

Borell's Generalized Prékopa-Leindler Inequality: A Simple Proof

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We present a simple proof of Christer Borell's general inequality in the Brunn-Minkowski theory. We then discuss applications of Borell's inequality to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang.

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1. Introduction

Let us denote by $\text{supp}(f)$ the support of a function f . In [6] Christer Borell proved the following inequality (see [6, Theorem 2.1]), which we will call the Borell-Brunn-Minkowski inequality.

Theorem 1.1 (Borell-Brunn-Minkowski inequality). *Let $f, g, h : \mathbb{R}^n \rightarrow [0, +\infty)$ be measurable functions. Let $\varphi = (\varphi_1, \dots, \varphi_n) : \text{supp}(f) \times \text{supp}(g) \rightarrow \mathbb{R}^n$ be a continuously differentiable function with positive partial derivatives, such that $\varphi_k(x, y) = \varphi_k(x_k, y_k)$ for every $x = (x_1, \dots, x_n) \in \text{supp}(f)$, $y = (y_1, \dots, y_n) \in \text{supp}(g)$. Let $\Phi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality*

$$h(\varphi(x, y))^{\prod_{k=1}^n \left(\frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right)} \geq \Phi(f(x)^{\prod_{k=1}^n \rho_k}, g(y)^{\prod_{k=1}^n \eta_k}) \quad (1)$$

holds for every $x \in \text{supp}(f)$, for every $y \in \text{supp}(g)$, for every $\rho_1, \dots, \rho_n > 0$

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and for every $\eta_1, \dots, \eta_n > 0$, then

$$\int h \geq \Phi \left(\int f, \int g \right).$$

C. Borell proved a slightly more general statement, involving an arbitrary number of functions. For simplicity, we restrict ourselves to the statement of Theorem 1.1.

Theorem 1.1 yields several important consequences. For example, applying Theorem 1.1 to indicators of compact sets (i.e. $f = 1_A$, $g = 1_B$, $h = 1_{\varphi(A,B)}$) yields the following generalized Brunn-Minkowski inequality.

Corollary 1.2 (Generalized Brunn-Minkowski inequality). *Let A, B be compact subsets of $\mathbb{R}^n$. Let $\varphi = (\varphi_1, \dots, \varphi_n) : A \times B \rightarrow \mathbb{R}^n$ be a continuously differentiable function with positive partial derivatives, such that $\varphi_k(x, y) = \varphi_k(x_k, y_k)$ for every $x = (x_1, \dots, x_n) \in A$, $y = (y_1, \dots, y_n) \in B$. Let $\Phi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality*

$$\Pi_{k=1}^n \left(\frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(\Pi_{k=1}^n \rho_k, \Pi_{k=1}^n \eta_k)$$

holds for every $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$, then

$$|\varphi(A, B)| \geq \Phi(|A|, |B|),$$

where $|\cdot|$ denotes Lebesgue measure and $\varphi(A, B) = \{\varphi(x, y) : x \in A, y \in B\}$.

The classical Brunn-Minkowski inequality (see e.g. [23], [12]) follows from Corollary 1.2 by taking $\varphi(x, y) = x + y$, $x \in A, y \in B$, and $\Phi(a, b) = (a^{1/n} + b^{1/n})^n$, $a, b \geq 0$. Although the Brunn-Minkowski inequality goes back to more than a century ago, it still attracts a lot of attention (see e.g. [20], [16], [13], [18], [9], [10], [11], [14], [17]).

Theorem 1.1 also allows us to recover the so-called Borell-Brascamp-Lieb inequality. Let us denote by $M_s^\lambda(a, b)$ the s -mean of the real numbers $a, b \geq 0$ with weight $\lambda \in [0, 1]$, defined as

$$M_s^\lambda(a, b) = ((1 - \lambda)a^s + \lambda b^s)^{\frac{1}{s}} \quad \text{if } s \notin \{-\infty, 0, +\infty\},$$

$M_{-\infty}^\lambda(a, b) = \min(a, b)$, $M_0^\lambda(a, b) = a^{1-\lambda}b^\lambda$, $M_{+\infty}^\lambda(a, b) = \max(a, b)$. We will need the following Hölder inequality (see e.g. [15]).

