\{f_n\}_n$ is a sequence in E' which is eventually zero at points of F but not at some point $y \in E$, then $H := \bigcap_n \left(f_n^\perp \bigcap F\right)$ is a closed countable-codimensional subspace of $G := F + \operatorname{sp}\{y\}$. The linear form $h \in G^*$ which is 1 at y and 0 on F is not continuous, even though it vanishes on H. Therefore G, a subspace between F and E, is not primitive by Theorem 2.4(*2*). $\square$

Fifth-row Theorems 2.1–2.3 and 3.11 quickly yield

Theorem 3.12. *If every subspace between F and E is primitive and E is non-S_σ, inductive, or φ-complemental, then so, respectively, is F.*

Proof. By Theorem 3.11 and the Hahn-Banach theorem, each pointwise eventually zero sequence S in F' or $\mathfrak{p}(F)$ is uniquely extended to a pointwise eventually zero sequence S^- in E' or $\mathfrak{p}(E)$, respectively, and S is equicontinuous if S^- is. Also, for $S \subset F'$, linear dependence of S^- implies linear dependence of S. □

Corollary 3.13. *If the subspaces between F and E share a fifth-row property, then they share every fifth-row property that E enjoys.*

The conclusion must remain in the fifth row: Metrizable non-barrelled primitive spaces F exist. Yet if we let E be the completion $\hat{F}$ of F, the spaces between F and E have all the fifth-row properties, and the Fréchet space E has all sixteen properties, but F has none above the fifth row (see [9], [53], Example 6.4).

At the fourth row we have

Theorem 3.14. *Assume either (a) F is dlc, or (b) E is dlc. The following two statements are equivalent.*

(*1*) *Every subspace between F and E is dlc.*
(*2*) *Every $\sigma(E', F)$-bounded set is $\sigma(E', E)$-bounded.*

Proof. We argue case (a) only.

(*2*) $\Rightarrow$ (*1*). It suffices to prove E itself is dlc. Suppose $\{f_k\}_k \subset E'$ with finite $M_x := \sup_k |f_k(x)|$ for each $x \in E$ and $\{\lambda_n\}_n$ is an absolutely summable scalar sequence. We must show that the pointwise sum f of the series $\sum_k \lambda_k f_k$ is continuous. Since F is dlc, there is a unique g in E' which agrees with f on F. Choose $m_1 < m_2 < \cdots$ such that $\sum_{k>m_n} |\lambda_k| \le 1/n$ for each $n \in \mathbb{N}$, and set $h_n = g - \sum_{k \le m_n} \lambda_k f_k$. For each $x \in F$ and for all n,

$$|h_n(x)| = \left| \sum\nolimits_{k>m_n} \lambda_k f_k(x) \right| \le M_x/n.$$

Therefore $\{nh_n\}_n$ is $\sigma(E', F)$-bounded and, by (*2*), $\sigma(E', E)$-bounded. It follows that $\{h_n\}_n$ is $\sigma(E', E)$-null, which implies $g = f$ is continuous.

$\neg$(*2*) $\Rightarrow \neg$(*1*). Let B be a $\sigma(E', F)$-bounded set which is not bounded at some point $y \in E$. Then $\operatorname{sp} B^\circ$ contains F and is a dense hyperplane of $\operatorname{sp} B^\circ + \operatorname{sp}\{y\}$. The latter subspace is not dlc by Theorem 2.6(*4*) and denies (*1*). □

Again, every metrizable non-barrelled primitive space shows that, in the next Theorem, the *dlc* hypothesis is optimal and cannot be replaced by any fifth-row property.

Theorem 3.15. *Suppose every subspace between F and E is dlc. If E is sBL or enjoys a given property not in the third row, then so is or does F, respectively.*

Proof. If $\{B_n\}_n$ is an absorbing sequence in F, so is $\{\overline{B}_n\}_n$ in E, by Theorems 3.5 and 3.14. The *sBL*, *BL*, *QB*, *non-S_σ* and *barrelled* cases follow immediately.

The *dlc* and *primitive* cases are trivial, and Theorem 3.12 covers the rest of the fifth row.

Theorem 3.14 surely suffices for the *property (C)*, ℓ^∞- and mc_0-*barrelled* cases, and with the Hahn-Banach theorem, implies that $\{p_n\}_n \subset \mathfrak{p}(E)$ is bounded at the points of E if bounded at the points of F, which suffices for the $\aleph_0$-*barrelled* and *property (L)* cases. □

Remark 3.16. Theorem 3.15 must exclude the entire third row.

Proof. Examples 6.16 and 6.18 present a space with all the third-row properties and a dense subspace with precisely the properties of rows 4 and 5. □

A Fig. 1.1 result of ours [55, Theorem 9] relies on

Corollary 3.17. *If every subspace between F and E has a given property in the right column of Fig.* 1.1 *and E has the stronger property immediately to the left, then so does F.*

The result holds *a fortiori* if we replace the columns of Fig. 1.1 with the third and forth columns of Fig. 1.3. Or, one may substitute the fourth and fifth columns:

Corollary 3.18. *If E has a given property in column 4 of Fig.* 1.3 *and every subspace between F and E has the property to the right in column 5, then F also has the given property.*

Corollary 3.10 extends well beyond the third row:

Theorem 3.19. *If all subspaces between F and E share a third-row property, then they are all sBL if E is, and share each of the sixteen properties that E may enjoy, no exceptions.*

Proof. Each third-row property implies *dlc*, so Theorem 3.15 applies in case E is sBL or enjoys properties outside row 3; Corollary 3.10 covers row 3. □

Corollary 3.20. *If F is c_0-barrelled, then F shares each of the sixteen properties enjoyed by its completion $\hat{F}$.*

Corollary 3.21 (Amemiya-Kōmura). *Metrizable barrelled spaces are Baire-like.*

4. Preserving properties

First- and second-column properties are not generally preserved by inductive limits, as any proper strict (LB)-space shows. We suspect, as the Novák example [29] suggests, that *property (C)* is not preserved even by finite products. For the remaining twelve properties we have

Theorem 4.1. *Let E be the inductive limit space of a system $\{(E_\iota, f_\iota) : \iota \in I\}$, where each f_ι is a linear map from space E_ι into E. If each E_ι enjoys a given property from columns 3-5 other than property (C), then so does E.*

Proof. Note that each f_ι is continuous, a seminorm p on E is continuous if each $p \circ f_\iota$ is continuous, and a sequence $\{p_n\}_n$ of seminorms is equicontinuous if each $\{p_n \circ f_\iota\}_n$ is equicontinuous. The same holds for linear forms on E. In all twelve cases (five in column 3, four in column 4, three in column 5) the seminorm/linear form proof is straightforward. We illustrate with two cases.

Suppose each E_ι has property (L). If $\{p_n\}_n$ is pointwise bounded in $\mathfrak{p}(E)$ and $\{a_n\}_n$ is a null scalar sequence, then each $\{|a_n|\, p_n \circ f_\iota\}_n$ is equicontinuous, hence $\{|a_n|\, p_n\}_n$ is equicontinuous, proving E has property (L).

If each E_ι has property (S), let $\{g_n\}_n$ be a pointwise Cauchy sequence in E'; let $g \in E^*$ be the pointwise limit of the sequence. Then each $g \circ f_\iota$ is the pointwise limit of the sequence $\{g_n \circ f_\iota\}_n$ in E'_ι, and so is continuous. Therefore g is continuous, and E has property (S). □

Theorem 4.2. *All sixteen properties except possibly* property (C) *are preserved by finite products.*

Proof. Theorem 4.1, [39, Theorem 2.9], and [47, Theorem 8 (and proof)]. □

Theorem 4.3. *All sixteen properties are preserved by (Hausdorff) quotients.*

Proof. Theorem 4.1 and (easy) proofs in [39, 47] leave only *property (C)*, also easily dispatched: If B is a $\sigma(E', E)$-bounded set whose members vanish on a closed subspace M of E, then so does any $\sigma(E', E)$-cluster point of B. □

The following lemma is essentially known (cf. [8, 1.2.6]).

Lemma 4.4. *Let $E = \prod_{\iota \in I} E_\iota$ be a product space, and for each $S \subset I$, canonically identify the product space $\prod_{\iota \in S} E_\iota$ and each E_ι as subspaces of E.*

(*1*) *If $\mathcal{F}$ is a pointwise bounded family in either E' or $\mathfrak{p}(E)$, then there exists a finite subset J of the indexing set I such that each member of $\mathcal{F}$ vanishes on $\prod_{\iota \in I \setminus J} E_\iota$.*

(*2*) *If $\{A_n\}_n$ is a closed absorbing sequence in E, then there exist $n \in \mathbb{N}$ and a finite subset J of I such that $A_n \supset \prod_{\iota \in I \setminus J} E_\iota$.*

Proof. We may assume each $E_\iota \neq \{0\}$ and take $E' = \oplus_{\iota \in I} E'_\iota$.

(*1*). In the case of linear forms, *pointwise bounded* ensures $\mathcal{F} \subset \oplus_{\iota \in J} E'_\iota$ for some finite $J \subset I$, implying the conclusion. From this and the Hahn-Banach theorem, the seminorm case also follows.

(*2*). Assume the result false. Then $A_1 \not\supset \prod_{\iota \in I} E_\iota$ implies there exists $\iota_1 \in I$ such that $A_1 \not\supset E_{\iota_1}$, since A_1 is absolutely convex and closed. Choose $x_1 \in E_{\iota_1} \setminus A_1$. The bipolar theorem provides $f_1 \in A_1^\circ$ with $|f_1(x_1)| > 1$. Since $A_2 \not\supset \prod_{\iota \neq \iota_1} E_\iota$, we similarly find $\iota_2 \neq \iota_1$, $x_2 \in E_{\iota_2}$ and $f_2 \in A_2^\circ$ with $|f_2(x_2)| > 1$. The sequence $f_1, f_2, \ldots$ we thus obtain is $\sigma(E', E)$-bounded, since $\{A_n\}_n$ is absorbing, yet it does not uniformly vanish on $\prod_{\iota \in I \setminus J} E_\iota$ for any finite $J \subset I$, since the ι_k's are distinct, $x_k \in E_{\iota_k}$, and $f_k(x_k) \neq 0$. But this contradicts (*1*). □

Theorem 4.5. *All sixteen properties except possibly* property (C) *are preserved by arbitrary products.*

Proof. Reduce to finite products via the Lemma, then apply Theorem 4.2. □

A *three-space property* is one which every space E must have whenever both F and E/F have the property for some closed subspace F of E. Half of the original fourteen are three-space properties [54]. Note that *φ-complemental* is not a three-space property [54, Example 2.3], but *mc_0-barrelled* is:

Theorem 4.6. *Let F be a closed subspace of a space E.*

(*A*) *If F and E/F are both mc_0-barrelled, then so is E.*
(*B*) *If F is mc_0-barrelled and E/F is dlc, then E is also dlc.*
(*C*) *If F is φ-complemental and E/F is dlc, then E is primitive.*

Proof. (*A*) Let $\{f_n\}_n$ be a $\sigma(E', E)$-bounded sequence and $\{a_n\}_n$ a null scalar sequence. Choose null scalar sequences $\{b_n\}_n$ and $\{c_n\}_n$ such that each $a_n = b_n c_n$; e.g., take $b_n = \sqrt{|a_n|}$ and $c_n = \sqrt{|a_n|}\,\mathrm{sgnm}(a_n)$. Now $\{c_n f_n|_F\}_n$ is equicontinuous on the mc_0-barrelled space F, and the Hahn-Banach Theorem provides $\{h_n\}_n \subset E'$, equicontinuous on E, such that each $h_n|_F = c_n f_n|_F$. Thus $\{c_n f_n - h_n\}_n$ is $\sigma(E', E)$-bounded and each of its members vanishes on F. Since E/F is mc_0-barrelled, $\{b_n(c_n f_n - h_n)\}_n = \{a_n f_n - b_n h_n\}_n$ is equicontinuous. Since $\{h_n\}_n$ is equicontinuous, so is $\{b_n h_n\}_n$, and so is the sequential sum $\{a_n f_n\}_n$.

(*B*) Let $\{f_n\}_n$ be a $\sigma(E', E)$-bounded sequence and $\{\lambda_n\}_n$ an absolutely summable scalar sequence. We must show that the pointwise sum f of $\sum_n \lambda_n f_n$ is in E'. Choose an absolutely summable sequence $\{b_n\}_n$ and a null sequence $\{c_n\}_n$ such that $\lambda_n = b_n c_n$ for each n (see proofs of Theorems 2.5, 2.8). By hypothesis, $\{c_n f_n|_F\}_n$ is equicontinuous on F, and the Hahn-Banach Theorem provides $\{h_n\}_n \subset E'$, equicontinuous on E, such that each $h_n|_F = c_n f_n|_F$. Let h be the weak sum in E' of the series $\sum_n b_n h_n$. Now $\{c_n f_n - h_n\}_n$ is $\sigma(E', E)$-bounded and each of its members vanishes on F. Since E/F is dlc, the pointwise sum of $\sum_n b_n (c_n f_n - h_n) = \sum_n \lambda_n f_n - \sum_n b_n h_n = f - h$ is in E'. Thus $f \in E'$.

(C) If $\{f_n\}_n$ is a pointwise eventually zero sequence in E', so is $\{2^n f_n|_F\}_n$ in F', and the latter is equicontinuous by hypothesis and Theorem 2.3. The Hahn-Banach theorem provides $\{h_n\}_n \subset E'$, equicontinuous on E, with each $h_n|_F = 2^n f_n|_F$. Let h be the weak sum in E' of $\sum_n 2^{-n} h_n$, and let f be the pointwise sum in E^* of $\sum_n f_n$. Now $\{2^n f_n - h_n\}_n$ is $\sigma(E', E)$-bounded and each of its members vanishes on F. Since E/F is dlc, the pointwise sum of $\sum_n 2^{-n}(2^n f_n - h_n) = f - h$ is in E'. Therefore so is f, which implies E is primitive. □

Part (B) optimally improves both parts of [54, Theorem 2.2]. Indeed, we will see that parts $(A$-$C)$ and the next two Theorems are all optimal: In the sixteen property context, the hypotheses cannot be relaxed, the conclusions cannot be strengthened.

Theorem 4.7. *If F is a closed φ-complemental subspace of E and E/F is mc_0-barrelled, then E is φ-complemental.*

Proof. If $\{f_n\}_n$ is a pointwise eventually zero sequence in E', so is $\{nf_n\}_n$. Restrictions of the latter to F are equicontinuous by Theorem 2.3. The Hahn-Banach theorem ensures an equicontinuous sequence $\{h_n\}_n$ in E' such that each $nf_n - h_n$ vanishes on F. As the difference of two $\sigma(E', E)$-bounded sequences, $\{nf_n - h_n\}_n$ is also weakly bounded; the hypothesis on E/F implies that $\{f_n - h_n/n\}_n$ is equicontinuous. Thus so is $\{f_n\}_n$, the sum of two such sequences. □

Theorem 4.8. *If F is a closed non-S_σ subspace of E and E/F is φ-complemental, then E is φ-complemental.*

Proof. Let $\{f_n\}_n$ be a pointwise eventually zero sequence in E'. Then $\{f_n|_F\}_n$ is $\sigma(F', F)$-bounded and is finite-dimensional by Theorem 2.1. The Hahn-Banach theorem provides a finite-dimensional $\sigma(E', E)$-bounded sequence $\{h_n\}_n$ in E' such that each $f_n - h_n$ vanishes on F. Since $G := \bigcap_n h_n^\perp$ is finite-codimensional in E, so is $(G+F)/F$ in E/F. Hence $(G+F)/F$ is φ-complemental (Theorem 4.11, below). Now $\{(f_n - h_n)|_{G+F}\}_n$ vanishes on F and is eventually zero at points of $G+F$, thus is equicontinuous by Theorem 2.3. Because $\{f_n - h_n\}_n$ is $\sigma(E', E)$-bounded and equicontinuous on a finite-codimensional subspace, it is equicontinuous on all of E, as is $\{h_n\}_n$. Therefore $\{f_n\}_n$ is equicontinuous. □

Let the ordered triple (a, b, c) denote the following statement, which could be true or false: *If E is any space with closed subspace F such that F has property a and E/F has property b, then E has property c*, where a, b, c are members of the set of sixteen properties. We determine the truth or falsity of all $16^3 = 4096$ statements of the form (a, b, c). This we have already done in the $14^3 = 2744$ cases for which a, b, c are in the older fourteen-property set [54], leaving only $4096 - 2744 = 1352$ cases to consider where at least one of a, b, c is in the doubleton set $\{mc_0\text{-}barrelled, \varphi\text{-}complemental\}$.

