

Ellipsoidal Cones in Normed Vector Spaces

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The characterization of ellipsoids is intimately tied to characterizing the Banach spaces that are Hilbert spaces. We give two characterizations of cones over ellipsoids in real normed vector spaces. Let C be a closed convex cone with nonempty interior such that C has a bounded section of codimension 1. We show that C is a cone over an ellipsoid if and only if every bounded section of C has a center of symmetry. We also show that C is a cone over an ellipsoid if and only if the affine span of $\partial C \cap \partial(a - C)$ has codimension 1 for every point a in the interior of C . These results generalize the finite-dimensional cases proved in [4].

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1. Introduction

In a landmark paper [7], W. Rudin and K. T. Smith answered a question of J. Korevaar by showing that, if X is a strictly convex real Banach space of dimension $n \neq 2$ and, for each finite-dimensional subspace π in X , the best approximation function P_π is linear, then X is a Hilbert space. Their theorem led to the following characterization of ellipsoids: *If K is a centrally symmetric compact convex body in n -space (where n is possibly infinite) such that, for every ν -dimensional subspace π with $0 < \nu < n - 1$, the union of the tangency sets of all support planes of K which are translates of π lies in a plane of dimension $n - \nu$, then K is an ellipsoid.* This result is a vivid example of how the characterization of ellipsoids is intimately tied to characterizing the Banach spaces that are Hilbert spaces.

In [4], the second author, with J. Jerónimo, showed that a finite-dimensional pointed cone in which every bounded section has a center of symmetry is an ellipsoidal cone. Hence these are exactly the cones over closed unit balls of Hilbert

spaces that have been translated away from the origin. The primary goal of this paper is to generalize this result to infinite-dimensional vector spaces. While this may seem like a result that would follow from a straightforward induction argument (at least for a separable Banach space), such an approach is elusive. We give an affirmative answer to the infinite-dimensional generalization by carefully using the following well-known fact: *Cones with bounded sections have dual cones with nonempty interiors* (Cf. [3, Theorem 3.8.4]).

Fix a real normed vector space V . A *cone* in V is a nonempty convex subset $C \subset V$ that is closed under nonnegative scalar multiplication. The cone C is *pointed* if it contains no line through the origin; that is, if $C \cap (-C) = \{0\}$.

Given a convex subset $K \subset V$, let $\text{aff}(K)$ denote the affine span of K , and let $\text{lin}(K)$ be the linear span of K . We write $\text{int}(K)$ for the interior of K in V with respect to the norm on V . The *relative interior* $\text{relint}(K)$ and the *relative boundary* ∂K are the interior and boundary, respectively, of K with respect to $\text{aff}(K)$. The *cone over* K , denoted $\text{cone}(K)$, is the intersection of all cones containing K . A *section* of K is the nonempty intersection of K with a closed affine subspace of V . We call a section S of K *proper* if S has codimension 1 with respect to $\text{aff}(K)$.

While the concept of an ellipsoid in finite dimensions is well known, in infinite-dimensional vector spaces this requires careful definition. We define a subset $E \subset V$ to be an *ellipsoid* if, for some $x \in \text{aff}(E)$, there exists an inner product on the translation $\text{aff}(E) - x$ such that $E - x$ is the closed unit ball corresponding to this inner product. An *ellipse* is a 2-dimensional ellipsoid. A cone $C \subset V$ is *ellipsoidal* if some proper section of C is an ellipsoid.

A subset $S \subset V$ is *centrally symmetric* if there exists a point $x \in V$ such that $S - x = -(S - x)$. In this case, x is a *center of symmetry* of S , and x is the unique center of symmetry of S if S is bounded.

We are now ready to define the main concept in this paper.

Definition 1.1 (CSS Cones). Let $C \subset V$ be a cone. We say that C satisfies the *centrally symmetric sections (CSS) property* if

- (1) C is closed and $\text{relint}(C) \neq \emptyset$,
- (2) there exists a bounded proper section of C , and
- (3) every bounded proper section of C is centrally symmetric.

