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The Mazur-Orlicz Theorem for Convex Functionals

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In this paper we extend the Mazur-Orlicz theorem from the sublinear functional to the convex functional.

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1. Introduction

In 1953, Mazur and Orlicz [1] published the famous theorem named after them, which is a useful tool in different branches of analysis and has a number of applications. In 2008, Simons [2] stated the following version of the Mazur-Orlicz theorem: If μ is a sublinear functional on a nonzero real vector space E and D is a nonempty convex subset of E, then there exists a linear functional L on E such that $L \leq \mu$ and

$$\inf_{d \in D} L(d) = \inf_{d \in D} \mu(d).$$

In this paper, inspired by the methods in [2], we show if ρ is a convex functional, there also exists an affine functional A satisfying $A \leq \rho$ and

$$\inf_{d \in D} A(d) = \inf_{d \in D} \rho(d).$$

2. Basic notions

Throughout this paper, E is a nonzero real vector space and $\mathbb{R}$ is the set of real numbers.

Definition 2.1. A convex functional on E is a map $\rho : E \to \mathbb{R}$ such that for any $x, y \in E$ and $\lambda \in [0, 1]$,

$$\rho\big(\lambda x + (1-\lambda)y\big) \leq \lambda\rho(x) + (1-\lambda)\rho(y).$$

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Definition 2.2. An affine functional on E is a map $A : E \to \mathbb{R}$ such that for any $x, y \in E$ and $\lambda \in [0, 1]$,

$$A\big(\lambda x + (1-\lambda)y\big) = \lambda A(x) + (1-\lambda)A(y).$$

Definition 2.3. A sublinear functional on E is a map $\mu : E \to \mathbb{R}$ such that for any $x, y \in E$ and $\lambda \in [0, 1]$,

$$\mu\big(\lambda x + (1-\lambda)y\big) \leq \lambda\mu(x) + (1-\lambda)\mu(y)$$

and for any $\tau \geq 0$,

$$\mu(\tau x) = \tau\mu(x).$$

Definition 2.4. A linear functional on E is a map $L : E \to \mathbb{R}$ such that for any $x, y \in E$ and $\lambda \in [0, 1]$,

$$L\big(\lambda x + (1-\lambda)y\big) = \lambda L(x) + (1-\lambda)L(y)$$

and for any $\tau \geq 0$,

$$L(\tau x) = \tau L(x).$$

Lemma 2.5. *A functional A is affine if and only if there exist a linear functional L and a constant $r \in \mathbb{R}$ such that for any $x \in E$,*

$$A(x) = L(x) + r.$$

Proof. $\Leftarrow$ This is obvious.

$\Rightarrow$ Let $L(x) := A(x) - A(0)$. Then L is an affine functional. For any $\lambda \in [0, 1]$, we have

$$\begin{aligned} L(\lambda x) &= A(\lambda x) - A(0) = \lambda A(x) + (1-\lambda)A(0) - A(0) \\ &= \lambda\big(A(x) - A(0)\big) = \lambda L(x). \end{aligned}$$

For any $\lambda > 1$, then $\frac{1}{\lambda} < 1$ and

$$L(\lambda x) = \frac{\lambda}{\lambda} L(\lambda x) = \lambda L(x).$$

Thus L is a linear functional and $A(x) = L(x) + A(0)$. $\square$

3. Main results

Lemma 3.1. *For a convex functional ρ, there exists an affine functional A such that $A(0) = \rho(0)$ and $A(x) \leq \rho(x)$ for all $x \in E$.*

Proof. Define $\rho'(x) = \rho(x) - \rho(0)$. Then ρ' is a convex functional and $\rho'(0) = 0$. Let $\mu(x) := \inf_{\tau>0} \frac{1}{\tau}\rho'(\tau x)$. We now show μ is a sublinear functional. Let $x \in E$. For all $\tau > 0$,

$$\rho'(\tau x) + \tau\rho'(-x) \geq (1+\tau)\rho'\left(\frac{1(\tau x) + \tau(-x)}{1+\tau}\right) = (1+\tau)\rho'(0) = 0.$$