Certain facts make the task manageable. Foremost is the integrity of Fig. 1.3. In Section 2 we saw that all its implication arrows are valid, and we prove later (Section 6) that a implies b only when indicated by Fig. 1.3. This visual aid is most useful in conjunction with Theorems 4.6-4.8 and the following facts:

($\mathfrak{T}$) If (a, b, c) is true, then $a, b \Rightarrow c$. (Principle 4.1 in [54]. Indeed, if $E_1 \times E_2$ has property c, then so must E_1 and E_2, by Theorem 4.3.)
($\mathfrak{F}$) For arbitrary values of b, c,
  (i) (*property(C)*$, b, c)$ is false [54, Example 2.2], and
  (ii) (*barrelled*, *non-*$S_\sigma, c)$ is false [54, Example 2.3].

We have marshalled enough material to decide all 1352 cases.

Theorem 4.9. *There are two broad categories.*

I. *The* mc_0-barrelled *cases:*
  ($\mathfrak{a}$) (mc_0-barrelled, b, c) *is true if and only if either*
    (i) $b \Rightarrow mc_0$-barrelled $\Rightarrow c$, *or*
    (ii) $b \Rightarrow dlc \Rightarrow c$;
  ($\mathfrak{b}$) (a, mc_0-barrelled, c) *is true if and only if either*
    (i) $a \Rightarrow mc_0$-barrelled $\Rightarrow c$, *or*
    (ii) $a \Rightarrow \varphi$-complemental $\Rightarrow c$;
  ($\mathfrak{c}$) (a, b, mc_0-barrelled) *is true if and only if* $a, b \Rightarrow mc_0$-barrelled.

II. *The* φ-complemental *cases:*
  ($\mathfrak{a}$) (φ-complemental, b, c) *is true if and only if either*
    (i) $b \Rightarrow dlc$ *and* $c =$ *primitive, or*
    (ii) $b \Rightarrow mc_0$-barrelled *and* φ-complemental $\Rightarrow c$;
  ($\mathfrak{b}$) (a, φ-complemental, c) *is true if and only if* φ-complemental $\Rightarrow c$ *and* $a \Rightarrow$ non-S_σ;
  ($\mathfrak{c}$) (a, b, φ-complemental) *is true if and only if either*
    (i) $a, b \Rightarrow mc_0$-barrelled, *or*
    (ii) $a \Rightarrow \varphi$*-complemental and* $b \Rightarrow mc_0$*-barrelled, or*
    (iii) $a \Rightarrow$ non-S_σ *and* $b \Rightarrow \varphi$-complemental.

Proof. I($\mathfrak{a}$): *Sufficiency* of *(i)* and *(ii)* is the import of Theorem 4.6(*A*) and (*B*), respectively. *Necessity*: Assume (mc_0*-barrelled*, b, c) is true for given b, c. Fact $\mathfrak{T}$ ensures mc_0*-barrelled* $\Rightarrow c$; i.e., c is in row 4 or 5 and in column 4 or 5. Since *barrelled* $\Rightarrow mc_0$*-barrelled*, our $\mathfrak{F}$(ii) ensures that *non-*$S_\sigma \not\Rightarrow b$; i.e., b is not in row 5. If b is in column 5, then $\mathfrak{T}$ ensures c is in column 5 and in row 4 or 5; i.e., *dlc* $\Rightarrow c$ and *(ii)* holds. Otherwise, b is in neither row 5 nor column 5, and *(i)* holds.

I($\mathfrak{b}$): *Sufficiency* of (i) and (ii) is the import of Theorems 4.6(A) and 4.7, respectively. *Necessity*: Assume $(a, mc_0\text{-}barrelled, c)$ is true for given a, c. By $\mathfrak{F}$(i), a is not in column 5; i.e., $a \Rightarrow$ *φ-complemental*. If c is in row 5, then $\mathfrak{T}$ ensures (ii) holds. If c is not in row 5, then neither is a, and (i) must hold by $\mathfrak{T}$.

I($\mathfrak{c}$): *Sufficiency* and *Necessity* are obvious from Theorem 4.6(A) and $\mathfrak{T}$.

II($\mathfrak{a}$): *Sufficiency* of (i) and (ii) is the import of Theorems 4.6(C) and 4.7, respectively. *Necessity*: Assume $(\varphi\text{-}complemental, b, c)$ is true for given b, c. Result $\mathfrak{F}$(ii) ensures b is not in row 5. Hence if b is in column 5, then $b \Rightarrow dlc$, and two applications of $\mathfrak{T}$ ensure that $c =$ primitive; i.e., (i) holds. On the other hand, if b is not in column 5, then, as we already know b is not in row 5, we have $b \Rightarrow$ *mc_0-barrelled*. Moreover, *φ-complemental* $\Rightarrow c$ by $\mathfrak{T}$; i.e., (ii) holds.

II($\mathfrak{b}$): *Sufficiency* and *Necessity* follow from Theorem 4.8, $\mathfrak{F}$(ii) and $\mathfrak{T}$.

II($\mathfrak{c}$): *Sufficiency* of (i), (ii) and (iii) follows from Theorems 4.6(A), 4.7 and 4.8, respectively. *Necessity*: Assume $(a, b, \varphi$-complemental$)$ is true for given a, b. Principle $\mathfrak{T}$ ensures that $a, b \Rightarrow$ *φ-complemental*; i.e., neither a nor b is in column 5. If neither is in row 5, then (i) holds. If b is in row 5, then $\mathfrak{F}$(ii) ensures $a \Rightarrow$ *non-S_σ*; i.e., (iii) holds. If a is in row 5 and b is not, then (ii) holds. □

Whereas c_0- and *C-barrelled* are not always preserved even by hyperplanes (Example 6.21), countable-codimensional subspaces F preserve the other fourteen properties. Two cases suffice: When F is closed, and when F is dense (since F is always dense in its closure). For F closed (Theorem 4.10), we consider the fifth-column properties, and then those to the left. For F dense (Theorem 4.11), we consider the fifth-row properties, then those above.

Theorem 4.10. *Closed countable-codimensional subspaces preserve all sixteen properties.*

Proof. Let F, G be algebraic complements in E, with F closed and $\dim G \leq \aleph_0$.

Case 1: E has a given property in column 5. Since E is primitive, sequences in F' can be extended to sequences in E' that vanish on G.

Case 2: E has a given property in columns 1–4. Since E is φ-complemental, F is a quotient of E, and Theorem 4.3 applies. □

Theorem 4.11. *Arbitrary countable-codimensional subspaces preserve all sixteen properties except c_0- and* C-barrelled.

Proof. Let F be a countable-codimensional subspace of E, where E has a given one of the properties, c_0- and *C-barrelled* excluded. By Theorem 4.10, we may assume that F is dense in E. Every subspace between F and E is primitive (Theorem 2.9) and shares every fifth-row property that E enjoys (Theorem 3.12). Similarly, Theorems 2.10 and 3.15 prove the remaining ten cases (those above row 5) except for *property (S)*, a case settled in [24, Theorem]: If E has property (S), Theorems 2.10 and 3.14 show that any $\sigma(E', F)$-Cauchy sequence is $\sigma(E', E)$-bounded and thus admits a $\sigma(E', E)$-Cauchy subsequence via diagonalization with respect to a (countable) cobasis for F. □

Garcia-Lafuente [10] gave the c_0-*barrelled* part of

Theorem 4.12. *Let E be respectively c_0- or C-barrelled. The following three assertions about E are equivalent.*

(*1*) *Every countable-codimensional subspace is respectively c_0- or C-barrelled.*
(*2*) *Every dense hyperplane is c_0-barrelled.*
(*3*) *E has property (S).*

Proof. (*1*) $\Rightarrow$ (*2*), trivially.

(*2*) $\Rightarrow$ (*3*). If there exists a $\sigma(E', E)$-Cauchy sequence $\{f_n\}_n$ whose pointwise limit f is discontinuous, then $F = f^{\perp}$ is a dense hyperplane in E. Since f is not continuous, $\{f_n\}_n$ is not equicontinuous, hence neither is $\{f_n|_F\}_n$. But $\{f_n|_F\}_n$ is $\sigma(F', F)$-null, so that F is not c_0-barrelled, which contradicts (*2*).

(*3*) $\Rightarrow$ (*1*). Let F be a countable-codimensional subspace of E. By (*3*) and Theorem 4.10, $\overline{F}$ has property (S) and is respectively c_0- or C-barrelled. By Theorem 4.11, every subspace between F and $\overline{F}$ has property (S). Corollary 3.10 implies F is respectively c_0- or C-barrelled; (*1*) holds. □

Kalton's closed graph theory [20] considers spaces E for which every $\sigma(E', E)$-Cauchy sequence is equicontinuous (see [26, 4.9.1]). This is easily equivalent to the condition that E be c_0-barrelled and have property (S). Theorem 4.12 ensures countable-codimensional subspaces preserve Kalton's two-part property, though separately preserving only one part. Indeed,

Corollary 4.13. *If E has property (S), then every countable-codimensional subspace of E preserves each property that E may enjoy from among the sixteen, with no exceptions.*

We have yet another (see [43]) short proof of

Corollary 4.14 (Valdivia). *If a Mackey space E is dlc and lacks property (S), then some dense hyperplane in E is nonMackey.*

Proof. Box 2 implies E is C-barrelled. Since E lacks property (S), there is a dense hyperplane F which is not C-barrelled. But F is dlc by Valdivia's Theorem 2.10, and therefore cannot be Mackey (box 2, again). □

The converse fails. For a Mackey space E, we do not know which of *barrelled* and $\aleph_0$-*barrelled* is the optimally weak property that ensures every dense hyperplane is Mackey, but it is certainly one of the two. Clearly, *barrelled* is sufficiently strong, since hyperplanes of barrelled spaces are barrelled, hence Mackey; and ℓ^∞-*barrelled* is too weak: Saxon [43, Theorem 3] provides a nice class of ℓ^∞-barrelled Mackey spaces in which *every* dense hyperplane is non-Mackey.

5. Density and countable enlargements

Let (E, ξ) be a space with dual E'. If H is an $\aleph_0$-dimensional subspace of E^* transverse to E' and η is the coarsest locally convex topology on E finer than ξ such that $(E, \eta)' = E' + H$, then we say (E, η) is a *countable enlargement* (CE) of (E, ξ). The original definition for barrelled spaces [63] is equivalent; see [55, 62]. The topology η satisfies the *Tweddle-Yeomans Criterion* if there is no linearly independent $\sigma(E' + H, E)$-bounded sequence transverse to E'.

The *CE problem* for a class of locally convex spaces is the question of whether, in the given class, every space which admits a CE must admit a CE that is also in the class. It is famously open for barrelled spaces (the BCE problem). It has a positive solution for metrizable barrelled spaces [42, 50] and for separable barrelled spaces [49, Cor. 3] (CEs always preserve metrizability and separability), for metrizable primitive spaces [55], for bornological spaces [62], and for quasibarrelled spaces that are not barrelled [62]. There are negative solutions for Mackey spaces [62, Ex. 4.1, 4.2], and for at least nine types of weak barrelledness [55].

However, we will provide positive solutions for all of *separable* weak barrelledness via dense subspaces. Our first three Theorems are the work of Saxon / Sánchez Ruiz / Tweddle (S/SR/T) [55, Theorems 5, 8, 15] extended to the larger context of Fig. 1.3. The third extension follows from the first two and the original.

Theorem 5.1 (S/SR/T). *When (E, ξ) enjoys a given fifth-row property, so does the CE (E, η) if and only if no linearly independent pointwise eventually zero sequence in $E' + H$ is transverse to E'. Thus when (E, ξ) is respectively non-S_σ, inductive, or φ-complemental, so is (E, η) if (and only if) (E, η) is primitive.*

Proof. The non-existence condition on sequences in $E' + H$ is a convenient form of the condition in [55, Theorem 8] that there exist no infinite-dimensional subset N of $E' + H$ transverse to E' such that $N^\perp$ is countable-codimensional in E. Thus the original [55] covers all but the φ-complemental case.

Necessity. Assume both ξ and η are primitive. If S is a pointwise eventually zero sequence in $E' + H$, then $S^\perp =: E_1$ is η-closed and countable-codimensional. Let E_2 and E_3 be subspaces such that, algebraically, $E = E_1 \oplus E_2 \oplus E_3$, with

$E_1 + E_2$ the ξ-closure of E_1. By η-primitivity, every linear form in E^* which vanishes on E_1 but not on all of E_2 is in $E' + H$ but not in E'. Thus if E_2 were infinite-dimensional, we would have

$$\aleph_0 = \dim\left(\left(E' + H\right)/E'\right) \geq \dim\left(\left(E_1 + E_2\right)/E_1\right)^* = \dim E_2^* \geq \mathfrak{c},$$

a contradiction. Hence $\infty > \dim E_2 = \dim E_2^* = \dim\left(E_1 + E_3\right)^{\perp}$. Trivially, $S \subset E_1^{\perp} \subset \left(E_1 + E_3\right)^{\perp} + \left(E_1 + E_2\right)^{\perp}$. By ξ-primitivity, $\left(E_1 + E_2\right)^{\perp} \subset E'$. If S is transverse to E', then $\dim S$ cannot exceed the finite $\dim\left(E_1 + E_3\right)^{\perp}$.

Sufficiency. Assume (E, ξ) is φ-complemental. Let $\{f_n\}_n \subset E'$ and $\{h_n\}_n \subset H$, with $\{f_n + h_n\}_n$ pointwise eventually zero. The condition ensures that $\{h_n\}_n$ is finite-dimensional, so $F := \bigcap_n h_n^{\perp}$ is a ξ-dense finite-codimensional subspace at whose points $\{f_n\}_n$ is eventually zero. Therefore $\{f_n\}_n$ is eventually zero at points of E (Theorems 2.9, 3.11), as is $\{h_n\}_n$, then, and $\{f_n\}_n$ must be ξ-, hence η-equicontinuous. Moreover, $\{h_n\}_n$ is η-equicontinuous, as is any finite-dimensional bounded subset of the weak dual. Consequently, $\{f_n + h_n\}_n$ is η-equicontinuous, and (E, η) is φ-complemental by Theorem 2.3. □

Theorem 5.2 (S/SR/T). *Suppose (E, ξ) has a given property in row 1-4 other than c_0- and* C-barrelled. *A CE (E, η) has the property if and only if η satisfies the Tweddle-Yeomans Criterion. Thus (E, η) has the (stronger) given property if and only if (E, η) is dlc.*

Proof. The original covers all but sufficiency for the case where (E, ξ) is mc_0-barrelled. Let $\{f_n\} \subset E'$ and $\{h_n\}_n \subset H$ with $\{f_n + h_n\}_n$ pointwise bounded, and let $\{a_n\}_n$ be a null scalar sequence. The condition ensures $\{h_n\}_n$ is finite-dimensional, so $F := \bigcap_n h_n^{\perp}$ is a ξ-dense finite-codimensional subspace on which $\{a_n f_n|_F\}_n$ is relatively ξ-equicontinuous (Theorem 4.11). By density, $\{a_n f_n\}_n$ is ξ-, hence η-equicontinuous. Also η-equicontinuous is the finite-dimensional bounded $\{a_n h_n\}_n$, and then so is $\{a_n\left(f_n + h_n\right)\}_n$. Thus (E, η) is mc_0-barrelled. □

Theorem 5.3 (S/SR/T). *If (E, ξ) has a given one of the sixteen properties and contains a dense $\mathfrak{c}$-codimensional subspace F such that all the subspaces between F and E also have the property, then some CE of (E, ξ) preserves the property.*

Using the last two sections and Saxon-Robertson's proof, one may generalize the generalization [61, Theorem] of the original [49, Theorem 1], and analogize:

Theorem 5.4 (Assume the Continuum Hypothesis). *Suppose a space E contains a subspace M with $\kappa = \dim M \geq \mathfrak{c} \geq \dim M'$. If E is primitive, dlc, or has property (S), respectively, then E contains a dense κ-codimensional subspace F such that the sequences in E' which are eventually zero, bounded, or null, respectively, at the points of F are so, respectively, at the points of E.*

Proof. We argue the first (*primitive-eventually zero*) case only. The other two arguments are symmetric, with analogous supporting theorems in Sections 3 and 4.