We call a cone with the CSS property a *CSS cone*.

Our main result is that the CSS cones in V are precisely the ellipsoidal cones.

Theorem 1.2 (proved on p. 804). *Let C be a cone in a normed vector space V . Then C is a CSS cone if and only if C is an ellipsoidal cone.*

The proof that finite-dimensional CSS cones are ellipsoidal appeared in [4].

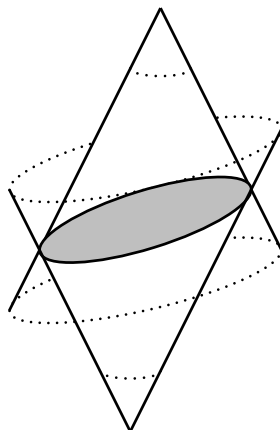


Figure 1.1: An FBI cone C and a translation of $-C$. The shaded region is the convex hull of the intersection of the boundaries. The FBI property implies that this convex hull is contained in a hyperplane.

That article also established another characterization of finite-dimensional ellipsoidal cones: Such a cone is ellipsoidal if and only if it is a so-called *FBI cone*.

Definition 1.3. Let $C \subset V$ be a cone. We say that C satisfies the *flat boundary intersections (FBI) property* if

- (1) C is closed and $\text{relint}(C) \neq \emptyset$,
- (2) there exists a bounded proper section of C , and
- (3) for each $a \in \text{relint}(C)$, some proper section of C contains $\partial C \cap \partial(a - C)$.

We call a cone with the FBI property an *FBI cone*. (See Figure 1.1.)

Theorem 1.4 (proved on p. 805). *Let C be a cone in a normed vector space V . Then C is a FBI cone if and only if C is an ellipsoidal cone.*

Unlike the proof of the CSS characterization of ellipsoidal cones, the proof of the FBI characterization carries over with very little change to the infinite-dimensional setting. We give this proof in Section 4. In [4], the proof that CSS cones in $\mathbb{R}^n$ are ellipsoidal proceeded by showing that CSS cones are FBI cones, so the proof that FBI cones are ellipsoidal was key to the argument. Unfortunately, the proof in [4] that CSS cones are FBI cones relied on the existence of a Lebesgue measure, so a different strategy is needed to prove that CSS cones are ellipsoidal in infinite-dimensional normed vector spaces.

As a corollary of Theorems 1.2 and 1.4, we get two characterizations of those normed vector spaces that are inner-product spaces.

Corollary 1.5. *Let V be a normed vector space. Then the following are equivalent.*

- (1) V is an inner product space.
- (2) V contains a CSS cone with nonempty interior.
- (3) V contains an FBI cone with nonempty interior.

In particular, if a Banach space X contains a CSS cone or an FBI cone, then X is a Hilbert space.

2. Cone lemmas

In the sections that follow, we will require some lemmas regarding cones, their duals, their sections, and the relationships between these concepts and central symmetry. Most of these lemmas are well known, though the property of central symmetry seems to be little-studied in the context of infinite-dimensional sections of cones. The seminal monograph of Kreĭn and Rutman [6] still provides an excellent introduction to cones in linear spaces. A more recent treatment may be found in [1].

Let V be a real normed vector space with norm $\|\cdot\|$, and let V^* be the dual space of continuous linear functionals on V under the operator norm, also denoted by $\|\cdot\|$. Let a cone $C \subset V$ be given. Recall that the *dual cone* of C is the cone $C^* \subset V^*$ defined by

$$C^* := \{\varphi \in V^* : \varphi(x) \geq 0 \text{ for all } x \in C\}.$$

Recall also the following well-known result regarding cones in normed vector spaces.