Thus $\frac{1}{\tau}\rho'(\tau x) \geq -\rho'(-x)$. Taking the infimum over τ gives $\mu(x) \geq -\rho'(-x) > -\infty$. For any $\lambda \geq 0$, taking $\tau' = \lambda\tau$, $\mu(\lambda x) = \inf_{\tau>0} \frac{1}{\tau}\rho'(\tau\lambda x) = \lambda \inf_{\tau'>0} \frac{1}{\tau'}\rho'(\tau' x) = \lambda\mu(x)$ when $\lambda > 0$ and $\mu(0x) = \rho'(0) = 0$ when $\lambda = 0$. On the other hand, let $x, y \in E$. For any $\tau_1, \tau_2 > 0$, write $\theta := \frac{1}{2\tau_1} + \frac{1}{2\tau_2} > 0$. Then

$$\begin{aligned} &\frac{1}{2}\left(\frac{1}{\tau_1}\rho'(\tau_1 x)\right) + \frac{1}{2}\left(\frac{1}{\tau_2}\rho'(\tau_2 y)\right) \\ &\geq \theta\rho'\left(\frac{x+y}{2\theta}\right) = \theta\rho'\left(\frac{1}{\theta}\left(\frac{1}{2}x + \frac{1}{2}y\right)\right) \geq \mu\left(\frac{1}{2}x + \frac{1}{2}y\right). \end{aligned}$$

Taking the infimum over τ_1 and τ_2 gives $\frac{1}{2}\mu(x) + \frac{1}{2}\mu(y) \geq \mu(\frac{1}{2}x + \frac{1}{2}y)$. Thus μ is a sublinear functional from E to $\mathbb{R}$.

By the classical Hahn-Banach theorem for sublinear functionals (See Lemma 1.2 of chapter 1 in [2]), there exists a linear functional L such that $L \leq \mu$. Since $\mu \leq \rho'$, $L \leq \rho'$. Note that L is linear implies $L(0) = 0$. Thus $A(x) := L(x) + \rho(0)$ is the one as required. $\square$

Remark 3.2. Lemma 3.1 is actually saying that the algebraic subdifferential of ρ at 0 is nonempty. This can also be deduced from Theorem 2.1.13 in [3].

Lemma 3.3. *For a convex functional ρ on E and a nonempty convex set D, let $\beta := \inf_{d\in D}\rho(d)$. If $\beta \in \mathbb{R}$, then*

$$\sigma(x) := \inf_{\tau>0, d\in D} (1+\tau)\rho\left(\frac{x+\tau d}{1+\tau}\right) - \tau\beta$$

is a convex functional on E, $\sigma \leq \rho$ and $\inf_{d\in D}\sigma(d) = \beta$.

Proof. Let $x \in E$, $d \in D$ and $\tau > 0$. Since $(1+\tau)\rho\big((x+\tau d)/(1+\tau)\big) \leq \rho(x) + \tau\rho(d)$, $\sigma \leq \rho$. Given $\bar{d} \in D$, for all $d \in D$ and $\tau > 0$, we have $(1+\tau)\rho\big((\bar{d}+\tau d)/(1+\tau)\big) - \tau\beta \geq (1+\tau)\beta - \tau\beta = \beta$. Thus $\sigma(\bar{d}) \geq \beta$. Taking the infimum over $\bar{d} \in D$ gives $\inf_{d\in D}\sigma(d) \geq \beta$. Since $\sigma \leq \rho$, $\inf_{d\in D}\sigma(d) = \beta$. We now prove that σ is a convex functional on E. Let $\bar{d} \in D$. For all $d \in D$ and $\tau > 0$, since $(2\bar{d} + \tau d)/(2+\tau) \in D$,

$$\begin{aligned} &(1+\tau)\rho\left(\frac{x+\tau d}{1+\tau}\right) + \rho(2\bar{d} - x) \\ &\geq (2+\tau)\rho\left(\frac{x+\tau d + 2\bar{d} - x}{2+\tau}\right) = (2+\tau)\rho\left(\frac{\tau d + 2\bar{d}}{2+\tau}\right) \geq (2+\tau)\beta. \end{aligned}$$