Let N be an algebraic complement of M in E, and note that there are only $\mathfrak{c} = \left(2^{\aleph_0}\right)^{\aleph_0}$ distinct sequences in M'.

Case I. $\dim M > \mathfrak{c}$. For each sequence in M', choose exactly one point in M at which the sequence is not eventually zero, when one or more exist. The collection D of all these points has cardinality $\leq \mathfrak{c}$, so $\operatorname{sp} D$ has codimension in M equal to $\dim M$ $[> \mathfrak{c}]$. Clearly, any sequence in E' which is eventually zero at the points of $F := (\operatorname{sp} D) + N$ must be eventually zero at all points of $M + N = E$, and the Hahn-Banach theorem ensures F is dense in E.

Case II. $\dim M = \mathfrak{c}$. By [49, Proposition 2], M contains a dense $\mathfrak{c}$-codimensional subspace M_0. Let B be a cobasis of M_0 in M. There is a collection $\mathcal{B}$ of $2^{\mathfrak{c}}$ subsets of B, each with cardinality $\mathfrak{c}$, such that $T_1 \bigcap T_2$ is countable for distinct T_1, T_2 in $\mathcal{B}$ [22, Theorem 1.3, p. 48]. The proof proceeds by contradiction: Suppose that, for each T in $\mathcal{B}$, $F_T := M_0 + \operatorname{sp}(B \setminus T) + N$ denies the conclusion, so that there exists a sequence S_T in E' eventually zero at points of F_T but not at all points of T; let R_T be the sequence of restrictions to M. The mapping $T \mapsto R_T$ cannot be one-to-one, since $2^{\mathfrak{c}} > \mathfrak{c}$. Let T_1, T_2 be distinct with $R_{T_1} = R_{T_2}$. Then R_{T_i} is eventually zero at points of $L := M_0 + \operatorname{sp}(B \setminus (T_1 \bigcap T_2))$, and so S_{T_i} is eventually zero at points of $L + N$, a dense countable-codimensional subspace of E. Every subspace between $L + N$ and E is primitive by Theorem 4.11. Therefore by Theorem 3.11, S_{T_i} is eventually zero at all points of E, a contradiction. □

Theorem 5.5 (Assume the Continuum Hypothesis). *If a separable space E with $E' \neq E^*$ is primitive or dlc, then so is, respectively, every subspace between E and some dense $\mathfrak{c}$-codimensional subspace F.*

Proof. Primitivity and $E' \neq E^*$ imply $\dim E \geq \mathfrak{c}$, and $\mathfrak{c} \geq \dim E'$ by separability. Theorem 5.4 applies, and then Theorem 3.11 or 3.14, respectively. □

Theorem 5.6 (Assume the Continuum Hypothesis). *If a separable space (E, ξ) has dual E' and $\dim E \geq \mathfrak{c}$, then there is a dense $\mathfrak{c}$-codimensional subspace F such that every $\sigma(E', F)$-null sequence is $\sigma(E', E)$-null.*

Proof. There are only $\mathfrak{c}$ sequences in E'. With $\leq \mathfrak{c}$ induction steps we construct the smallest subspace $E^\sharp$ of E^* such that $E' \subset E^\sharp$ and $E^\sharp$ is $\sigma\left(E^\sharp, E\right)$-sequentially complete. Clearly, $\mathfrak{c} = \mathfrak{c} \cdot \mathfrak{c} \geq \dim E^\sharp$. Thus Theorem 5.4 yields a $\sigma\left(E, E^\sharp\right)$-dense $\mathfrak{c}$-codimensional subspace F of E such that every $\sigma\left(E^\sharp, F\right)$-null sequence is $\sigma\left(E^\sharp, E\right)$-null. It follows that F is $\sigma(E, E')$-dense, hence ξ-dense, and every $\sigma(E', F)$-null sequence is $\sigma(E', E)$-null. □

Theorem 5.7 (Assume the Continuum Hypothesis). *Suppose* (E,ξ) *is a separable space that enjoys one or more of the sixteen properties, and* $E' \neq E^*$. *There is a CE* (E,η) *that enjoys every property enjoyed by* (E,ξ).

Proof. Theorems 5.5 and 5.3 allow us to choose a CE (E,η) which is primitive, since (E,ξ) is, and which also enjoys *dlc* if (E,ξ) does. Thus (E,η) enjoys every fifth-row property that (E,ξ) enjoys, by Theorem 5.1, and enjoys every property above row 5 that (E,ξ) enjoys, with the possible exceptions of c_0- and *C-barrelled*, by Theorem 5.2. If (E,ξ) is neither c_0- nor C-barrelled, the argument is complete.

Assume (E,ξ) is c_0- or C-barrelled. Theorem 5.6 provides a dense $\mathfrak{c}$-codimensional subspace F which is c_0-barrelled, as is, then, every subspace G between F and E. Thus if E is C-barrelled, so is every such G, by Corollary 3.10. From Theorem 5.3, some CE (E,η) is c_0- or C-barrelled, respectively. And hence is dlc, enjoying any other properties (E,ξ) may enjoy. □

Corollary 5.8. *Under the Continuum Hypothesis, there is a positive solution to the CE problem for the class of separable spaces having a given one of the sixteen properties.*

Something like separability is necessary. In [55, Theorem 19] we show that if $\aleph_1 \neq \mathfrak{b}$, a ZFC-consistent strong *denial* of the Continuum Hypothesis, then there are non-separable spaces E with $E' \neq E^*$ that have any given one of the nine properties in rows 2-4 and do *not* admit a property-preserving CE. The positive separable CE solutions, aside from the *barrelled* case, came only after we (very recently) discovered Theorems 3.11, 3.14, 5.4.

The next Theorem uses and implies most of Theorem 5.7, since $M = \{0\}$ is always mc_0-barrelled.

Theorem 5.9. [Assume the Continuum Hypothesis.] *Let* M *be a closed subspace of a dlc space* (E,ξ) *with* E/M *separable and* $(E/M)' \neq (E/M)^*$.

(a) *If* M *is* mc_0*-barrelled, then some CE of* (E,ξ) *enjoys each property that* (E,ξ) *enjoys, with the possible exceptions of* c_0*- and* C-barrelled.

(b) *If* M *is* φ*-complemental, then some CE of* (E,ξ) *enjoys each fifth-row property that* (E,ξ) *enjoys.*

Proof. Let $q : E \to E/M$ denote the quotient map and ξ_q the quotient topology on $(E,\xi)/M$, a dlc space by Theorem 4.3. By Theorem 5.7 there is an $\aleph_0$-dimensional subspace H_q of $(E/M)^*$ transverse to $\left(E/M,\xi_q\right)'$ such that the corresponding CE η_q of ξ_q is dlc. Set $H = \{f \circ q : f \in H_q\}$ and note that H is an $\aleph_0$-dimensional subspace of E^* transverse to E'. The corresponding CE for E has topology η such that η and ξ induce the same topology on M $(\subset H^{\perp})$ and the quotient topology on $(E,\eta)/M$ coincides with η_q. By Theorem 4.6, the

CE (E, η) is dlc if M is mc_0-barrelled, and is primitive if M is φ-complemental. Therefore (a) and (b) hold by Theorems 5.1 and 5.2. $\square$

A most famous question is whether every infinite-dimensional Banach space has an infinite-dimensional separable quotient. Familiar Banach and barrelled spaces admit such quotients, and every closed subspace M of a Banach space is certainly mc_0-barrelled. Theorem 5.9 begs one to solve both the BCE and the Banach separable quotient problems by proving that every barrelled space E with $E' \neq E^*$ admits a properly separable [30, 45] quotient E/M with M a closed mc_0-barrelled subspace. Alas, one cannot. No GM-space admits a properly separable quotient, and the nonempty class of infinite-dimensional quasi-Baire GM-spaces consists of those GM-spaces E with $E' \neq E^*$ which have no infinite-dimensional separable quotients whatsoever [19]. Still, if we assume $\mathfrak{d} = \aleph_1$, a condition weaker than the Continuum Hypothesis, then *every* GM-space E with $E' \neq E^*$ admits a barrelled CE [57, Theorem 5].

The chances of a suitable CE seem greater when $\dim E^*/E'$ is large; nonexistent when $\dim E^*/E'$ is finite. Robertson-Tweddle-Yeomans [32] proved that if E is a barrelled space with $E' \neq E^*$, then $\dim E^*/E'$ is uncountable. Robertson-Saxon-Robertson [31] gave a new proof of Dierolf's result [3] that any barrelled space E with $E' \neq E^*$ must have a dense infinite-codimensional subspace, and thus $\dim E^*/E' \geq \mathfrak{c}$. Saxon [43] recently relaxed the hypothesis from "barrelled" to "quasibarrelled". Assuming $\mathfrak{d} = \aleph_1$, Saxon-Tweddle [57] proved that E barrelled with $E' \neq E^*$ implies E contains a dense $\aleph_1$-codimensional subspace, so that $\dim E^*/E' \geq 2^{\aleph_1}$. Tweddle-Catalan proved [61] that *if E has property (S) with $E' \neq E^*$ and $\dim E$ nonmeasurable, then $\dim E^*/E'$ is uncountable.* Theorem 5.13, part I, will strengthen the conclusion and weaken the hypothesis from *property (S)* to *primitive*, a concept born to accommodate large-codimensional dense subspaces; see [41, Section 1], [57, Theorems 1, 4].

Given a space E and Hamel basis B, set

$$C(B) = \{x \in B : \text{the coefficient functional corresponding to } x \text{ is continuous}\},$$
$$D(B) = \{x \in B : \text{the corresponding coefficient functional is discontinuous}\}.$$

When there is no ambiguity, we may simply write C and D. Clearly, E is the algebraic direct sum of $E_C := \operatorname{sp} C$ and $E_D := \operatorname{sp} D$, where $\operatorname{sp} \emptyset = \{0\}$. The Saxon-Tweddle splitting theorem [56] asserts that if B is an arbitrary basis for a Mackey $\aleph_0$-barrelled space E, then E is the topological direct sum $E_{C(B)} \oplus E_{D(B)}$ and $E_{C(B)}$ has its strongest locally convex topology.

Lemma 5.10. *A primitive space E has a dense infinite-codimensional subspace F if and only if E has a basis B for which $D(B)$ is infinite.*

Proof. Given F, choose a basis B for E with $B \bigcap F$ a basis for F, so that $D(B)$ contains the infinite cobasis $B \backslash F$.

Conversely, if B is given with $D(B)$ infinite, let D_0 be a countably infinite subset of $D(B)$ and set $H = \mathrm{sp}(B \backslash D_0)$. We prove $\dim \overline{H}/H$ is infinite. Otherwise, $\overline{H} \subset H + \mathrm{sp}\, L$ for some finite $L \subset D_0$ and the (infinite) set $D_0 \backslash L$ contains a point x. By definition, the corresponding coefficient functional h is discontinuous and vanishes on $H + \mathrm{sp}\, L$, hence on $\overline{H}$; but this contradicts Theorem 2.4. Therefore H is, indeed, infinite-codimensional in $\overline{H}$. Let N be an algebraic complement of $\overline{H}$ in E and set $F = H + N$. □

Theorem 5.11. *Every Mackey $\aleph_0$-barrelled space E with $E' \neq E^*$ has an infinite-codimensonal dense subspace.*

Proof. Let B be a basis for E. If the corresponding D were finite, then both E_C and E_D would have their strongest locally convex topology, as would $E = E_C \oplus E_D$ [56], so that $E' = E^*$, a contradiction. Lemma 5.10 applies. □

This relaxes properly the Dierolf hypothesis that E be barrelled [56, Example 2.1]. An obvious question intrigues: How far, if at all, can one relax $\aleph_0$-*barrelled* within the sixteen property context? We find two potential answers, one equivalent to the existence, the other to the nonexistence of strongly inaccessible cardinals.

Recall that a countably additive measure μ on the set of all subsets of a set S is an *Ulam measure* on S if μ assumes only the values 0 and 1, and is 0 at singletons and is 1 at S. The cardinal $|S|$ is *measurable* if there exists an Ulam measure on S. No one knows if measurable (equivalently, if strongly inaccessible) cardinals possibly exist, but if they do, they must be "inconceivably large" [71].

Lemma 5.12. *Let B be a basis for a primitive space E having no dense infinite-codimensional subspace. If $C_0 \subset C(B)$ with $|C_0|$ nonmeasurable, then each linear form f that vanishes on $B \backslash C_0$ is continuous.*

Proof. For each $A \subset C_0$, let f_A be the linear form that agrees with f on A and vanishes on $B \backslash A$. Define a function μ on the subsets A of C_0 so that $\mu(A)$ is 0 if f_A is continuous and is 1 if f_A is discontinuous. Thus $\mu(A)$ is 0 if A is empty or a singleton, by hypothesis.

Suppose f is discontinuous. Then $\mu(C_0) = 1$, and C_0 must contain disjoint subsets A_1 and B_1 such that $\mu(A_1) = \mu(B_1) = 1$. Indeed, if such A_1, B_1 do not exist and $S_1, S_2, \ldots$ are disjoint subsets of C_0, then $\mu(S_k) = 0$ for all but possibly one value of k. If every $\mu(S_k) = 0$, set $A = \bigcup_k S_k$ and note, by primitivity, that $f_A = \sum_k f_{S_k}$ is continuous, so that $\mu(A) = 0 = \sum_k \mu(S_k)$; if some $\mu(S_p) = 1$, then $f_A = f_{A \backslash S_p} + f_{S_p}$ is discontinuous, since precisely one of the summands is, and therefore $\mu(A) = 1 = 0 + 1 = \sum_{k \neq p} \mu(S_k) + \mu(S_p)$. Thus μ is countably additive and is, in fact, an Ulam measure on C_0 that contradicts hypothesis. Therefore A_1 and B_1 exist as desired. Since $|B_1| \leq |C_0|$, we have $|B_1|$ is also nonmeasurable, and the same argument produces disjoint subsets

A_2, B_2 of B_1 such that f_{A_2} and f_{B_2} are both discontinuous. This process yields disjoint subsets $A_1, A_2, \ldots$ in C_0 such that each f_{A_k} is discontinuous. Simple linear algebra produces a basis B' for E for which each f_{A_k} is a coefficient functional. Thus $|D(B')|$ is infinite, and Lemma 5.10 contradicts hypothesis. We must conclude that f is continuous. $\square$

Part *III* of the following theorem is crucial to our proof in [55, Theorem 3] that *if a space E admits a CE, it admits a CE that is* not *primitive.* This generalizes [61, Proposition 5]. The conclusion of *III* also holds when E is primitive, $E' \neq E^*$, and each dense hyperplane in E is Mackey [43, Theorem 6].