Proposition 2.1. *Let C be a cone in a normed vector space V . Then, under the canonical embedding $V \hookrightarrow V^{**}$, we have that $\text{int}(C) \hookrightarrow \text{int}(C^{**})$. Moreover, the closure of C under this embedding is C^{**} .*

We will find the following notation to be convenient: Given points $x \in V$ and $\varphi \in V^*$, let $S_x(C^*)$ and $S_\varphi(C)$ be the sections of C^* and C , respectively, defined as follows

$$\begin{aligned} S_x(C^*) &:= \{\psi \in C^* : \psi(x) = 1\}, \\ S_\varphi(C) &:= \{y \in C : \varphi(y) = 1\}. \end{aligned}$$

More generally, we will occasionally need a canonical affine subspace in V or V^* that is perpendicular to a given affine subspace of V^* or V , respectively. Recall that, if $L \subset V$ and $M \subset V^*$ are linear subspaces, then the *annihilators* of L and M are defined by

$$\begin{aligned} L^\perp &:= \{\varphi \in V^* : \varphi(x) = 0 \text{ for all } x \in L\}, \\ {}^\perp M &:= \{y \in V : \psi(y) = 0 \text{ for all } \psi \in M\}. \end{aligned}$$

Analogously, given an affine subspace $A \subset V$, respectively $B \subset V^*$, bounded away from the origin, define the *perpendicular affine spaces*

$$A^\perp := \{\varphi \in V^* : \varphi(x) = 1 \text{ for all } x \in A\},$$

$${}^\perp B := \{y \in V : \psi(y) = 1 \text{ for all } \psi \in B\}.$$

It is well known that, if $L \subset V$ is a closed linear subspace and if $M \subset V^*$ is a weak*-closed linear subspace, then the annihilators satisfy a duality relation: ${}^\perp(L^\perp) = L$ and $({}^\perp M)^\perp = M$.

Lemma 2.2. *Let V be a normed vector space. Let $A \subset V$ and $B \subset V^*$ be closed affine subspaces such that $0 \notin A$, $0 \notin B$, and $\text{codim}(A)$ and $\dim(B)$ are finite. Then ${}^\perp(A^\perp) = A$ and $({}^\perp B)^\perp = B$.*

Proof. Fix $x_0 \in A$, $\varphi_0 \in A^\perp$, $\psi_0 \in B$, and $y_0 \in {}^\perp B$. Let $L := A - x_0$ and $M := B - \psi_0$. Write $x_0^\perp$ and ${}^\perp \psi_0$ for the annihilator of the linear span of x_0 and ψ_0 , respectively. Observe that $A^\perp = (L^\perp \cap x_0^\perp) + \varphi_0$ and ${}^\perp B = ({}^\perp M \cap {}^\perp \psi_0) + y_0$. The equations to be proved now follow from the analogous facts about annihilators mentioned above. $\square$

Definition 2.3. A convex set $K \subset V$ is *linearly bounded* if the intersection of K with every affine line is a bounded line segment. Equivalently, a linearly bounded convex set contains no ray.

The following lemma establishes a natural relationship between the nonempty linearly bounded sections of C and the dual cone C^* . In particular, the dual cone of C is the set of elements of V^* that are positive on a linearly bounded section of C .

Lemma 2.4. *Let C be a pointed cone in a normed vector space V , and let $\varphi \in V^*$ be such that $S_\varphi := S_\varphi(C)$ is nonempty and linearly bounded. Then $C^* = \{\psi \in V^* : \psi(S_\varphi) \geq 0\}$.*

Proof. We first show that $\varphi(C \setminus \{0\}) > 0$. For, suppose otherwise. Then, since C is pointed, there exists a $v \in C \setminus \{0\}$ such that $\varphi(v) = 0$. Fix $w \in S_\varphi$. Then, for every $\lambda \geq 0$, we have that $w + \lambda v \in C$ and $\varphi(w + \lambda v) = 1$, so $w + \lambda v \in S_\varphi$, which implies that S_φ is not linearly bounded.