Thus $(1+\tau)\rho\big((x+\tau d)/(1+\tau)\big)-\tau\beta \geq 2\beta-\rho(2\bar{d}-x)$. Taking the infimum over d and τ gives $\sigma(x) \geq 2\beta - \rho(2\bar{d}-x) > -\infty$. For any $x, y \in E$ and $\lambda \in [0,1]$, let $d_1, d_2 \in D$ and $\tau_1, \tau_2 > 0$ be arbitrary. Write $\theta := \lambda\tau_1 + (1-\lambda)\tau_2 > 0$, $z := \lambda x + (1-\lambda)y \in E$ and $d := \big(\lambda\tau_1 d_1 + (1-\lambda)\tau_2 d_2\big)/\theta \in D$. Then

$$\begin{aligned}
&\lambda(1+\tau_1)\rho\left(\frac{x+\tau_1 d_1}{1+\tau_1}\right) - \lambda\tau_1\beta + (1-\lambda)(1+\tau_2)\rho\left(\frac{y+\tau_2 d_2}{1+\tau_2}\right) - (1-\lambda)\tau_2\beta \\
&\geq (1+\theta)\rho\left(\frac{\lambda x + \lambda\tau_1 d_1 + (1-\lambda)y + (1-\lambda)\tau_2 d_2}{1+\theta}\right) - \theta\beta \\
&= (1+\theta)\rho\left(\frac{z+\theta d}{1+\theta}\right) - \theta\beta \\
&\geq \sigma(z) = \sigma\big(\lambda x + (1-\lambda)y\big).
\end{aligned}$$

Taking the infimum over $d_1, d_2 \in D$ and $\tau_1, \tau_2 > 0$ gives $\lambda\sigma(x) + (1-\lambda)\sigma(y) \geq \sigma\big(\lambda x + (1-\lambda)y\big)$. Thus σ is a convex functional. $\square$

Lemma 3.4. *For a convex functional ρ on E and a nonempty convex set D, let $\beta := \inf_{d\in D}\rho(d)$. If $\beta \in \mathbb{R}$, then there exists a convex functional π on E such that $\pi \leq \rho$ and for all $d \in D$, $2\pi(0) - \pi(-d) \geq \beta$.*

Proof. Let Δ stand for the set of all convex functionals δ on E such that $\delta \leq \rho$ and $\inf_{d\in D}\delta(d) = \beta$. Since $\rho \in \Delta$, $\Delta \neq \emptyset$. We first prove every nonempty totally ordered subset Σ of Δ has a lower bound in Δ. Let $\delta \in \Delta$ and $\bar{d} \in D$. For all $x \in E$,

$$\frac{1}{2}\delta(x) + \frac{1}{2}\delta(2\bar{d}-x) \geq \delta\left(\frac{x+2\bar{d}-x}{2}\right) = \delta(\bar{d}) \geq \beta.$$

Thus $\delta(x) \geq 2\beta - \delta(2\bar{d}-x) \geq 2\beta - \rho(2\bar{d}-x)$. Let $\widehat{\sigma}(x) := \inf\{\sigma(x) : \sigma \in \Sigma\}$. For all $x \in E$,

$$\widehat{\sigma}(x) \geq \inf_{\delta\in\Delta}\delta(x) \geq 2\beta - \rho(2\bar{d}-x) > -\infty.$$