Theorem 5.13. *Let E be a primitive space.*

I. *If $E' \neq E^*$ and $\dim E$ is nonmeasurable, then E has a dense infinite-codimensional subspace (and thus $\dim E^*/E' \geq 2^{\aleph_0}$).*

II. [Tweddle-Catalan]. *If $\dim E$ is measurable, there is a Mackey topology $\mathcal{T}$ under which E is ℓ^∞-barrelled and $\dim E^*/(E, \mathcal{T})' = 1$.*

III. *In all cases, if $\dim E^*/E' \geq \aleph_0$, then E has a dense infinite-codimensional subspace (and $\dim E^*/E' \geq 2^{\aleph_0}$).*

Proof. *I*. Suppose $E' \neq E^*$, $\dim E$ is nonmeasurable, and E does not have a dense infinite-codimensional subspace. Let B be a Hamel basis for E. By Lemma 5.10, D is finite. If $f \in E^*$, the Hahn-Banach theorem provides $g \in E'$ with $f|_D = g|_D$. Since $f - g$ vanishes on $B \backslash C$ and $|C| = |B|$ is nonmeasurable, Lemma 5.12 implies $f - g$ is continuous; i.e., $(f - g) + g = f \in E'$, proving $E^* = E'$, a contradiction.

II. (See the Remark of [61, p. 59].) If $\dim E$ is measurable, the Mackey space $(E^*, \sigma(E^*, E))$ is not bornological (Mackey-Ulam), and admits a discontinuous bounded linear form u. Set $E' = u^{-1}(0)$ and give E the Mackey topology $\mathcal{T} := \mu(E, E')$. Let Q be a countable $\sigma(E', E)$-bounded set in E'. Under $\sigma(E^*, E)$, the closure G of the span of Q is minimal and separable in E^*, so G is isomorphic to the product of no more than $\mathfrak{c}$ copies of $\mathbb{K}$ and is bornological. Therefore the bounded linear form $u|_G$ is continuous, and since it vanishes on Q, is 0; i.e., $G \subset E'$, so that the absolutely convex hull of Q has the same closure $\overline{\mathrm{acx}}Q$ under both $\sigma(E', E)$ and $\sigma(E^*, E)$. We deduce from Tichonov's theorem that $\overline{\mathrm{acx}}Q$ is compact. Thus it and its subset Q are $\mathcal{T}$-equicontinuous.

III. Let $f_1, f_2, \ldots$ form a basis for a subspace H of E^* transverse to E', let A be a basis for $F := \bigcap_k f_k^{\perp}$, and let B be a cobasis for F in E. Since E/F is algebraically embeddable in the space of scalar sequences, $|B| \leq \mathfrak{c}$ is nonmeasurable. Assume E has no dense infinite-codimensional subspace. Then Lemma 5.10 implies $L := D(A \bigcup B)$ is finite. As $|B \backslash L|$ is nonmeasurable, every linear form vanishing on $A \bigcup L$ is continuous by Lemma 5.12. If the integer p exceeds $|L|$, then the restrictions of $f_1, \ldots, f_p$ to the span of L are linearly dependent, and there exists a non-trivial linear combination f of $f_1, \ldots f_p$ that vanishes on L. Since f also vanishes on A, it must be continuous, so that $0 \neq f \in H \bigcap E'$, a contradiction. $\square$

Remark 5.14. If a Mackey space E is ℓ^∞-barrelled and $\dim E^*/E' = 1$ (see part II), then E enjoys all those properties implied either by *ℓ^∞-barrelled*, or by *C-barrelled*, as box 2 confirms. Obversely, E enjoys only those properties.

Proof. It suffices to show that E is neither $\aleph_0$-barrelled nor non-S_σ. It cannot be $\aleph_0$-barrelled, since Theorem 5.11 would then yield an infinite-codimensional dense subspace, forcing the contradiction $\dim E^*/E' \geq \mathfrak{c}$. Now $\dim E$ is infinite since $E' \neq E^*$, so there is an increasing covering sequence $\{E_n\}_n$ of subspaces with each $\dim E_{n+1}/E_n = \dim E$. To avert the previous contradiction, each $\overline{E_n}$ must be proper in E, proving E is an S_σ space. $\square$

Corollary 5.15. *The following two unproven assertions are equivalent.*

(*1*) *Strongly inaccessible cardinals exist.*
(*2*) *Theorem* 5.11 *is optimal within the purview of Fig.* 1.3.

Indeed, if (*1*) holds, the $\aleph_0$-*barrelled* hypothesis in Theorem 5.11 cannot be replaced by any or all of the strictly weaker properties, as pointed out by part II and the Remark. On the other hand, if (*1*) fails, the hypothesis can be relaxed to the extreme of *primitive*, as part *I* shows.

Corollary 5.16. *The following two unproven assertions are equivalent.*

(*1*) *Strongly inaccessible cardinals do not exist.*
(*2*) *Every primitive space E with $E' \neq E^*$ admits an infinite-codimensional dense subspace.*

The Remark's proof allows us to replace the $\aleph_0$-*barrelled* hypothesis in Theorem 5.11 with *non-S_σ*. This non-comparable hypothesis is also optimal if and only if strongly inaccessible cardinals exist.

6. Distinguishing examples

Borrowing from this section, our [54] and Section 4 answer more than four thousand ($16^3 = 4096$) nuanced three-space-property questions. Here we answer more than four billion ($2^{16} \cdot 2^{16} = 4294\,967\,296$) questions of the form: If a space enjoys all the properties in set S_1, must it enjoy at least one of the properties in set S_2?

More basically, we need to know if distinct names represent distinct notions. The seminorm counterparts to *properties (C)* and *(S)* do not define new classes of spaces, but characterize $\aleph_0$-*barrelled* instead [38], and *W-barrelled* [70] is but an alias for Baire spaces [40].

Let $\mathcal{P}$ denote the set of sixteen distinct property names arrayed in Fig. 1.3. If $a, b \in \mathcal{P}$ are identically the same, then we say *a is equal to b* and write $a = b$. If $a_1, \ldots, a_n$ $(n \geq 2)$ are distinct points in $\mathcal{P}$, and if for $i = 1, \ldots, n-1$ there is a vertical or horizontal implication arrow in Fig. 1.3 pointing from a_i to a_{i+1} with no element of $\mathcal{P}$ in between, then we say a_1 *is greater than* a_n, and write $a_1 > a_n$ (equivalently, $a_n < a_1$). E.g., either of two paths in Fig. 1.3 justifies *QB* $>$ *inductive*, but *C-barrelled* $\not>$ *property (S)*. We write $a \geq b$ to mean a is greater than or equal to b (equivalently, $b \leq a$).

A set $S \subset \mathcal{P}$ is *radial* if $a \leq b \in S$ implies $a \in S$. The extremes $\mathcal{P}$ and $\emptyset$ are radial. For every property b, the smallest radial set containing b is ${}^b s = \{a : a \leq b\}$, and the largest radial set not containing b is $L_b = \{a : a \not\geq b\}$. If b is in the i^{th} row and j^{th} column, we also denote ${}^b s$ and L_b by ${}^{i,j}s$ and $L_{i,j}$, respectively. We denote by $\ulcorner b = \ulcorner(i, j)$ the set of points of $\mathcal{P}$ having position (i', j') with $i' \geq i$ and $j' \geq j$. For fourteen values of b, the sets $\ulcorner b$ and ${}^b s$ coincide, and $\ulcorner b$ is radial for all sixteen. For each $S \subset \mathcal{P}$ let $\langle S \rangle$ denote the smallest radial set containing S.

Given a space E, the set $\mathbf{P}(E) = \{b \in \mathcal{P} : E \text{ enjoys property } b\}$ is radial, since all the arrows of Fig. 1.3 are valid (Section 2). However, $L_{1,2}$ is a radial set *not* of the form $\mathbf{P}(E)$, since $[\{\textit{non-}S_\sigma, \textit{barrelled}\} \subset \mathbf{P}(E)] \Rightarrow [QB \in \mathbf{P}(E)]$ by definition of *QB*. While trivial, this implication is *not* indicated by Fig. 1.3. Question: Might there be nontrivial choices of $S_1, S_2 \subset \mathcal{P}$ such that the statement

$$\text{For } E \text{ an arbitrary space, } [S_1 \subset \mathbf{P}(E)] \Rightarrow [\mathbf{P}(E) \bigcap S_2 \neq \emptyset] \qquad (\dagger)$$

holds, but is not indicated by Fig. 1.3? We infer *No* via the main result of this section, Theorem 6.35: *All radial subsets of $\mathcal{P}$ except $L_{1,2}$ are of the form* $\mathbf{P}(E)$. Indeed, suppose we are given S_1 and S_2 which verify (†). If $\langle S_1 \rangle \neq L_{1,2}$, the main result and (†) provide some $b \in \langle S_1 \rangle \bigcap S_2$; hence $a \geq b$ for some $a \in S_1$, and the arrows in Fig. 1.3 clearly indicate (†) holds. And if $\langle S_1 \rangle = L_{1,2} \subsetneq \ulcorner(1, 2)$, the main result and (†) yield some $b \in \ulcorner(1, 2) \bigcap S_2$. Either (i) $b = QB$, or (ii) $b = \textit{non-}S_\sigma$, or (iii) $b \leq \textit{barrelled}$. Necessarily, *non-*$S_\sigma$, *barrelled* $\in S_1$, so implication (†) is a trivial consequence of the definition of *QB* in case (i), utterly trivial in case (ii), and obviously indicated by Fig. 1.3 in case (iii). *Thus (S_1, S_2) satisfies (†) if and only if it is obvious from the definition of QB or from a glance at Fig.* 1.3.

Another corollary to the main result: *The sixteen names in $\mathcal{P}$ represent distinct notions.* Indeed, if $b_1 \neq b_2$, then for some $i \in \{1, 2\}$ we have $b_i \notin \ulcorner b_{3-i}$. A space E exists with $\mathbf{P}(E) = \ulcorner b_{3-i}$ (Corollary 6.33). Thus E enjoys b_{3-i} and not b_i, so that b_1 and b_2 are logically distinct.

Many supporting examples and results are required.

Theorem 6.1. *The following two statements hold for arbitrary spaces E and F.*

(*i*) $\mathbf{P}(E \times F)$ *equals either* $\mathbf{P}(E) \bigcap \mathbf{P}(F)$ *or* $\mathbf{P}(E) \bigcap \mathbf{P}(F) \setminus \{$*property (C)*$\}$.

(*ii*) $\mathbf{P}(\varphi \times F) = \mathbf{P}(F) \setminus \{BL, QB, \text{ non-}S_\sigma\}$.

Proof. Theorems 4.2 and 4.3 yield (*i*). If F enjoys *property (C)*, so does $\varphi \times F$, for in the weak dual $\mathbb{K}^{\mathbb{N}}$ of φ, bounded sets are relatively sequentially compact. Therefore (*i*) implies (*ii*). □

Theorem 6.2. *If E is S_σ and barrelled, then $F := (E, \sigma(E, E'))$ enjoys precisely the fifth-column properties.*

Proof. By duality invariance, F enjoys the fifth-column properties and contains a closed $\aleph_0$-codimensional subspace. But, because of its weak topology, F contains no copy of φ, and cannot be φ-complemental. Thus Fig. 1.3 denies F the properties outside column 5. □

Theorem 6.3. *A Fréchet space E has an infinite-dimensional separable quotient if and only if it contains a dense subspace F which enjoys precisely the fifth-row properties.*

Proof. E has such a quotient if and only if it contains a non-barrelled dense subspace F dominated by a Fréchet space [48, 58]. □

Dominating Banach or (LB)-spaces essentially double Examples 6.4-6.30, for each uncountable-dimensional (E, ξ), thus dominated, admits a CE (E, η) satisfying the Tweddle-Yeomans Criterion with $\mathbf{P}(E, \eta) = \mathbf{P}(E, \xi)$ (see [55]), yet the two spaces are very different: The Criterion denies domination of (E, η) by a Banach or (LB)-space.

Example 6.4. According to [58], inside each infinite-dimensional separable Banach space E lies a separable dense non-barrelled, equiv., non-dlc subspace E_1 dominated by a Banach space, hence non-S_σ. Therefore $\mathbf{P}(E_1) = {}^{5,2}s = L_{4,5}$.

Example 6.5. According to Theorem 6.2, a separable proper strict (LB)-space endowed with its weak topology is a separable space E_2 that enjoys precisely the fifth-column properties; i.e., $\mathbf{P}(E_2) = {}^{2,5}s = L_{5,4}$.

Example 6.6. If E is any infinite-dimensional Banach space endowed with its weak topology $\sigma(E, E')$, then E has precisely the properties in the union of the fifth row and fifth column; i.e., $\mathbf{P}(E) = L_{4,4}$.

Proof. By duality invariance, $(E, \sigma(E, E'))$ has property (C) and is non-S_σ, thus enjoys all the fifth-column and fifth-row properties. We must show it is not mc_0-barrelled. Let $\{f_n\}_n$ be a linearly independent sequence in E' with each $\|f_n\| = 1$. Then $\left\{\frac{1}{n}f_n : n \in \mathbb{N}\right\}^\circ$ is not a weak neighborhood of zero, since it contains no finite-codimensional subspace. □

Definition 6.7. If the normed space $(E, \|\quad\|)$ dominates (E, τ), we let $\|\tau\|$ denote the finest locally convex topology on E that agrees with τ on the norm unit ball B. A base of 0-neighborhoods for $\|\tau\|$ consists of all sets $W = \bigcap_n (t_n B + V_n)$, where $\{t_n\}_n$ is a sequence of positive scalars diverging to ∞ and each V_n is a τ-neighborhood of 0 [26, 8.1.12(i)]. *(See also* [11, 33]*).*

A seminorm p on E is $\|\tau\|$-continuous if and only if the restriction $p|_B$ is τ-continuous. Thus τ is coarser than $\|\tau\|$ which, in turn, is coarser than the normed topology since any basic W given as above contains tB, where $t = \min_n t_n > 0$.

Theorem 6.8. *If a normed space $(E, \|\quad\|)$ is infinite-dimensional with dual E', then $(E, \|\sigma(E, E')\|)$ is not c_0-barrelled.*

Proof. The Josefson-Nissenzweig theorem ([14], [25]) provides a $\sigma(E', E)$-null sequence $\{f_n\}_n$ with each $\|f_n\| = 1$, and, passing to a subsequence if necessary, we may assume there is a sequence $\{x_n\}_n$ in E with $\|x_n\| = 1$ such that $f_n(x_n) > 2/3$ and $|f_m(x_n)| < 1/3$ for all $m > n$ $(n = 1, 2, \ldots)$. We show that $U = \{f_n : n \in \mathbb{N}\}^\circ$ contains no W of the form $W = \bigcap_n (t_n B + V_n)$, where B is the unit ball, $\{t_n\}_n$ is a sequence of positive scalars diverging to ∞ and each V_n is a weak 0-neighborhood. Choose r such that $t_n \geq 6$ for $n > r$. Then $\left(\bigcap_{n \leq r} V_n\right) \bigcap 6B$ is contained in W but, as we shall see, not in U.

Let $\{g_1, \ldots, g_k\}$ be a finite subset of E' whose polar A is contained in the weak neighborhood $\bigcap_{n \leq r} V_n$; it is enough to show that $A \bigcap 6B \not\subset U$. The sequence $\{(g_1(x_n), \ldots, g_k(x_n))\}_n$ in Euclidean k-space is bounded, since the sequence of i^{th} coordinates is numerically bounded by $\|g_i\|$ for $i = 1, \ldots, k$. The Bolzano-Weierstrass theorem ensures a Cauchy subsequence, and we fix suitably large p and q with $p < q$ such that $|g_i(x_q) - g_i(x_p)| \leq \frac{1}{3}$ for $1 \leq i \leq k$. Hence $y = 3(x_q - x_p) \in A \bigcap 6B$. But $|f_q(y)| > 3(2/3 - 1/3) = 1$, so $y \notin U$. □

An alternate proof uses Ruess' [37, Proposition 2.6] and, again, the Josefson-Nissenzweig result, which seems essential, being easily proved by Theorem 6.8 itself. Indeed, the Theorem ensures a $\sigma(E', E)$-null sequence $\{f_n\}_n$ that is not $\|\sigma(E, E')\|$-equicontinuous, with each $f_n \neq 0$. Clearly,

$$(\{f_n\}_n)^\circ = \bigcap_n f_n^\circ \supset \bigcap_n \left(\|2f_n\|^{-1} B + (2f_n)^\circ\right),$$

and non-equicontinuity implies $\left\{\|f_n\|^{-1}\right\}_n$ fails to diverge to ∞. Equivalently, some subsequence is bounded. Therefore the corresponding subsequence of $\{f_n / \|f_n\|\}_n$ is $\sigma(E', E)$-null, verifying Josefson-Nissenzweig.