Now, suppose that $\psi(S_\varphi) \geq 0$, and let $y \in C$ be given. If $y = 0$, then we immediately have that $\psi(y) \geq 0$. If $y \in C \setminus \{0\}$, then $\varphi(y) > 0$, so $\frac{1}{\varphi(y)}y \in S_\varphi$. Thus, $\psi(\frac{1}{\varphi(y)}y) \geq 0$, so $\psi(y) \geq 0$. Therefore, $\psi \in C^*$, as claimed. Since the converse claim is immediate, the lemma is proved. $\square$

Furthermore, there is a well-known relationship between the bounded sections of C and the interior points of C^* . (Cf. [3, Theorem 3.8.4].) We provide the proof for completeness and because it indicates the point at which our

arguments require the local convexity of V . Given a point v in V or V^* , we write $B_r(v)$ to denote the closed ball of radius r centered at v .

Lemma 2.5. *Let C be a pointed cone in a normed vector space V . Given a nonzero functional $\varphi \in V^*$, the section $S_\varphi := S_\varphi(C)$ is bounded if and only if $\varphi \in \text{int}(C^*)$.*

Proof. Suppose that S_φ is bounded. Let $r > 0$ be such that $S_\varphi \subset B_r(0)$, and put $\varepsilon := 1/r$. Let $\psi \in B_\varepsilon(\varphi)$ be given. Observe that, for all $x \in S_\varphi$, we have that $|1 - \psi(x)| = |\varphi(x) - \psi(x)| \leq \|\varphi - \psi\| \|x\| \leq \varepsilon r = 1$, and hence $\psi(x) \geq 0$. Since S_φ is linearly bounded, it follows from Lemma 2.4 that $\psi \in C^*$, so $\varphi \in \text{int}(C^*)$.

Conversely, suppose that $\varphi \in \text{int}(C^*)$. Let $\varepsilon > 0$ be such that $B_\varepsilon(\varphi) \subset C^*$, and put $r := 1/\varepsilon$. Fix $x \in S_\varphi$. By the Hahn–Banach theorem, there exists a functional $\nu \in V^*$ such that $\nu(x/\|x\|) = 1$ and $\nu(B_1(0)) \leq 1$. That is, $\nu(x) = \|x\|$ and $\|\nu\| = 1$. Set $\psi := \varphi - \varepsilon\nu$. Then $\|\varphi - \psi\| = \varepsilon$, so $\psi \in C^*$, and hence $\psi(x) \geq 0$. Thus, $\varepsilon\|x\| = \varphi(x) - \psi(x) = 1 - \psi(x) \leq 1$, so $\|x\| \leq 1/\varepsilon = r$, yielding the claim. $\square$

It is well known that if X is a separable Banach space, then every pointed cone C has a base, i.e. there is a closed convex subset $B \subset X$ such that, for every $x \in C \setminus \{0\}$, there exist unique $\lambda > 0$ and $y \in B$ such that $x = \lambda y$. The separability assumption is indispensable, as demonstrated by the standard example of the cone C of all non-negative real-valued functions on $X = \ell^2(I)$ with respect to the counting measure, where I is an uncountable set. For this case, the set of all positive continuous linear functionals on C is isometrically isomorphic to C , but C^* has no strictly positive linear functionals, and so C^* has no interior. It readily follows that C has no base [3]. It is also worth noting that pointed cones in separable Banach spaces may not have a bounded base. For example, if $X = C[0, 1]$ with its usual uniform topology, then X is separable. If C is the cone of nonnegative valued functions in X , then C has a base, but, since C^* is the set of regular positive Borel measures on $[0, 1]$, C^* has an empty interior. Thus C has no bounded base.

We do not assume that our normed vector space V is separable, so there may exist pointed cones without bounded bases. However, condition (2) in the definition of CSS cones (Definition 1.1) guarantees that this is not the case with a CSS cone, because having a bounded proper section implies having a bounded base, as may be seen in the proof of Lemma 2.4. (The converse is also true. See [3, p. 34].) In particular, the dual of a CSS cone C always has a nonempty interior. As seen in the next lemma, it follows from a series of results due to Borwein and Lewis that the functionals in the interior of C^* are precisely the functionals that are strictly positive on $C \setminus \{0\}$.