Lemma 1.3 (Generalized Hölder inequality). *Let $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ such that $\beta + \gamma \geq 0$ and $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$. Then, for every $a, b, c, d \geq 0$ and $\lambda \in [0, 1]$,*

$$M_\alpha^\lambda(ac, bd) \leq M_\beta^\lambda(a, b) M_\gamma^\lambda(c, d).$$

Corollary 1.4 (Borell-Brascamp-Lieb inequality). *Let $\gamma \geq -\frac{1}{n}$, $\lambda \in [0, 1]$ and $f, g, h : \mathbb{R}^n \rightarrow [0, +\infty)$ be measurable functions. If the inequality*

$$h((1 - \lambda)x + \lambda y) \geq M_\gamma^\lambda(f(x), g(y))$$

holds for every $x \in \text{supp}(f), y \in \text{supp}(g)$, then

$$\int_{\mathbb{R}^n} h \geq M_{\frac{\gamma}{1+\gamma n}}^\lambda \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right).$$

Corollary 1.4 follows from Theorem 1.1 by taking $\varphi(x, y) = (1 - \lambda)x + \lambda y$, $x \in \text{supp}(f), y \in \text{supp}(g)$, and $\Phi(a, b) = M_{\frac{\gamma}{1+\gamma n}}^\lambda(a, b)$, $a, b \geq 0$. Indeed, using Lemma 1.3, one obtains that for every $x \in \text{supp}(f), y \in \text{supp}(g)$, and for every $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$,

$$\begin{aligned} h(\varphi(x, y)) \Pi_{k=1}^n \left(\frac{\partial \varphi}{\partial x_k} \rho_k + \frac{\partial \varphi}{\partial y_k} \eta_k \right) &= h((1 - \lambda)x + \lambda y) \Pi_{k=1}^n ((1 - \lambda)\rho_k + \lambda\eta_k) \\ &\geq M_\gamma^\lambda(f(x), g(y)) M_{\frac{1}{n}}^\lambda(\Pi_{k=1}^n \rho_k, \Pi_{k=1}^n \eta_k) \\ &\geq M_{\frac{\gamma}{1+\gamma n}}^\lambda(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k) \\ &= \Phi(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k). \end{aligned}$$

Corollary 1.4 was independently proved by Borell (see [6, Theorem 3.1]), and by Brascamp and Lieb [8].

Another important consequence of the Borell-Brunn-Minkowski inequality is obtained when considering φ to be nonlinear. Let us denote for $\mathbf{p} = (p_1, \dots, p_n) \in [-\infty, +\infty]^n$, $x = (x_1, \dots, x_n) \in [0, +\infty]^n$ and $y = (y_1, \dots, y_n) \in [0, +\infty]^n$,

$$M_{\mathbf{p}}^\lambda(x, y) = (M_{p_1}^\lambda(x_1, y_1), \dots, M_{p_n}^\lambda(x_n, y_n)).$$

Corollary 1.5 (Nonlinear extension of the Brunn-Minkowski inequality). *Let $\mathbf{p} = (p_1, \dots, p_n) \in [0, 1]^n$, $\gamma \geq -(\sum_{i=1}^n p_i^{-1})^{-1}$, $\lambda \in [0, 1]$, and $f, g, h : [0, +\infty)^n \rightarrow [0, +\infty)$ be measurable functions. If the inequality*

$$h(M_{\mathbf{p}}^\lambda(x, y)) \geq M_\gamma^\lambda(f(x), g(y))$$

holds for every $x \in \text{supp}(f), y \in \text{supp}(g)$, then

$$\int_{[0, +\infty)^n} h \geq M_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}}^\lambda \left(\int_{[0, +\infty)^n} f, \int_{[0, +\infty)^n} g \right).$$

Corollary 1.5 follows from Theorem 1.1 by taking $\varphi(x, y) = M_{\mathbf{p}}^\lambda(x, y)$, $x \in \text{supp}(f), y \in \text{supp}(g)$, and $\Phi(a, b) = M_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}}^\lambda(a, b)$, $a, b \geq 0$. Indeed,

using Lemma 1.3, one obtains that for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$, and for every $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$,

$$\begin{aligned} & h(\varphi(x, y)) \Pi_{k=1}^n \left(\frac{\partial \varphi}{\partial x_k} \rho_k + \frac{\partial \varphi}{\partial y_k} \eta_k \right) \\ &= h(M_{\mathbf{p}}^\lambda(x, y)) \Pi_{k=1}^n M_{\frac{p_k}{1-p_k}}^\lambda(x_k^{1-p_k}, y_k^{1-p_k}) M_1(x_k^{p_k-1} \rho_k, y_k^{p_k-1} \eta_k) \\ &\geq M_\gamma^\lambda(f(x), g(y)) \Pi_{k=1}^n M_{p_k}^\lambda(\rho_k, \eta_k) \\ &\geq M_\gamma^\lambda(f(x), g(y)) M_{(\sum_{i=1}^n p_i^{-1})-1}^\lambda(\Pi_{k=1}^n \rho_k, \Pi_{k=1}^n \eta_k) \\ &\geq M_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})-1}^\lambda(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k) \\ &= \Phi(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k). \end{aligned}$$

In the particular case where $\mathbf{p} = (0, \dots, 0)$, Corollary 1.5 was rediscovered by Ball [1]. In the general case, Corollary 1.5 was rediscovered by Uhrin [24].