Definition 6.9. Given a space (E, τ), the L *topology associated with* τ is the coarsest topology τ^L finer than τ such that $\left(E, \tau^L\right)$ has property (L).

Always, one may construct τ^L in $\aleph_1$ inductive steps. Moreover,

Theorem 6.10. *If τ is a coarser locally convex topology for a normed space $(E, \|\ \ \|)$, then $\|\tau\|$ is coarser than τ^L. If $(E, \|\ \ \|)$ is barrelled, then $\|\tau\| = \tau^L$.*

Proof. If a seminorm p on E is $\|\tau\|$-continuous, then the restriction $p|_B$ to the unit ball B in $(E, \|\ \ \|)$ is τ-continuous, hence continuous for the finer topology τ^L. Property (L) ensures $\tau^L = \tau^L_B$ (Ruess; cf. [51, 2.2(2)]). Therefore p is τ^L-continuous, and $\|\tau\|$ is coarser than τ^L.

Assuming $(E, \|\ \ \|)$ is barrelled, we must prove $\|\tau\|$ is finer than τ^L. By definition of τ^L, since $\|\tau\|$ is finer than τ, it suffices to prove $(E, \|\tau\|)$ has property (L). Let A be a $\|\tau\|$-barrel. If p is a seminorm on E and $p|_A$ is $\|\tau\|$-continuous, then so is $p|_{nA}$ for all $n \in \mathbb{N}$. Since $\|\tau\|$ is coarser than the norm topology, A is a norm-barrel, so $mA \supset B$ for some suitably large m, which implies $p|_B$ is $\|\tau\|$-continuous. Since $\|\tau\|$ agrees with τ on B by definition, $p|_B$ is τ-continuous, and thus p is $\|\tau\|$-continuous. Since p and A were arbitrarily given, $(E, \|\tau\|)$ has property (L) by Theorem 2.5(6). □

Example 6.11. If E is any infinite-dimensional Banach space with norm $\|\ \ \|$ and dual E', then $(E, \|\sigma(E, E')\|)$ is dominated by a Banach space, has properties (C) and (L), but is not c_0-barrelled. Hence $(E, \|\sigma(E, E')\|)$ enjoys precisely the properties that are in either row 4, 5, or column 5; i.e., in $L_{3,4}$.

Proof. By duality invariance and Theorems 6.8, 6.10. □

A space E is c_0-barrelled if and only if, given any $\sigma(E', E)$-null sequence $\{g_n\}_n$, the linear mapping $x \mapsto \{g_n(x)\}_n$ from E into c_0 is continuous, since the preimage of the unit ball of c_0 is the polar set $A = \{g_n : n \in \mathbb{N}\}^\circ$. The mapping (onto its image) is one-to-one and open if and only if A is bounded. On the other hand, if E is a subspace of c_0, then the coordinate functionals restricted to E form a $\sigma(E', E)$-null sequence whose polar is bounded, being the unit ball in E.

In summary, *a c_0-barrelled space E is isomorphic to a subspace of c_0 if and only if there exists a $\sigma(E', E)$-null sequence with bounded polar.*

Let each E_n be a separable Banach space with unit ball B_n such that infinitely many of the E_n cannot be embedded in c_0. (Such examples are well known; e.g., ℓ^1 cannot be embedded in c_0 since the strong dual of ℓ^1 is non-separable.) Let H denote the direct sum $\oplus_n E_n$ with dual $H' \approx \prod_n E'_n$ and consider the vector topology η on H with 0-neighborhood base all sets U of the form

$$U = \left[\epsilon\left(B_1 \oplus \cdots \oplus B_k\right) \oplus \left(\oplus_{n>k} E_n\right)\right] \bigcap D$$

where $\epsilon > 0$, $k \in \mathbb{N}$, and D is the polar of a $\sigma(H', H)$-null sequence.

Example 6.12. (H, η) is a separable c_0-barrelled space compatible with a strict (LB)-space, but is not inductive and therefore is not ℓ^∞-barrelled via box 1. Thus it enjoys precisely the properties of columns 4 and 5 with ℓ^∞-*barrelled* excluded; i.e., $\mathbf{P}(H, \eta) = \langle$*property (C)*, c_0-*barrelled*$\rangle$, a radial set generated by two points.

Proof. In its direct sum topology, H is barrelled and each D of the above form is a neighborhood of 0, proving (H, η) is dominated by the strict inductive limit of its Banach subspaces $E_1 \oplus \cdots \oplus E_n \quad (n \in \mathbb{N})$. Thus η is compatible with the pairing $\langle H, H' \rangle$, whence (H, η) is c_0-barrelled. To see it is not inductive, we note that the domination is strict: $\oplus_n B_n$ is a 0-neighborhood in the inductive limit topology, but contains no set U as above, since $U \bigcap E_p = E_p \bigcap D$ is unbounded in norm, where p is the first integer larger than k such that E_p cannot be embedded in c_0. □

Observe that if all but finitely many of the E_n could be embedded in c_0, then η would be just the (barrelled) direct sum topology.

In the next two examples, let E be an infinite-dimensional Banach space with an unconditional Schauder basis $\{x_n\}_n$ such that c_0 is not (isomorphically) contained in E. Again, there are many well known examples, including $\ell^p \ (1 \leq p < \infty)$. [Reflexive ℓ^p easily qualify, because the strong bidual of any Banach space containing c_0 is non-separable.]

We review some pertinent facts. Since c_0 embeds in its infinite-dimensional closed subspaces, non-containment of c_0 is stronger than the reverse non-containment we used for H. We may assume that the norm of each of the basis elements x_n is 1. There is the sequence $\{f_n\}_n \subset E'$ of *coefficient functionals* such that, for each $x \in E$, the series $\sum_n f_n(x)\, x_n$ converges unconditionally in E to x. For each σ in the family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ we define the continuous linear map $T_\sigma : E \to E$ so that $T_\sigma(x) = \sum_{n\in\sigma} f_n(x)\, x_n$. Unconditionality ensures each $\{T_\sigma(x)\}_{\sigma\in\mathcal{F}}$ is bounded, and Banach-Steinhaus yields equicontinuity of $\{T_\sigma\}_{\sigma\in\mathcal{F}}$. Thus there is an equivalent norm $\|\quad\|$ with unit ball B such that

$$\|x\| = \sup_{\sigma\in\mathcal{F}} \left\| \sum_{n\in\sigma} f_n(x)\, x_n \right\| \tag{1}$$

for all $x \in E$. Hence each $\|x_n\| = \|f_n\| = 1$. Let S_k denote $T_{\{1,\ldots,k\}}$.

It is known that for $b_1, \ldots, b_n$ scalars, $\sum_{i\leq n} |b_i| \leq \pi \max_{\sigma\subset\{1,\ldots,n\}} \left|\sum_{i\in\sigma} b_i\right|$. Given $\sigma \in \mathcal{F}$, choose $h \in E'$ with $\|h\| = 1$ and $h\left(\sum_{n\in\sigma} x_n\right) = \left\|\sum_{n\in\sigma} x_n\right\|$.

Then

$$\begin{aligned}\left\|\sum_{n\in\sigma} a_n x_n\right\| &= \max_{\sigma'\subset\sigma}\|h\|\left\|\sum_{m\in\sigma'} a_m x_m\right\| \geq \max_{\sigma'\subset\sigma}\left|\sum_{m\in\sigma'} a_m h(x_m)\right| \\ &\geq \frac{1}{\pi}\sum_{n\in\sigma}|a_n h(x_n)| \geq \frac{1}{\pi}\left(\min_{n\in\sigma}|a_n|\right)\sum_{n\in\sigma}|h(x_n)| \\ &\geq \frac{1}{\pi}\left(\min_{n\in\sigma}|a_n|\right)\left|h\left(\sum_{n\in\sigma} x_n\right)\right| = \frac{1}{\pi}\left(\min_{n\in\sigma}|a_n|\right)\left\|\sum_{n\in\sigma} x_n\right\|. \qquad (2)\end{aligned}$$

We must establish some facts about $\sigma(E', E)$-null sequences. Obviously, $\{f_n\}_n$ is a very special case; we denote its polar by P and observe that if $a > 0$, then

$$B + aP \ \text{ is closed.} \qquad (3)$$

Indeed, suppose $\{u_n\}_n \subset B$ and $\{v_n\}_n \subset aP$ with $\{u_n + v_n\}_n$ convergent to some x in E. Fix $k \in \mathbb{N}$ such that $R_k(x) := x - S_k(x)$ has norm $\leq a$. Thus $|f_j(R_k(x))|$ is 0 if $j \leq k$ and is $\leq \|f_j\|\,\|R_k(x)\| \leq 1 \cdot a$ for $j > k$. Since $|f_j(u_n)| \leq \|f_j\|\,\|u_n\| \leq 1$ and $|f_j(v_n)| \leq a$ by definition of P, k-dimensional Bolzano-Weierstrass subsequences allow us to assume that $\{S_k(u_n)\}_n$ and $\{S_k(v_n)\}_n$ respectively converge to some u and v in E. Each $S_k(v_n)$ is obviously in aP and $S_k(u_n)$ is in B by (1). Hence $u \in B$ and $v \in aP$. In fact, since $v = S_k(v)$, it is clear that $v + R_k(x) \in aP$. Continuity implies

$$S_k(x) = S_k\left(\lim_n (u_n + v_n)\right) = \lim_n (S_k(u_n) + S_k(v_n)) = u + v.$$

Therefore $x = S_k(x) + R_k(x) = u + (v + R_k(x)) \in B + aP$; i.e., $B + aP$ is closed.

Now if $\{h_n\}_n$ is an arbitrary $\sigma(E', E)$-null sequence, there are positive integer sequences $\{p(n)\}_n$ and $\{q(n)\}_n$ with $p(n) < q(n) < p(n+1)$ such that for each n, k with $y_k = \sum_{j\leq k}\left[x_{q(j)} - x_{p(j)}\right]$,

$$|h_n(y_k)| < M := 1 + 2\sup\{\|h_i\| : i \in \mathbb{N}\}, \qquad (4)$$

as we now prove. [M is finite, since E is barrelled.] We arbitrarily choose $p(1) < q(1)$ and note that each $\left|h_n\left(x_{q(1)} - x_{p(1)}\right)\right| \leq \|h_n\|\left\|x_{q(1)} - x_{p(1)}\right\| \leq 2\|h_n\| < M$. Suppose we have found $p(1) < q(1) < \ldots < p(m) < q(m)$ so that (4) holds for all n, k with $k \leq m$. As $\lim_n h_n(y_m) = 0$, we find r such that $n > r$ implies $|h_n(y_m)| < 1$. Now, no matter how $q(m+1) > p(m+1) > q(m)$ are eventually chosen, $n > r$ will imply that $|h_n(y_{m+1})| = \left|h_n(y_m) + h_n\left(x_{q(m+1)} - x_{p(m+1)}\right)\right| < 1 + 2\|h_n\| \leq M$. Only for $n \leq r$ need we prove our choices produce y_{m+1} verifying $|h_n(y_{m+1})| < M$. Let $\delta = M - \max_{n\leq r}|h_n(y_m)|$, a positive scalar. The bounded sequence

$\{(h_1(x_j), \ldots, h_r(x_j))\}_j$ in Euclidean r-space has a Cauchy subsequence, and we choose $q(m+1) > p(m+1) > q(m)$ such that

$$\left|h_n\left(x_{q(m+1)} - x_{p(m+1)}\right)\right| < \delta \ \text{ for } 1 \leq n \leq r.$$

Consequently, $n \leq r$ implies that $|h_n(y_{m+1})| \leq |h_n(y_m)| + \left|h_n\left(x_{q(m+1)} - x_{p(m+1)}\right)\right| < (M-\delta) + \delta = M$, which completes the induction step and establishes (4).

The formal series $\sum_j \left[x_{q(j)} + x_{p(j)}\right]$ is norm divergent, since each of its terms has norm at least 1, and thus is surely not unconditionally convergent. According to Lemma 15.8 in [59], since E does not contain c_0, the divergent series cannot be weakly unconditionally Cauchy; equivalently ([59], Corollary 15.1, 3°), $\left\{\left\| \sum_{j\in\sigma}\left[x_{q(j)} + x_{p(j)}\right]\right\| : \sigma \in \mathcal{F}\right\}$ is unbounded. We fix $n > M$, set $\epsilon = \frac{1}{M} - \frac{1}{n}$, and fix $\sigma \in \mathcal{F}$ such that $\left\| \sum_{j\in\sigma}\left[x_{q(j)} + x_{p(j)}\right]\right\| > \frac{n\pi}{\epsilon}$. Let k denote the largest member of σ; then (1) implies $\left\| \sum_{j\leq k}\left[x_{q(j)} + x_{p(j)}\right]\right\| > \frac{n\pi}{\epsilon}$. Now (4) implies $z = y_k/M$ is in the polar D of $\{h_n\}_n$. Set $\theta = \{p(j)\}_{j\leq k} \bigcup \{q(j)\}_{j\leq k}$. If w is any member of $n^{-1}P$, then (1) yields $\|z - w\| \geq \left\|\sum_{i\in\theta} f_i(z-w)\, x_i\right\|$. For each $i \in \theta$ we have $|f_i(z-w)| \geq |f_i(z)| - |f_i(w)| \geq \frac{1}{M} - \frac{1}{n} = \epsilon$, and so (2) yields $\|z - w\| \geq \frac{\epsilon}{\pi}\left\|\sum_{i\in\theta} x_i\right\| > n$; i.e., $z - n^{-1}P$ misses nB.

We conclude that if D is the polar of an arbitrary $\sigma(E', E)$-null sequence, then each suitably large integer n admits a corresponding z with

$$z \in D \setminus \left(nB + n^{-1}P\right). \tag{5}$$

Now repeat the above construction choosing n and $p(1)$ so that $p(1) > n > 2M$. Let l be the least integer such that $\left\| \sum_{j\leq l}\left[x_{q(j)} + x_{p(j)}\right]\right\| > \frac{\pi}{\epsilon}$, and define z and θ as above, but with l replacing k. Since $p(1) > n$, for every $i \in \theta$ we have $i > n$, and thus $|f_i(z-w)| \geq \frac{1}{M} - \frac{1}{n} = \epsilon$ when w is any point in $n^{-1}P + \bigcap_{i>n} f_i^{\perp}$. Now (1) and (2) imply $\|z-w\| \geq \frac{\epsilon}{\pi}\left\| \sum_{j\leq l}\left[x_{q(j)} + x_{p(j)}\right]\right\| > 1$, so that $z - \left[n^{-1}P + \bigcap_{i>n} f_i^{\perp}\right]$ misses B. Since $n > 2M$, we have $\epsilon = \frac{1}{M} - \frac{1}{n} > \frac{1}{2M}$. By definition of l, then,

$$\left\|\sum_{j\leq l}\left[x_{q(j)} + x_{p(j)}\right]\right\| \leq \frac{\pi}{\epsilon} + \left\|x_{q(l)} + x_{p(l)}\right\| \leq \frac{\pi}{\epsilon} + 2 < 2M\pi + 2.$$

Therefore

$$\begin{aligned}\|z\| &\leq \frac{1}{M}\left(\left\|\sum_{j\leq l} x_{q(j)}\right\| + \left\|\sum_{j\leq l} x_{p(j)}\right\|\right) \leq \frac{2}{M}\left\|\sum_{j\leq l}\left[x_{q(j)} + x_{p(j)}\right]\right\| \\ &< \frac{2}{M}(2M\pi + 2) \leq 4\pi + 4.\end{aligned}$$

We conclude that if D is the polar of an arbitrary $\sigma(E', E)$-null sequence, then to every suitably large n there corresponds some z with

$$z \in \left(D \bigcap 17B\right) \setminus \left(B + n^{-1}P + \bigcap_{i>n} f_i^{\perp}\right). \tag{6}$$

Consider the vector topology ξ on E having 0-neighborhood base all polars D of $\sigma(E', E)$-null sequences.