Lemma 2.6. *Let C be a closed pointed cone in a normed vector space V such that the dual cone C^* has nonempty interior, and let $\varphi \in V^*$. Then $S_\varphi := S_\varphi(C)$ is bounded if and only if φ is strictly positive on $C \setminus \{0\}$.*

Proof. By Lemma 2.5, S_φ is bounded if and only if $\varphi \in \text{int}(C^*)$. Since $\text{int}(C^*) \neq \emptyset$, it follows from [2, Corollary 2.14] and [2, Theorem 3.10] that $\varphi \in \text{int } C^*$ if and only if $\varphi(C \setminus 0) > 0$. $\square$

Lemma 2.7. *Let C be a pointed cone in a normed vector space V . If S^* is a bounded finite-dimensional section of C^* that intersects the interior of C^* , then the section*

$$S := \{x \in C : \text{for all } \varphi \in S^*, \varphi(x) = 1\}$$

of C is bounded, finite codimensional, and intersects the interior of C .

Proof. We first prove that $S \cap \text{int}(C) \neq \emptyset$. Let $B = \text{aff}(S^*)$. Observe that $S = {}^\perp B \cap C$, and suppose, to get a contradiction, that ${}^\perp B \cap \text{int}(C) = \emptyset$. Then there exists a supporting hyperplane H of C containing ${}^\perp B$. Let $\psi \in C^* \setminus \{0\}$ be such that $H = \ker \psi$. Fix $\varphi \in S^*$. Then $\varphi + \lambda\psi \in C^*$ for all $\lambda \geq 0$. Moreover, $\varphi + \lambda\psi \in ({}^\perp(B))^\perp = B$ by Lemma 2.2. Hence, $\varphi + \lambda\psi \in S^*$ for all $\lambda \geq 0$, so S^* is not bounded, a contradiction.

Let $n := \dim(S^*)$. Since S^* intersects the interior of C^* , there exist linear functionals $\varphi_0, \dots, \varphi_n \in \text{int}(C^*)$ that affinely span $\text{aff}(S^*)$. It is easy to see that $S = \bigcap_{i=0}^n S_{\varphi_i}$. Indeed, since $S \cap \text{int}(C) \neq \emptyset$, the affine span of S is the intersection of the affine spans of the S_{φ_i} :

$$\text{aff}(S) = \bigcap_{i=0}^n \text{aff}(S_{\varphi_i})$$

Therefore, S is of co-dimension $\dim(S^*)+1$. Finally, S is bounded by Lemma 2.5. $\square$

By Lemma 2.5, a bounded proper section S of a pointed cone C corresponds to an interior point of C^* . This correspondence interchanges dimension and codimension, since it is the correspondence between a hyperplane and a vector normal to that hyperplane. However, if the bounded proper section S is centrally symmetric, then there is a canonical corresponding *proper* section of C^* , of the *same* co-dimension, which is also centrally symmetric.

Lemma 2.8. *Let C be a pointed cone in a normed vector space V . Suppose that $x \in C$ and $\varphi \in C^*$ are such that $S_\varphi := S_\varphi(C)$ is bounded and centrally symmetric about x . Then $S_x := S_x(C^*)$ is centrally symmetric about φ .*

Proof. Let $\psi \in S_x$ be given. We want to show that $\varphi - (\psi - \varphi) = 2\varphi - \psi \in S_x$. Since it is clear that $(2\varphi - \psi)(x) = 1$, it remains only to show that $2\varphi - \psi \in C^*$. By Lemma 2.4, it suffices to show that $(2\varphi - \psi)(S_\varphi) \geq 0$, or, equivalently, $\psi(S_\varphi) \leq 2$. To this end, let $y \in S_\varphi$ be given. Since S_φ is centrally symmetric about x , we have that $2x - y \in S_\varphi \subset C$. Thus, $\psi(2x - y) \geq 0$, or, equivalently, $\psi(y) \leq 2$, as desired. $\square$