We now prove $\widehat{\sigma}$ is convex. Let $x, y \in E$, $\sigma_1, \sigma_2 \in \Sigma$ be arbitrary. For any $\lambda \in [0,1]$, if $\sigma_1 \geq \sigma_2$, then

$$\begin{aligned}
&\lambda\sigma_1(x) + (1-\lambda)\sigma_2(y) \geq \lambda\sigma_2(x) + (1-\lambda)\sigma_2(y) \\
&\geq \sigma_2\big(\lambda x + (1-\lambda)y\big) \geq \widehat{\sigma}\big(\lambda x + (1-\lambda)y\big),
\end{aligned}$$

while if $\sigma_2 \geq \sigma_1$, then

$$\begin{aligned}
&\lambda\sigma_1(x) + (1-\lambda)\sigma_2(y) \geq \lambda\sigma_1(x) + (1-\lambda)\sigma_1(y) \\
&\geq \sigma_1\big(\lambda x + (1-\lambda)y\big) \geq \widehat{\sigma}\big(\lambda x + (1-\lambda)y\big).
\end{aligned}$$

In either case, $\lambda\sigma_1(x) + (1-\lambda)\sigma_2(y) \geq \widehat{\sigma}\big(\lambda x + (1-\lambda)y\big)$. Taking the infimum over Σ gives $\lambda\widehat{\sigma}(x) + (1-\lambda)\widehat{\sigma}(y) \geq \widehat{\sigma}\big(\lambda x + (1-\lambda)y\big)$. $\inf_{d\in D}\widehat{\sigma}(d) = \beta$ comes from $\inf_{d\in D}\sigma(d) = \beta$ for any $\sigma \in \Sigma$. Thus $\widehat{\sigma} \in \Delta$.

From Zorn's lemma, there exists a minimal element π of Δ. Let

$$\pi'(x) := \inf_{\tau>0, d\in D}(1+\tau)\pi\left(\frac{x+\tau d}{1+\tau}\right) - \tau\beta.$$

From Lemma 3.3, $\pi' \leq \pi$ and $\inf_{d\in D}\pi'(d) = \beta$. Thus $\pi' \in \Delta$. Since π is the minimal element in Δ, $\pi' = \pi$. Thus

$$\pi(x) = \inf_{\tau>0, d\in D}(1+\tau)\pi\left(\frac{x+\tau d}{1+\tau}\right) - \tau\beta.$$

For any $d \in D$, taking $\tau = 1$ gives $\pi(-d) \leq 2\pi(0) - \beta$. $\square$

We now give the main result of this paper.

Theorem 3.5 (A convex version of the Mazur-Orlicz theorem). *Let ρ be a convex functional on E and D be a nonempty convex subset of E. Then there exists an affine functional A on E such that $A \leq \rho$ and*

$$\inf_{d\in D} A(d) = \inf_{d\in D}\rho(d).$$

Proof. Let $\beta := \inf_{d\in D}\rho(d)$. If $\beta = -\infty$, the result comes directly from Lemma 3.1. We can suppose $\beta \in \mathbb{R}$. Define the auxiliary convex functional π as in Lemma 3.4. By Lemma 3.1, there exists an affine functional A such that $A(0) = \pi(0)$ and $A \leq \pi$. Since $\pi \leq \rho$, $A \leq \rho$. Let $d \in D$. Since $\frac{1}{2}A(d) + \frac{1}{2}A(-d) = A\left(\frac{1}{2}d + \frac{1}{2}(-d)\right) = A(0) = \pi(0)$, $A(d) + A(-d) = 2\pi(0)$ and so

$$A(d) = 2\pi(0) - A(-d) \geq 2\pi(0) - \pi(-d) \geq \beta.$$

Taking the infimum over $d \in D$,

$$\inf_{d\in D} A(d) \geq \beta = \inf_{d\in D}\rho(d).$$

On the other hand, since $A \leq \rho$,

$$\inf_{d\in D} A(d) \leq \inf_{d\in D}\rho(d).$$ $\square$

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References

[1] S. Mazur, W. Orlicz: Sur les espaces métriques linéaires II, Stud. Math. 13 (1953) 137–179.

[2] S. Simons: From Hahn-Banach to Monotonicity, Springer, Berlin (2008).

[3] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, River Edge (2002).