Example 6.13. (E, ξ) is a c_0-barrelled space with the weak dual of a separable Banach space, thus is mc_0-barrelled, separable, and has all the fifth-row and fifth-column properties. It has none of the other properties; i.e., $\mathbf{P}(E, \xi) = L_{2,4} \bigcap L_{4,3}$.

Proof. The definition of ξ and compatibility with $(E, \|\ \ \|)$ ensure c_0-barrelledness, separability, the fifth-row and fifth-column properties, and the same absolutely convex closed sets, so that by (3), each $U_n = B + n^{-2}P$ is an absolutely convex closed 0-neighborhood in (E, ξ). The intersection contains B and so is absorbing. But $U = \bigcap_n nU_n = \bigcap_n (nB + n^{-1}P)$ is not a ξ-neighborhood of 0 since, by (5), it contains no basic ξ-neighborhood D. Theorem 2.5 denies (E, ξ) property (L). Therefore it cannot be C-barrelled, $\aleph_0$-barrelled, barrelled, QB, or BL, and, being separable, cannot be ℓ^∞-barrelled by box 1. $\square$

Example 6.14. $(E, \|\xi\|)$ is separable with all the properties that are in either row 4, 5, or column 5, and is c_0-barrelled. It has none of the other properties, so $\mathbf{P}(E, \|\xi\|) = L_{2,4} \bigcap L_{3,3}$.

Proof. The previous argument and Theorem 6.10 only leave us to exclude the other properties via box 1 and the following proof that $(E, \|\xi\|)$ is not C-barrelled: Each $U_n = B + n^{-1}P + \left(\bigcap_{i>n} f_i^{\perp}\right)$ is a ξ-, hence a $\|\xi\|$-neighborhood of 0, absolutely convex and closed in light of (3) and [51, 2.9]. Each point of E is contained in almost all of the U_n. It suffices to show that $U = \bigcap_n U_n$ does not contain any set $W = \bigcap_n (t_n B + D_n)$, where $\{t_n\}_n$ is a sequence of positive scalars diverging to ∞ and each D_n is the polar of a $\sigma(E', E)$-null sequence. Choose r such that $t_k \geq 17$ for all $k > r$. Then W contains $D \bigcap 17B$, where $D = \bigcap_{k \leq r} D_k$ is, itself, the polar of a $\sigma(E', E)$-null sequence. Our (6) implies $D \bigcap 17B$, and hence W, is not contained in U_n for suitably large n. Therefore $W \not\subset U$. $\square$

Let E be a non-reflexive Banach space (such as ℓ^1, c_0 or ℓ^∞) whose dual E' is $\sigma(E', E)$-separable, and let $\mathcal{L}$ be the Mackey space $(E', \mu(E', E))$, obviously separable and dlc, hence C-barrelled by box 2. Since the unit ball in E' is a barrel that is not a 0-neighborhood in $\mathcal{L}$, the separable $\mathcal{L}$ is not barrelled, hence not ℓ^∞-barrelled by box 1.

Example 6.15. Obviously dominated by a Banach space, $\mathcal{L}$ is a separable C-barrelled space that is not ℓ^∞-barrelled.

Jarchow's universal generator $(\ell^\infty, \mu(\ell^\infty, \ell^1))$ for the variety of Schwartz spaces (cf. [4] and [13, 10.5.3]) is a particular case in point. Let us denote it by J. It is known that a 0-neighborhood base for J is the family of all sets of the form $U_t = \{x \in \ell^\infty : |x(n)| \le t(n) \text{ for all } n\}$, where $t = \{t(n)\}_n$ is a sequence of positive scalars tending to ∞. Let $\|\ \|$ be the usual sup norm on ℓ^∞ with unit ball B and note that $U_t = \bigcap_n [(t(n)/2)B + (t(n)/2) \cdot e_n^\circ]$, where e_n is the n^{th} canonical unit vector in ℓ^1. Thus $\mu(\ell^\infty, \ell^1)$, finer than $\sigma(\ell^\infty, \ell^1)$, is coarser than the associated L topology $\|\sigma(\ell^\infty, \ell^1)\|$. Therefore $\mu(\ell^\infty, \ell^1) = \|\sigma(\ell^\infty, \ell^1)\|$.

In [24, pp. 100, 101] , Levin-Saxon noted J lacks property (C): The unit vectors e_n have no $\sigma(\ell^1, \ell^\infty)$-cluster point. Banach proved J has property (S). Consequently,

Example 6.16. J enjoys precisely the properties in rows 3-5.

Lemma 6.17. *Let F be a subspace of J between c_0 and ℓ^∞. Clearly, F is dense in J with dual ℓ^1 and is separable. At the least, F enjoys all the properties in rows 4 and 5. Its inherited topology is $\|\sigma(F, \ell^1)\|$.*

Proof. Since J enjoys *non-S_σ* and *property (L)*, so does F; indeed, every $\sigma(\ell^1, c_0)$-bounded set is $\sigma(\ell^1, \ell^\infty)$-bounded, and Theorems 3.14(b) and 3.15 apply.

By definition, $\|\sigma(\ell^\infty, \ell^1)\|$ induces on F a topology coarser than $\|\sigma(F, \ell^1)\|$, and also finer, by Theorem 6.10. □

Example 6.18. The subspace c of J has precisely the properties in rows 4 and 5.

Proof. The Cauchy sequence $\{e_n\}_n$ in $(\ell^1, \sigma(\ell^1, c))$ proves c lacks property (S). But J has property (S), and Theorem 3.8 proves c is not c_0-barrelled. □

The Lemma yields

Example 6.19. The subspace c_0 of J is just $(c_0, \|\sigma(c_0, \ell^1)\|)$, a special case of Example 6.11, with precisely the properties in $L_{3,4}$.

For $i = 1, 2, 3$, separable $(LF)_i$-spaces exist [47, 48], and an $(LF)_{4-i}$-space has precisely the properties in columns i through 5. We do not know if separable spaces exist with precisely the properties in $L_{2,4}$. All dense subspaces of J are separable, but many (perhaps all) lack one or more properties in $L_{2,4}$.

Example 6.20. If F is a subspace of J and $F \bigcap c_0$ is dense in c_0, equivalently, in J, then either F is not c_0-barrelled or F lacks property (C).

Proof. If $F \subset c_0$, then Example 6.19 and Theorem 3.1 imply F is not c_0-barrelled.

If $F \not\subset c_0$, there exists $x \in F$ with $|x(n)| = |\langle x, e_n\rangle| \geq 3$ for all n in some infinite subset S of $\mathbb{N}$. Let $v \in \ell^1$. Choose $u \in F \bigcap c_0$ with $|\langle x - u, v\rangle| \leq 1$. Now $|\langle u, e_n\rangle| < 1$ for almost all $n \in \mathbb{N}$, so for almost all $n \in S$ we have $|\langle x - u, e_n - v\rangle| > 3 - 1 - 1 = 1$. Therefore v is not a cluster point of the $\sigma(\ell^1, F)$-bounded set $\{e_n : n \in S\}$, and F lacks property (C). □

Nevertheless, we have examples enough to prove that separability partitions $\mathcal{P}$ into singleton equivalence classes, except for the tripleton, box 1, whose members are separably equivalent by [26, 8.2.20]. Fig. 1.3 tells the entire separable story. For instance, Examples 6.5, 6.12, 6.16, 6.18 respectively prove that c_0-*barrelled* is not separably equivalent to any property on its right, its left, above, or below, and hence must stand alone. Each of the other singletons may be established in similar fashion with four or fewer examples. For $\{BL\}$, just one separable $(LF)_2$-space suffices (e.g., ℓ^{p-} defined in Section 7). For $\{$*property (C)*$\}$, two Examples, 6.5 and 6.16, suffice.

Finite-codimensional subspaces may fail to preserve C- and c_0-*barrelled*.

Example 6.21 (See [65], [26, 5.2.1], [43, Theorem 1]). The space c with the Mackey topology $\mu(c, \ell^1)$ is C-barrelled without property (S). Its dense hyperplane c_0 is non-Mackey and fails to be even c_0-barrelled.

Proof. To Example 6.18 apply duality invariance and box 2. The dense hyperplane c_0 inherits a topology strictly coarser than the usual sup norm topology. Since the latter coincides with the Mackey topology $\mu(c_0, \ell^1)$, the former does not. Nor does it make the $\sigma(\ell^1, c_0)$-null sequence $\{e_n\}_n$ equicontinuous, since it would then be finer than the norm topology. □

Another particular case of Example 6.15, $G := (\ell^1, \mu(\ell^1, c_0))$ does not have property (S): The unit vectors e_n in c_0 are not weakly summable (cf. [24, §6], [69, §4], [26, 8.2.21(b)]).

Example 6.22. G is separable, C-barrelled, dominated by a Banach space, and without property (S). Thus G enjoys precisely the properties in $L_{3,5}$.

Example 6.4, $(LF)_i$-spaces, and [52] prove Fig. 1.3 tells the entire Mackey space story. And nearly completes the story for *separable* Mackey spaces, where there are exactly three non-singleton equivalence classes: Boxes 2 and 4 and the interval $\{b \in \mathcal{P} : \textit{barrelled} \geq b \geq \textit{property (C)}\}$. The interval is from Ferrando-Sánchez Ruiz [9]: *A separable Mackey space is barrelled if (and only if) it has property (C).* The rest follows from our separable Mackey examples and [52].

For instance, Examples 6.16 and 6.22 prove the singleton $\{$*property (S)*$\}$ is an equivalence class.

The interval implies no separable Mackey space enjoys precisely the properties in $L_{2,4}$, and our next Examples 6.25-6.30 are of necessity non-separable.

Let ϑ be the locally convex topology on ℓ^∞ that has a base of neighborhoods of zero all sets of the form $(U_t \bigcap c_0) + \epsilon B$, notation as above, and define $M = (\ell^\infty, \vartheta)$. Apparently, ϑ is coarser than the usual norm topology, and is finer than that of J since, given t, we may take $\epsilon = \min_n t(n)/2$ and define $t'(n) = t(n) - \epsilon$ so that $(U_{t'} \cap c_0) + \epsilon B \subset U_{t'} + \epsilon B = U_t$. Readily, M and J induce the same topology on c_0. Also, $M' = (\ell^\infty, \| \ \|)'$: If $f \in (\ell^\infty, \| \ \|)'$ then $f|_{c_0}$ is canonically identified with a member of ℓ^1 which, in turn, is identified with some $g \in J'$. Thus g is bounded both on B and on some U_t, and the norm-continuous $f - g$ is bounded on $U = (U_t \bigcap c_0) + B$, since it vanishes on c_0. Hence $f = (f - g) + g$ is bounded on U, proving ϑ is compatible with the usual norm topology, though clearly strictly coarser. The latter fact implies that the coordinate functionals are not ϑ-equicontinuous, and M is not ℓ^∞-barrelled. To show that M is C-barrelled, we need the following result (cf. [27] or [21], § 31.2 (3)), used by Sobczyk [60] to observe that c_0 is not complemented in ℓ^∞.

Theorem 6.23 (Phillips). *If $\{f_n\}_n$ is a $\sigma\left((\ell^\infty)', \ell^\infty\right)$-null sequence, then $\{f_n|_{c_0}\}_n$ converges to 0 in the ℓ^1-norm $\| \ \|_1$.*

The Hahn-Banach theorem yields

Corollary 6.24. *If $\{p_n\}_n$ is a pointwise null sequence in $\mathfrak{p}(\ell^\infty, \| \ \|)$, then $\lim_n \sup\{p_n(x) : x \in B \cap c_0\} = 0$.*

Example 6.25. M is C-barrelled but not ℓ^∞-barrelled. Compatibility with the Banach space ℓ^∞ensures that M has property (C) and is non-S_σ; it has precisely the properties in $L_{2,4}$.

Proof. Let $\{p_n\}_n$ be a pointwise null sequence in $\mathfrak{p}(M) \subset \mathfrak{p}(\ell^\infty, \| \ \|)$. We must show that $\{p_n\}_n$ is uniformly bounded on some 0-neighborhood in M. Since the Banach space $(\ell^\infty, \| \ \|)$ is barrelled, $\{p_n\}_n$ is uniformly bounded on B. It remains to show that $\{p_n\}_n$ is uniformly bounded on some set of the form $U_t \cap c_0$. Put $a_n = 1/n + \sup\{p_n(x) : x \in B \cap c_0\}$. By the Corollary, $\lim_n a_n = 0$, and by definition $\{a_n^{-1} p_n\}_n$ is uniformly bounded by 1 on $B \cap c_0$. Because J and M induce the same property (L) topology on c_0, Theorem 2.5(*4*) implies $\{p_n|_{c_0}\}_n = \{a_n(a_n^{-1} p_n|_{c_0})\}$ is equicontinuous, and is uniformly bounded on some set of the form $U_t \cap c_0$, as required. □

Obversely, M proves the Phillips result: If $\{f_n\}_n$ is $\sigma\left((\ell^\infty)', \ell^\infty\right)$-null, its polar contains some $U_t \cap c_0$, since M is c_0-barrelled. We may assume $t(n+1) \geq t(n)$ for each n. If $A_n = t(n) \cdot B \cap \operatorname{sp}\{e_1, \ldots, e_n\}$, then $t(n) \cdot B \cap c_0 = A_n + C_n$ for a unique set C_n, and $C_n \subset U_t \cap c_0$. For all sufficiently large i we have f_i numerically bounded by 1 on both A_n and C_n; hence $\|f_i|_{c_0}\|_1 \leq 2/t(n)$.

An independent proof that M has property (L): Since J and M induce the same property (L) topology on c_0, and since by definition M/c_0 coincides with the Banach quotient space ℓ^∞/c_0, our M must have the three-space property (L).

M is vital to our proof in [54] that C- and c_0-*barrelled* are *not* three-space properties.

A space E is ℓ^∞-barrelled if and only if, given any $\sigma(E', E)$-bounded sequence $\{g_n\}_n$, the mapping $x \mapsto \{g_n(x)\}_n$ from E into ℓ^∞ is continuous. As in the c_0-*barrelled* analog, we note that *an ℓ^∞-barrelled space E is isomorphic to a subspace of ℓ^∞ if and only if there exists a $\sigma(E', E)$-bounded sequence with bounded polar.*

Since any separable Banach space E is an ℓ^∞-barrelled space whose unit ball is the (bounded) polar of a $\sigma(E', E)$-bounded sequence, we have the familiar fact that *every separable Banach space is (isomorphically) contained in ℓ^∞.* Similarly, *the strong dual of any separable Banach space is contained in ℓ^∞.*

Let each E_n be a Banach space with unit ball B_n such that infinitely many of the E_n cannot be embedded in ℓ^∞. [Such examples are well known; e.g., $c_0(I)$ or $\ell^p(I)$ $(1 < p < \infty)$, where I is an uncountable indexing set. We recall $\ell^1(\mathbb{R})$ *is* isomorphic to a subspace of ℓ^∞.] Let K denote the direct sum $\oplus_n E_n$ with dual $K' \approx \prod_n E'_n$ and consider the vector topology ζ on K with 0-neighborhood base all sets U of the form

$$U = [\epsilon(B_1 \oplus \cdots \oplus B_k) \oplus (\oplus_{n>k} E_n)] \bigcap C$$

where $\epsilon > 0$, $k \in \mathbb{N}$, and C is the polar of a $\sigma(K', K)$-bounded sequence.

Example 6.26. Dominated by a strict (LB)-space, ℓ^∞-barrelled, and not inductive, (K, ζ) thus enjoys precisely the properties of columns 4 and 5.