3. CSS cones are ellipsoidal cones

Fix a CSS cone C with nonempty interior in a normed vector space V . As mentioned in the introduction, the proof that C is an ellipsoidal cone in the finite-dimensional case appeared in [4]. Somewhat surprisingly, there does not seem to be a straightforward transfinite-induction argument that extends this result to the infinite-dimensional case. We will instead take a detour through the dual cone C^* .

We call a section C' of C a *sectional subcone* of C if C' contains the origin. We will give an argument by finite induction below showing that every finite-codimensional sectional subcone of C is also CSS. Had we been able to extend this induction argument to a transfinite-induction argument, we could have applied the finite-dimensional CSS characterization of ellipsoidal cones to prove that every finite-dimensional sectional subcone of C is ellipsoidal. The conclusion that C itself is ellipsoidal would then have followed from the Jordan–von Neumann characterization of inner-product spaces.

Unfortunately, a direct argument by transfinite induction for the claim that all sectional subcones of CSS cones are CSS cones eludes us. Our strategy instead will be as follows. To show that C is ellipsoidal, we will show that its dual C^* is ellipsoidal. That C^* is ellipsoidal will follow from the Jordan–von Neumann characterization of ellipsoids once we show that every finite-dimensional sectional subcone of C^* is ellipsoidal. To prove this, we will need to show that every bounded finite-dimensional section S^* of C^* is centrally symmetric and then apply the finite-dimensional CSS characterization of ellipsoidal cones.

Thus, we need to find a center of symmetry φ for a given finite-dimensional section S^* of C^* . To do this, we look at the perpendicular section

$$S := \{y \in C : \psi(y) = 1 \text{ for all } \psi \in S^*\}$$

of C . It follows from Lemma 2.7 that S is a bounded section of C with finite codimension. Our finite induction argument will thus suffice to show that S is centrally symmetric, with a center of symmetry x . Furthermore, S is contained in a proper section T of C with co-dimension 1, which will determine a dual vector $\varphi \in C^*$ via $\varphi(T) = 1$. Finally, the central symmetry of S will imply that S^* is centrally symmetric about φ by Lemma 2.8, establishing the result.

Lemma 3.1. *Let C be a CSS cone with nonempty interior in a normed vector space V . Let S be a bounded section of C such that $\text{codim}(S) < \infty$ and $S \cap \text{int}(C) \neq \emptyset$. Then S is centrally symmetric.*

Indeed, S is contained in a proper section T of C such that the center of symmetry of T lies on S .

Proof. We begin with the case where $\text{codim}(S) = 2$. We will show that there is a bounded proper section T of C containing S whose center of symmetry (which exists by the CSS property) lies on S .

Fix $y \in S$, and let $L := \text{aff}(S) - y$. The image of C under the quotient map $Q: V \rightarrow V/L$ is a 2-dimensional pointed cone in which $Q(S)$ contains a single point. Put $C' := Q(C)$ and $\{s\} := Q(S)$. Since quotient maps are open, $s \in \text{int}(C')$. Let $\ell \subset V/L$ be the affine line through s such that s is the midpoint of $\ell \cap C'$. Let $H := Q^{-1}(\ell)$, and put $T := H \cap C$. We claim that T is a bounded proper section of C containing S whose center of symmetry is on S .

It is clear that $S \subset T$. To see that T is bounded, observe that $H - y$ strictly supports C at 0. That is, $(H - y) \cap C = \{0\}$. For, suppose that $z \in (H - y) \cap C$. Then, since $(\ell - s) \cap C' = \{0\}$, we have that $Q(z) = 0$, and so $z \in (S - y) \cap C$. If z were nonzero, then λz would also be in $(S - y) \cap C$ for all $\lambda > 0$, so S would be unbounded, contrary to our hypothesis. Hence, $z = 0$. Thus, by Lemma 2.6, T is bounded, so T has a center of symmetry x , which must map to the center of symmetry of $\ell \cap C'$ under Q . That is, $Q(x) = s$, so $x \in S$, as desired. Thus, S is a section of a centrally symmetric set that contains the center of symmetry of that set. Therefore, S itself is centrally symmetric.

If $n := \text{codim}(S) \geq 3$, fix a codimension-1 linear subspace $W \subset V$ containing S . By the preceding argument, $C \cap W$ is a CSS cone in which S is a bounded codimension- $(n - 1)$ section intersecting the interior of $C \cap W$. The theorem now follows from the induction hypothesis applied to $C \cap W$. $\square$

Lemma 3.2. *Let C be a CSS cone with nonempty interior in a normed vector space V . Let S^* be a bounded section of C^* such that $\dim(S^*) < \infty$ and $S^* \cap \text{int}(C^*) \neq \emptyset$. Then S^* is centrally symmetric.*

Proof. Let $S := \{y \in C : \psi(y) = 1 \text{ for all } \psi \in S^*\}$. By Lemma 2.7, we have that S is bounded, intersects the interior of C , and has finite codimension. Thus, by Lemma 3.1, there exists a proper section T of C containing S whose center of symmetry x is in S . Let $\varphi \in C^*$ be such that $S_\varphi := S_\varphi(C) = T$. Then, by Lemma 2.8, φ is the center of symmetry of $S_x := S_x(C^*)$. Note that $S^* \subset S_x$, so it remains only to show that $\varphi \in S^*$.

Indeed, since $S \cap \text{int} C \neq \emptyset$, we have that $\text{aff}(S) = {}^\perp(\text{aff } S^*)$. Thus, $\text{aff}(S^*) = (\text{aff } S)^\perp$ by Lemma 2.2. In particular, $\varphi \in \text{aff}(S^*)$. Since $\varphi \in C^*$, we conclude that $\varphi \in S^*$, as desired. $\square$

The previous Lemma motivates the following definition.

Definition 3.3. Let C be a closed pointed cone in a normed vector space V with nonempty interior. We call C *co-CSS* if every bounded finite-dimensional section of C intersecting the interior of C is centrally-symmetric.

Thus, Lemma 3.2 says that the dual of a CSS cone is co-CSS.

Lemma 3.4. *Let C be a co-CSS cone in a normed vector space V . Fix a finite-dimensional subspace $L \subset V$ such that $L \cap \text{int } C \neq \emptyset$. Then the cone $L \cap C$ is ellipsoidal.*

Proof. Since C is pointed, the finite-dimensional cone $L \cap C$ is also pointed. Since C is co-CSS, every bounded proper section of $L \cap C$ is centrally symmetric. In particular, $L \cap C$ is CSS. Therefore, by the finite-dimensional CSS characterization of ellipsoidal cones [4, Theorem 1.4], $L \cap C$ is ellipsoidal. $\square$

Lemma 3.5. *Let C be a CSS cone in a normed vector space V . Then the dual cone C^* is ellipsoidal.*

Proof. Since C is CSS, there exists a bounded proper section $S_\varphi := S_\varphi(C)$ of C , where, by Lemma 2.5, $\varphi \in \text{int}(C^*)$. Let $x \in C$ be the center of symmetry of S_φ . Then, by Lemma 2.8, the section $S_x := S_x(C^*)$ is centrally symmetric about φ . In addition, S_x is a proper section of C^* because it contains the point $\varphi \in \text{int}(C^*)$. Furthermore, S_x is bounded by Lemma 2.5 because $x \in \text{int}(C) \hookrightarrow \text{int}(C^{**})$ under the canonical embedding. Finally, by Lemma 3.4, every finite-dimensional section of S_x through φ is an ellipsoid centered at φ . Hence, by the Jordan–von Neumann characterization of inner-product spaces [5], S_x is an ellipsoid. Therefore, C^* is ellipsoidal. $\square$

We are now ready to prove our main result.