Proof. Repeat the proof of Example 6.12 word-for-word with K, C, ζ and ℓ^∞ replacing H, D, η and c_0, respectively. □

Observe that if all but finitely many of the E_n could be embedded in ℓ^∞, then ζ would just be the (barrelled) direct sum topology. Schmets is credited [26, 8.2.38] with an early, more intricate example of a non-inductive ℓ^∞-barrelled space. Any such example is not even quasi-inductive, since [*quasi-inductive* $\wedge$ *primitive*] $\Rightarrow$ *inductive.* (See [54] for a brief discussion of weak quasibarrelledness.)

The strong dual E' of a non-distinguished Fréchet space E is $\aleph_0$-barrelled but not barrelled. However, we desire an example dominated by a Banach space, and E' never is: $\mathbb{K}^{\mathbb{N}}$ is a quotient of the non-normable E (Eidelheit), hence E' contains an $\aleph_0$-dimensional subspace on which every linear form is continuous, which disallows metrizable domination.

For such an example, let E be a Banach space satisfying the following two conditions: (1) *E is not embedded in ℓ^∞*, and (2) *If A is a separable subset of $(E', \sigma(E', E))$, then so is every $D \subset A$.* Give E the topology γ with 0-neighborhood base all polars C of $\sigma(E', E)$-bounded sequences S. For instance, one could take E to be any non-separable reflexive Banach space, as in the example at the end of §29.4 in [21]: None of the polar sets S° can be bounded, for then $S^{\circ\circ}$ would be absorbing and strongly separable in E' by reflexivity, making the strong dual itself separable, which contradicts non-separability of E [71, Problem 4, p. 144]. This verifies (1). Similar reasoning shows the set $A^{\circ\circ}$ is separable in the metrizable topology $\beta(E', E)$, thus so is every subset D, thus D is also $\sigma(E', E)$-separable; i.e., (2) holds. We affirm [26, 8.3.27]: (E, γ) is a (DF)-space that is not quasibarrelled. However, example [26, 8.3.6(d)] is false unless emended so as to have E satisfy (1).

The non-reflexive Banach space $c_0(I)$ $(|I| > \aleph_0)$ also satisfies (1) and (2).

Example 6.27. Dominated by a Banach space, (E, γ) is a non-barrelled $\aleph_0$-barrelled space, and thus enjoys precisely the properties of rows 2-5.

Proof. Condition (1) is [necessary and] sufficient to ensure (E, γ) is not barrelled.

Suppose $U = \bigcap_n U_n$ is absorbing, where each U_n is an absolutely convex closed neighborhood in (E, γ). Each $U_n = U_n^{\circ\circ}$ contains the polar C_n of a $\sigma(E', E)$-bounded sequence, and since the closed absolutely convex hull of any countable set is separable, C_n° is $\sigma(E', E)$-separable, as is the subset U_n° by (2). Thus there is a countable set S_n with $S_n^{\circ\circ} = U_n^\circ$, so that $S_n^\circ = S_n^{\circ\circ\circ} = U_n^{\circ\circ} = U_n$, which shows that $U = \bigcap_n S_n^\circ = (\bigcup_n S_n)^\circ$ is a basic γ-neighborhood of 0. □

Given its Mackey topology, (E, γ) becomes our original Banach space, but Saxon-Tweddle [56] managed to find a non-barrelled $\aleph_0$-barrelled *Mackey* space dominated by a Banach space.

Let E be a Banach space with unit ball B and with sequences of closed subspaces $\{E_n\}_n$ and $\{F_n\}_n$ such that $E = E_1 \oplus F_1$ with $E_1 \bigcap F_1 = \{0\}$, and such that each $F_n = E_{n+1} \oplus F_{n+1}$ with $E_{n+1} \bigcap F_{n+1} = \{0\}$. We further stipulate that no E_n can be embedded in ℓ^∞. There are many possible choices; for instance, E could be $\ell^1(I)$ with $|I| > \mathfrak{c}$ and each $E_n \approx F_n \approx \ell^1(I)$.

Let $\mathcal{T}$ denote the vector topology on E with 0-neighborhood base all sets V of the form

$$V = [\epsilon B + F_k] \bigcap C,$$

where $\epsilon > 0$, $k \in \mathbb{N}$, and C is the polar of a countable $\sigma(E', E)$-bounded set.

Example 6.28. $(E, \mathcal{T})$ is an ℓ^∞-barrelled space compatible with a Banach space, but does not have property (L). It has precisely the properties in $L_{4,3}$.

Proof. Compatibility with the pairing $\langle E, E' \rangle$ is clear and ensures ℓ^∞-barrelledness by definition of $\mathcal{T}$. It also ensures the same closed absolutely convex sets for $\mathcal{T}$ as for the norm topology, so that the $\mathcal{T}$-closure U_n of $B + F_n$ is contained in $2B + F_n$, and $\{U_n\}_n$ is a sequence of closed absolutely convex $\mathcal{T}$-neighborhoods of 0 whose intersection is absorbing (contains B). Set $U = \bigcap_n nU_n$. Let V be of the above form for arbitrarily given ϵ, k and C, and fix $n > k$. Topological complementation in the Banach space E ensures $B \subset A + D$ for suitable *norm-bounded* sets $A \subset E_1 + \cdots + E_{n-1} + F_n$ and $D \subset E_n$. Clearly,

$$U \bigcap E_n \subset nU_n \bigcap E_n \subset (2nB + F_n) \bigcap E_n \subset (2nA + 2nD + F_n) \bigcap E_n = 2nD$$

is norm-bounded, while $V \bigcap E_n = C \bigcap E_n$ is not, by hypothesis on E_n. Therefore $V \not\subset U$, and Theorem 2.5 implies $(E, \mathcal{T})$ does not have property (L). □

Now let $E, E', B, \{E_n\}_n, \{F_n\}_n, \mathcal{T}$ be as in the last example, with the additional requirement that each point of E be in almost all of the sets $Q_n = E_1 + \cdots + E_n + (B \bigcap E_{n+1}) + F_{n+1}$. Again, there are many valid choices, including $E = \ell^1(I) \approx E_n \approx F_n$ $(|I| > \mathfrak{c})$. Let $\|\ \ \|$ denote the norm on E.

Example 6.29. $(E, \|\mathcal{T}\|)$ is an ℓ^∞-barrelled space which is dominated by a Banach space and has property (L), but is not C-barrelled. It has precisely the properties in $L_{3,3}$.

Proof. With E' as dual, $(E, \|\mathcal{T}\|)$ is ℓ^∞-barrelled, Banach space dominated, and has property (L).

Each Q_n is absolutely convex and, by complementation and compatibility, is a closed 0-neighborhood for $\mathcal{T}$, and thus also for $\|\mathcal{T}\|$. By hypothesis each point is in almost all the Q_n, and thus is absorbed by $Q = \bigcap_n Q_n$. To prove $(E, \|\mathcal{T}\|)$ is not C-barrelled, it suffices to show that $Q = \bigcap_n Q_n$ does not contain $W = \bigcap_n (t_n B + V_n)$ for $\{t_n\}_n$ a sequence of positive scalars diverging to ∞ and each V_n a $\mathcal{T}$-neighborhood of 0. Choose r such that $n > r$ implies $t_n \geq 2$. Then $W \supset \left(\bigcap_{n \leq r} V_n\right) \bigcap 2B \supset V \bigcap 2B$ for some basic $\mathcal{T}$-neighborhood V of 0. For some k and some polar C of a $\sigma(E', E)$-bounded sequence, then, $W \supset F_k \bigcap C \bigcap 2B$. Therefore

$$W \bigcap E_{k+1} \supset E_{k+1} \bigcap C \bigcap 2B.$$

Since $E_{k+1} \bigcap C$ is unbounded and balanced, it contains points of norm 2, and thus so does $W \bigcap E_{k+1}$. But $Q \bigcap E_{k+1} \subset Q_k \bigcap E_{k+1} = B \bigcap E_{k+1}$ contains no points exceeding 1 in norm, and $W \not\subset Q$ follows. □

Levin-Saxon [24] gave a class of non-barrelled Mackey ℓ^∞-barrelled spaces which, reasoning as below, are not $\aleph_0$-barrelled. Valdivia provided another such class (cf. [26], 8.2.39). By our answer to the Ferrando/López Pellicer question ([52], or box 2), these are classes of ℓ^∞-barrelled spaces which are C-barrelled but not $\aleph_0$-barrelled. Again, however, none of these spaces is dominated by a Banach space, since they all contain $\aleph_0$-dimensional subspaces that make every linear form continuous. Our adaptation below *does* admit Banach space domination.

Let $E = \ell^1(I)$, I uncountable, and identify E' with $\ell^\infty(I)$ in the usual manner. Let F be those points of E' having at most countably many non-zero coordinates.

Example 6.30. Dominated by a Banach space, $(E, \mu(E, F))$ is ℓ^∞-barrelled and, being Mackey, is C-barrelled, but is not $\aleph_0$-barrelled. It has precisely the properties in $L_{2,3}$.

Proof. Since a countable set in F has the same bipolar for both pairings $\langle E, F\rangle$ and $\langle \ell^1(I), \ell^\infty(I)\rangle$, the $\mu(E, F)$ topology is ℓ^∞-barrelled.

We know the unit ball B in $\ell^1(I)$ is $\sigma(\ell^1(I), c_0(I))$-compact. But B is also a subset of F, and $E = \ell^1(I)$ is a subset of $c_0(I)$, so B is compact in the coarser $\sigma(F, E)$ topology. For each n,

$$M_n = \{x \in F : \text{precisely } n \text{ coordinates of } x \text{ are } 1, \text{ the rest are } 0\} \subset nB.$$

Hence the absolutely convex $\sigma(F, E)$-closed hull $M_n^{\circ\circ}$ of M_n is $\sigma(F, E)$-compact, and M_n° is an absolutely convex closed $\mu(E, F)$-neighborhood of 0. Furthermore, $U = \bigcap_n M_n^\circ = \left(\bigcup_n M_n\right)^\circ$ is absorbing in E, for it contains B. Yet U is not a $\mu(E, F)$ neighborhood of 0, for if it were, then $U^\circ = \left(\bigcup_n M_n\right)^{\circ\circ}$ would be $\sigma(F, E)$-compact, and the $\sigma(F, E)$-closure of $\bigcup_n M_n$ would coincide with its $\sigma(\ell^\infty(I), E)$-closure. Coincidence fails, since the latter closure contains the point x with all coordinates 1, and $x \notin F$. Therefore the space fails to be $\aleph_0$-barrelled. $\square$

We begin our final assault on the main result.

Lemma 6.31. *For $i = 1, \ldots, 5$ there exists a space $\mathcal{R}_i$ that enjoys precisely the properties in rows i-5.*

Proof. Let $\mathcal{R}_1$ be any BL space. Choose $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$ as in Example 6.27, 6.16, 6.18, 6.4, respectively. $\square$

Lemma 6.32. *For $i = 1, \ldots, 5$ there exists a space $\mathcal{C}_i$ that enjoys precisely the properties in columns i-5.*

Proof. For $i = 1, 2, 3$ let $\mathcal{C}_i$ be any $(LF)_{4-i}$-space [47]. Choose $\mathcal{C}_4$, $\mathcal{C}_5$ as in Example 6.26, 6.5, respectively. $\square$

Corollary 6.33. *For each $b \in \mathcal{P}$ there exists a space $\mathcal{E}(\ulcorner b)$ which enjoys precisely the properties in $\ulcorner b$.*

Proof. Let (i,j) denote the position of b. For $(i,j) = (2,5)$, take $\mathcal{E}(\ulcorner b) = \mathcal{C}_5$, and for the fifteen other possible values of (i,j), take $\mathcal{E}(\ulcorner b) = \mathcal{R}_i \times \mathcal{C}_j$, applying the Lemmas and Theorem 6.1(i). □

Lemma 6.34. *Suppose property b is in row i and column j. There exists a space ${}^bE$, also denoted ${}^{i,j}E$, which enjoys precisely the properties in ${}^bs$ $(= {}^{i,j}s)$. And if $b \neq QB$, there exists a space E_b, also denoted $E_{i,j}$, which enjoys precisely the properties in L_b $(= L_{i,j})$.*

Proof. I. *Choices for ${}^bE$.*

- If $(i,j) \neq (3,3), (3,4)$, then ${}^bs = \ulcorner b$; set ${}^bE = \mathcal{E}(\ulcorner b)$.
- Since ${}^{3,3}s = \ulcorner(1,3) \bigcap L_{3,5}$, set ${}^{3,3}E = \mathcal{C}_3 \times G$ (Example 6.22, Theorem 6.1).
- Since ${}^{3,4}s = \ulcorner(2,4) \bigcap L_{3,5}$, set ${}^{3,4}E = \mathcal{C}_4 \times G$.

II. *Choices for E_b, given $(i,j) \neq (1,2)$.*

- If b is in the fifth row, then $i = 5$ and $2 \leq j \leq 5$. For $j = 2,3,4$, take $E_b = \mathcal{C}_{j+1}$. If $i = j = 5$, then $L_b = \emptyset$ and we let E_b be any non-primitive space; e.g., any metrizable $\aleph_0$-dimensional space.
- If b is in the fifth column and not in the fifth row, then $j = 5$ and $2 \leq i \leq 4$. For $i = 2,4$, take $E_b = \mathcal{R}_{i+1}$. For $i = 3$, set $E_b = G$ (Example 6.22).
- If b $(\neq QB)$ is in the first row, then either $b = BL$ and we take $E_b = \mathcal{C}_2$, or $b =$ *barrelled* and we take $E_b = \mathcal{R}_2$.
- For b in the second row and not the fifth column, set $E_{2,3} =$ the space of Example 6.30 and $E_{2,4} =$ the space M of Example 6.25.
- For b in the third row and not the fifth column, set $E_{3,3} =$ the space of Example 6.29 and $E_{3,4} = (E, \|\sigma(E,E')\|)$ with E any infinite-dimensional Banach space (Example 6.11).
- For b in the fourth row and not the fifth column, set $E_{4,3} =$ the space in Example 6.28 and $E_{4,4} =$ any of the spaces provided in Example 6.6. □

Theorem 6.35. *If $S \neq L_{QB}$ is a radial set, then there is a space $\mathcal{E}(S)$ that enjoys precisely the properties in S.*

Proof. Choices for $\mathcal{E}(\ulcorner b)$ may differ from some in the proof of Corollary 6.33. Let the choices for ${}^bE, E_b, \mathcal{R}_i, \mathcal{C}_j$ be fixed as above. Our general strategy: Either S is empty, or there is a largest element in $S \bigcap \mathcal{P}_0$, where $\mathcal{P}_0$ is the linearly ordered subset of $\mathcal{P}$ consisting of rows 1 and 2 and column 5. The following cases are exhaustive and mutually exclusive.

A. For $S = \emptyset$, let $\mathcal{E}(S) = E_{5,5}$.

B. If S contains a point in the first row, then S ($\neq L_{QB}$) is $\ulcorner BL$, $\ulcorner QB$, or $\ulcorner$*barrelled* and we may choose $\mathcal{E}(S)$ as in Corollary 6.33.

C. If S contains no points of the first row but contains $\aleph_0$-*barrelled*, then S is either $L_{1,3}$ or ${}^{2,3}s$ and we choose $\mathcal{E}(S)$ to be $E_{1,3}$ or ${}^{2,3}E$, respectively.

D. If S contains no point a of the interval $BL \geq a \geq \aleph_0$-*barrelled* but does contain ℓ^∞-*barrelled*, then either $S = L_{i,3}$ $(2 \leq i \leq 5)$ and we set $\mathcal{E}(S) = E_{i,3}$; or $S = L_{i,3} \backslash \{non\text{-}S_\sigma\}$ $(2 \leq i \leq 4)$, and we set $\mathcal{E}(S) = \varphi \times E_{i,3}$, using Theorem 6.1($ii$).

E. Suppose S contains no point a of the interval $BL \geq a \geq \ell^\infty$-*barrelled* but does contain *property (C)* <u>and</u> *non-*S_σ. Then $L_{2,4} \supset S \supset L_{4,4}$ and there are six possible subcases, each determined by the four or fewer points in $S \backslash L_{4,4}$.

(1) $S = L_{4,4}$: Take $\mathcal{E}(S) = E_{4,4}$.
(2) $S = L_{3,4}$: Take $\mathcal{E}(S) = E_{3,4}$.
(3) $S = L_{2,4}$: Take $\mathcal{E}(S) = E_{2,4}$.
(4) $S = L_{2,4} \bigcap L_{4,3}$: Take $\mathcal{E}(S) = (E, \xi)$ as in Example 6.13.
(5) $S = L_{2,4} \bigcap L_{3,3}$: Take $\mathcal{E}(S) = (E, \|\xi\|)$ as in Example 6.14.
(6) $S = L_{3,4} \bigcap L_{4,3}$: Take $\mathcal{E}(S) = E_{3,4} \times E_{4,3}$. The weak dual is that of a Banach space, so *property (C)* is preserved and Theorem 6.1(i) applies.

F. If *non-*$S_\sigma \notin S$ and $S^+ := S \bigcup \{non\text{-}S_\sigma\}$ is one of the sets considered in subcases (1-6), so that $\mathcal{E}(S^+)$ has already been chosen, then $\mathcal{E}(S) := \varphi \times \mathcal{E}(S^+)$ enjoys precisely the properties in S (Theorem 6.1(ii)).

Note that (A-F) includes all cases with $\{$*property (C)*, *inductive*$\} \subset S$.

G. If $S = \langle$*property (C)*, c_0-*barrelled*$\rangle$, take $\mathcal{E}(S) = (H, \eta)$ (Example 6.12).

H. If $S = \langle$*property (C)*, mc_0-*barrelled*$\rangle$, take $\mathcal{E}(S) = (H, \eta) \times (E, \|\sigma(E, E')\|)$, where E is any infinite-dimensional Banach space. Then the weak dual is that of an (LB)-space, so that $\mathcal{E}(S)$ has property (C); in fact, has precisely the properties in S, by Theorem 6.1(i).

I. If $S = \langle$*property (C)*, φ-*complemental*$\rangle$, take $\mathcal{E}(S) = (H, \eta) \times (E, \sigma(E, E'))$, using previous argument and choices of E and H (see Examples 6.6 and 6.12).

J. If $S = \langle$*property (C)*$\rangle$, take $\mathcal{E}(S) = \mathcal{C}_5$.

This exhausts the cases with *property (C)* $\in S$.

K. If *property (C)* $\notin S$ and *property (S)* $\in S$, then $S^+ := S \bigcup \{$*property (C)*$\}$ is one of the cases considered in (E-J) and $\mathcal{E}(S^+)$ has already been chosen; we set $\mathcal{E}(S) = E_{2,5} \times \mathcal{E}(S^+)$ and apply Theorem 6.1(i).

L. If *property (S)* $\notin S$ and $dlc \in S$, then $S^+ := S \bigcup \{$*property (S)*$\}$ fits the previous case where $\mathcal{E}(S^+)$ was chosen, and we set $\mathcal{E}(S) = E_{3,5} \times \mathcal{E}(S^+)$.

M. If $dlc \notin S$ and $S \neq \emptyset$, then $S = \ulcorner(5, j)$ with $2 \leq j \leq 5$: Take $\mathcal{E}(S)$ as in Corollary 6.33. $\square$

Let us relate Kalton's G-barrelled spaces [20] to Fig. 1.3. Recall that a space E is *G-barrelled* if every G-barrel is a neighborhood of 0, where a barrel U in E is a *G-barrel* if the normed space $E_{(U)}$ is separable [26, 4.1.24]. For Mackey spaces there is nothing left to prove, since we have [26, 4.1.26]: *A Mackey space is G-barrelled if and only if it has property (S).* Or, in light of Fig. 1.3, *A Mackey space E is G-barrelled* $\Leftrightarrow \mathbf{P}(E) \supset {}^{\ulcorner}(3,3)$. For general spaces E the best we can write is

$$\mathbf{P}(E) \supset {}^{\ulcorner}(2,4) \implies E \text{ is G-barrelled} \implies \mathbf{P}(E) \supset {}^{\ulcorner}(3,4)\,.$$

Both implications are essentially known. The first follows since G-barrels in E are polars of $\sigma(E', E)$-bounded sequences. For the second, observe that (i) if U is the polar of a $\sigma(E', E)$-Cauchy sequence, then U is a G-barrel, since $E_{(U)}$ is routinely norm-isomorphic to a subspace of the separable Banach space c, and (ii) [*Every $\sigma(E', E)$-Cauchy sequence is equicontinuous*] $\Leftrightarrow$ [*E is c_0-barrelled and has property (S)*] $\Leftrightarrow$ [$\mathbf{P}(E) \supset {}^{\ulcorner}(3,4)$].

We are left to prove the radial sets ${}^{\ulcorner}(2,4)$ and ${}^{\ulcorner}(3,4)$ are optimal. As products preserve *G-barrelled*, $E := J \times \mathcal{C}_4$ is G-barrelled, with $\mathbf{P}(E) = {}^{\ulcorner}(3,4)$ by Theorem 6.1(i). Therefore, in the display, no larger subset of $\mathcal{P}$ can replace ${}^{\ulcorner}(3,4)$. Nor can a properly smaller radial set S replace ${}^{\ulcorner}(2,4)$: Clearly, $L_{2,4} \bigcap L_{4,3} \supset S$, and the space (E, ξ) of Example 6.13 verifies $\mathbf{P}(E, \xi) = L_{2,4} \bigcap L_{4,3}$ but is not G-barrelled because, transparently, the unit ball of the strictly dominating Banach space is a G-barrel that is not a ξ-neighborhood of zero. Thus the displayed results are best possible; G-barrelled spaces in general, unlike the Mackey case, are not characterized by the property set $\mathbf{P}(E)$ alone.

7. About sBL

The metrizable spaces X_1 and X_2 of [16, p. 59] show that *sBL* does not belong in either of the large boxes 3 and 5. Moreover, Corollary 3.13 would fail if *sBL* were designated a fifth-row property, as our next (counter)Example shows.

Fix a finite $p > 1$, define $p_n = \frac{np+1}{n+1}$ for each $n \in \mathbb{N}$, and let ℓ^{p-} denote the inductive limit of the Banach spaces ℓ^{p_n} (see [47, 48]). The barrelled space ℓ^{p-} is non-S_σ, for it contains a dense subspace dominated by the Banach space ℓ^1, and is not sBL, as evidenced by the closed absorbing sequence $\{B_n\}_n$, where B_n denotes the unit ball in ℓ^{p_n}. Let F denote ℓ^{p-} with the metrizable topology it inherits from the Banach space ℓ^p. The above arguments show that F is non-S_σ and not sBL; in fact,

Example 7.1. The Banach space ℓ^p is sBL, and every subspace between it and the dense F (as above) is non-S_σ. Nevertheless, F is not sBL.

A third reason to omit *sBL* from Fig. 1.3: Basic Theorem 5.1 would broadly fail if *sBL* were a sanctioned fifth-row property.

Example 7.2. Every infinite-dimensional Fréchet space (E, ξ) has a CE (E, η) which is not sBL but is non-S_σ.

Proof. As $\dim(E) \geq \mathfrak{c} = \aleph_0 \cdot \mathfrak{c}$, the Baire space (E, ξ) is covered by an increasing sequence of $\mathfrak{c}$-codimensional subspaces one of which, let us call it M, is both dense and Baire, hence dense and non-S_σ. If N is an algebraic complement of M, there is an algebraic isomorphism θ from N onto the space $F = \bigcup_n \ell^{p_n}$ of the previous Example. Define $h_i \in E^*$ so that (1) $h_i(M) = \{0\}$ and (2) $h_i|_N = \phi_i \circ \theta$, where ϕ_i is the i^{th} coordinate functional on F $(i = 1, 2, \dots)$. Since each B_n is the polar in F of a set in $\mathrm{sp}\{\phi_i\}_i$, each $\theta^{-1}(B_n) + M$ is the polar in E of the corresponding set in $\mathrm{sp}\{h_i\}_i := H$, so $\theta^{-1}(B_n) + M$ is $\sigma(E, E' + H)$-closed. Now H is transverse to E' by (1); let (E, η) be the corresponding CE. The closed absorbing sequence $\left\{\theta^{-1}(B_n) + M\right\}_n$ shows that (E, η) is not sBL.

By construction, $M = H^\perp$ is a closed non-S_σ (Baire, even) subspace of (E, η), and $(E, \eta)/M$ is a continuous linear image of the non-S_σ space F, hence is non-S_σ. Therefore the third space (E, η) is non-S_σ [54]. $\square$

Saxon and Narayanaswami [47] proved that *a barrelled space is QB if and only if it does not contain a complemented copy of* φ. Saxon [39] proved that *a barrelled space is BL if it does not contain a copy of* φ (see [15, 23, 26]). The space (E, γ) of Example 6.27 shows that, in both these results, the *barrelled* hypothesis is optimal. Indeed, (E, γ) enjoys the properties in $L_{1,3}$, and, dominated by a Banach space, cannot contain any copy of φ whatsoever, yet fails to be either BL or QB because it is not barrelled.

However, Banach domination also implies (E, γ) is sBL and non-S_σ, begging the optimality question when *BL* and *QB* are replaced by *sBL* and *non-S_σ*, respectively. The latter case is easily dispatched.

Theorem 7.3. *A φ-complemental space E is non-S_σ if and only if it does not contain a complemented copy of φ.*

Proof. E contains a complemented copy of φ if and only if E contains a closed $\aleph_0$-codimensional subspace, if and only if E is S_σ. $\square$

Certainly, *φ-complemental* is the unique optimal hypothesis: Since its topology is so weak, $\mathcal{C}_5$ contains no copy of φ.

Finally, we determine the best hypothesis for the result *if E is barrelled, then E is sBL provided E does not contain (a copy of) φ.* Initially, Saxon [39, Corollary 2.2] relaxed the *barrelled* hypothesis to *ℓ^∞-barrelled*, then Kąkol relaxed it further to *c_0-barrelled*, Ruess replaced it with *property (L)*. The last two versions are optimal in the old fourteen-property context. The current expanded context allows a single best version, easily derived from Saxon's original.

Theorem 7.4. *If E is mc_0-barrelled and does not contain φ, then E is sBL.*

Proof. Suppose E is not sBL. Then there exists a closed absorbing sequence $\{B_n\}_n$ and a sequence $\{x_n\}_n$ such that each $x_n \in B_{n+1} \setminus \operatorname{sp}(B_n)$. Let p be an arbitrary seminorm defined on the span S of $\{x_n\}_n$ and define $A_n = nB_n$, so that $\{A_n\}_n$ is a closed absorbing sequence and each $x_n \in A_{n+1} \setminus \operatorname{sp}(A_n)$. The Lemma internal to the proof of [39, Theorem 2.1] provides a sequence $\{f_n\}_n \subset E'$ which easily satisfies the following two conditions: (i) Each f_n is in A_n°, hence $\{f_n\}_n$ is $\sigma(E', E)$-bounded and defines a seminorm q on E given by $q(x) = \sup_n |f_n(x)|$ for all $x \in E$; and (ii) $q(x) \geq p(x)$ for each $x \in S$. It follows that S has its strongest locally convex topology and is isomorphic to φ provided we can show that q is continuous, and for this we just need equicontinuity of $\{f_n\}_n$.

But $f_n \in A_n^\circ$ means that $nf_n \in B_n^\circ$. Therefore $\{nf_n\}_n$ is $\sigma(E', E)$-bounded, and $\{f_n\}_n = \{n^{-1}nf_n\}_n$ is equicontinuous on the mc_0-barrelled space E. □

Example 7.5. Let E be the space obtained by giving ℓ^{p-} its weak topology. Then E cannot contain φ, yet by duality invariance is not sBL and retains the properties in $L_{4,4}$. Consequently, of the sixteen properties, mc_0-*barrelled* is the one and only optimal hypothesis for the space of the previous Theorem.

Both hypothesis and conclusion are duality invariant in

Corollary 7.6. *If E is dlc and contains no infinite-dimensional subspace F with $F' = F^*$, then E is sBL.*

Proof. E is mc_0-barrelled under its Mackey topology (box 2), and $\varphi' = \varphi^*$. □

Since $F' \neq F^*$ for every infinite-dimensional metrizable space F, either of Examples 7.1 and 7.2 shows that the Corollary fails when *dlc* is replaced by any fifth-row property.

8. Conclusion

Having demonstrated the efficacy of Fig. 1.3 over and again, we could review how happily the Saxon-Levin-Valdivia theorem generalized to Theorem 4.11. Or, behold now the quite recent and felicitous marriage with P-spaces [5, 6, 44].

Earlier collaborations showed how the two-dimensional picture of weak barrelledness simplifies when all spaces considered are Mackey [52], or metrizable [53], or Fréchet-Urysohn [17], or (LF)-spaces [47], or of the form $C_p(X)$ or $C_c(X)$ for X a Tichonov space [18]. E.g., $C_p(X)$ spaces partition $\mathcal{P}$ into two equivalence classes, *row 5* and $\mathcal{P}\setminus$ *row 5*, reducing Fig. 1.3 to the display [BL $\Rightarrow$ non-S_σ].

But for $L_m(X)$ spaces, Mackey duals of $C_p(X)$, three equivalence classes exist: $\mathfrak{E}_1 = \{$*BL*, *QB*, *non-S_σ*$\}$, $\mathfrak{E}_2 = \{$*barrelled*, *$\aleph_0$-barrelled*$\}$, and $\mathfrak{E}_3 =$

$\mathcal{P} \setminus (\mathfrak{C}_1 \bigcup \mathfrak{C}_2)$ [5, 44]. Fig. 1.3 becomes [BL $\Rightarrow$ barrelled $\Rightarrow \ell^\infty$-barrelled]. Moreover, $L_m(X)$ has (i) the properties in $\mathfrak{C}_1$ if and only if X is finite [5]; (ii) the properties in $\mathfrak{C}_2$ if and only if X is discrete; (iii) the (eleven equivalent) properties in $\mathfrak{C}_3$ if and only if X is a P-space (= G_δ sets are open).

The weak dual of $L_m(X)$ is $C_p(X)$. If infinite, X admits a disjoint sequence $\{U_n\}_n$ of nonempty open sets; each U_n admits $0 \neq f_n \in C_p(X)$ with $f_n(X \setminus U_n) = \{0\}$. Hence $L_m(X)$ is an S_σ-space by Theorem 2.1(3), proving (i).

Buchwalter and Schmets proved X is discrete if and only if $C_p(X)$ is quasicomplete [2], equivalently, $L_m(X)$ is barrelled [21, §23.6(4)], equivalently, $L_m(X)$ is $\aleph_0$-barrelled (by Saxon's $\mathfrak{C}_2$), which led to the proof that X is discrete if and only if every bounded σ-compact subset of $C_p(X)$ is relatively compact [6].

The fifth-column properties (in ${}^{2,5}s \subset \mathfrak{C}_3$) translate into classic P-space characterizations [6]. Residual properties in $\mathfrak{C}_3 \setminus {}^{2,5}s$ trigger new characterizations and related results [6, 44], including: X is a P-space if and only if $C_p(X)$ is σ-relatively sequentially complete; if relatively sequentially complete *bounded* sets countably cover $C_p(X)$, then X is finite. Corollaries: X is finite when $C_p(X)$ is σ-relatively countably compact (Ferrando/Kąkol); σ-countably compact (Tkachuk/Shakmatov); or σ-compact (Velichko) (see [15, Theorem 9.13]).

May such marital bliss abound yet more, its fruit blessed by Fig. 1.3, the nascent periodic table of weak barrelledness.

Acknowledgements. Thanks for the referee's helpful comments and a shorter proof of Theorem 6.5.

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