Proof of Theorem 1.2. Every bounded proper section of an ellipsoidal cone is an ellipsoid, and ellipsoids are centrally symmetric. Hence, ellipsoidal cones are CSS.

To prove the converse, let a CSS cone of dimension ≥ 2 be given. By Lemma 3.5, C^* is ellipsoidal. Fix an ellipsoidal proper section S^* of C^* , and let $\varphi \in S^*$. Thus, we have an inner product on the codimension-1 linear subspace $M := \text{aff}(S^*) - \varphi$. Since S^* is bounded, $\text{aff}(S^*)$ does not contain the origin. Hence, we can complete the inner product on M to an inner product on all of V^* . Indeed, since V^* is already a dual space, it is in fact a Hilbert space. The dual of an ellipsoidal cone in a Hilbert space is ellipsoidal [6, p. 51], so C^{**} is ellipsoidal. Since C^{**} is the closure of C under the canonical embedding $V \hookrightarrow V^{**}$, we conclude that C itself is ellipsoidal in V . $\square$

It follows that if a normed vector space V contains a CSS cone with nonempty interior, then V is an inner product space. In particular, if a Banach space X contains a full-dimensional CSS cone, then X is a Hilbert space.

4. FBI cones are ellipsoidal cones

We conclude by proving that every cone in a normed vector space V that satisfies the FBI property (Definition 1.3) is ellipsoidal.

Lemma 4.1. *Let C be a cone in a normed vector space V such that $\text{int}(C)$ and $\text{int}(C^*)$ are both nonempty. Then, for each $a \in \text{int}(C)$, the intersection $C \cap (a - C)$ is bounded.*

Proof. Since $\text{int}(C^*) \neq \emptyset$, there exists a functional $\varphi \in C^*$ such that $S_\varphi := S_\varphi(C)$ is a bounded base of C by Lemma 2.4. In particular, $\ker(\varphi)$ strictly supports C at 0 by Lemma 2.6. By suitably normalizing φ , we also have that S_φ strictly supports $a - C$ at a . Thus,

$$C \cap (a - C) \subset \{y \in C : \varphi(y) \leq 1\}.$$

Since S_φ is bounded and is a base for C , it follows that $C \cap (a - C)$ is also bounded. $\square$

Proof of Theorem 1.4. Without loss of generality, suppose that $\text{int } C \neq \emptyset$. Fix $x \in \text{int}(C)$, and set $S := \text{conv}(\partial C \cap \partial(2x - C))$. By the FBI property and Lemma 4.1, S is a bounded section of C . Moreover, S is centrally symmetric about x . We show that every 2-dimensional section of S through x is an ellipse. Let E be such a section. Observe that $0 \notin \text{aff } E$, because $0 \in \text{aff } E \subset \text{aff } S$ would imply that $\text{aff } S = \text{lin } S$, contrary to the boundedness of S . Therefore, $C' := \text{cone}(E)$ is a 3-dimensional cone. We claim that $\text{cone}(E)$ is an FBI cone. To prove this, let $y \in \text{relint}(C)'$, and let $\Gamma := \partial(C') \cap \partial(y - C')$. On the one hand, Γ is contained in $\text{lin } E$. On the other hand, let H be the affine hyperplane in V containing $\partial C \cap \partial(y - C)$. As above, $0 \notin H$. Since $0 \in \text{lin } E$, it follows that $\Gamma \subset (\text{lin } E) \cap H$, but $\text{lin } E \not\subset H$. Hence, Γ is contained in a 2-dimensional affine subspace of $\text{lin } E$. That is, C' is an FBI cone. It follows from [4, Theorem 1.2] that E is an ellipse. Therefore, by the Jordan–von Neumann characterization of inner-product spaces [5], S is an ellipsoid. $\square$

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