Notice that the condition on $\mathbf{p}$ in Corollary 1.5 is less restrictive in dimension 1. It reads as follows:

Corollary 1.6 (Nonlinear extension of the Brunn-Minkowski inequality on the line). *Let $p \leq 1$, $\gamma \geq -p$, and $\lambda \in [0, 1]$. Let $f, g, h : [0, +\infty) \rightarrow [0, +\infty)$ be measurable functions such that for every $x \in \text{supp}(f)$, $y \in \text{supp}(g)$,*

$$h(M_p^\lambda(x, y)) \geq M_\gamma^\lambda(f(x), g(y)).$$

Then,

$$\int_0^{+\infty} h \geq M_{(\frac{1}{p} + \frac{1}{\gamma})-1}^\lambda \left(\int_0^{+\infty} f, \int_0^{+\infty} g \right).$$

A simple proof of Corollary 1.6 was recently given by Bobkov et al. [4].

In Section 2, we present a simple proof of Theorem 1.1, based on mass transportation. In Section 3, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang. We also prove an equivalence between the log-Brunn-Minkowski inequality and its possible extensions to convex measures (see Section 3 for precise definitions).

2. A simple proof of the Borell-Brunn-Minkowski inequality

In this section, we present a simple proof of Theorem 1.1.

Proof of Theorem 1.1. The proof is done by induction on the dimension. To prove the theorem in dimension 1, we use a mass transportation argument.

Step 1: (In dimension 1). First let us see that if $\int f = 0$ or $\int g = 0$, then the result holds. Let us assume, without loss of generality, that $\int g = 0$. By taking

$\rho = 1$, by letting η go to 0 and by using continuity and homogeneity of Φ in the condition (1), one obtains

$$h(\varphi(x, y)) \frac{\partial \varphi}{\partial x} \geq \Phi(f(x), 0) = f(x) \Phi(1, 0).$$

It follows that, for fixed $y \in \text{supp}(g)$,

$$\begin{aligned} \int h(z) dz &\geq \int_{\varphi(\text{supp}(f), y)} h(z) dz = \int_{\text{supp}(f)} h(\varphi(x, y)) \frac{\partial \varphi}{\partial x} dx \\ &\geq \int f \Phi(1, 0) = \Phi \left(\int f, \int g \right). \end{aligned}$$

A similar argument shows that the result holds if $\int f = +\infty$ or $\int g = +\infty$. Thus we assume thereafter that $0 < \int f < +\infty$ and $0 < \int g < +\infty$.

Let us show that one may assume that $\int f = \int g = 1$. Let us define, for $x, y \in \mathbb{R}$ and $a, b \geq 0$,

$$\begin{aligned} \tilde{f}(x) &= f \left(\Phi \left(\int f, 0 \right) x \right) \Phi(1, 0), & \tilde{g}(x) &= g \left(\Phi \left(0, \int g \right) x \right) \Phi(0, 1), \\ \tilde{h}(x) &= h \left(\Phi \left(\int f, \int g \right) x \right), & \tilde{\varphi}(x, y) &= \frac{\varphi(\Phi(\int f, 0)x, \Phi(0, \int g)y)}{\Phi(\int f, \int g)}, \\ \tilde{\Phi}(a, b) &= \Phi \left(a \frac{\int f}{\Phi(\int f, \int g)}, b \frac{\int g}{\Phi(\int f, \int g)} \right). \end{aligned}$$

Let $x \in \text{supp}(\tilde{f})$, $y \in \text{supp}(\tilde{g})$, and let $\tilde{\rho}, \tilde{\eta} > 0$. One has,

$$\begin{aligned} &\tilde{h}(\tilde{\varphi}(x, y)) \left(\frac{\partial \tilde{\varphi}}{\partial x} \tilde{\rho} + \frac{\partial \tilde{\varphi}}{\partial y} \tilde{\eta} \right) \\ &\geq \Phi \left(f \left(\Phi \left(\int f, 0 \right) x \right) \frac{\Phi(\int f, 0)}{\Phi(\int f, \int g)} \tilde{\rho}, g \left(\Phi \left(0, \int g \right) y \right) \frac{\Phi(0, \int g)}{\Phi(\int f, \int g)} \tilde{\eta} \right) \\ &= \tilde{\Phi}(\tilde{f}(x) \tilde{\rho}, \tilde{g}(y) \tilde{\eta}). \end{aligned}$$

Notice that the functions $\tilde{\varphi}$ and $\tilde{\Phi}$ satisfy the same assumptions as the functions φ and Φ respectively, and that $\int \tilde{f} = \int \tilde{g} = 1$. If the result holds for functions of integral one, then

$$\int \tilde{h}(w) dw \geq \tilde{\Phi}(1, 1) = 1.$$

The change of variable $w = z / \Phi(\int f, \int g)$ leads us to

$$\int h(z) dz \geq \Phi \left(\int f, \int g \right).$$

Assume now that $\int f = \int g = 1$. By standard approximation, one may assume that f and g are compactly supported positive Lipschitz functions (relying on the fact that Φ is continuous and increasing in each coordinate, compare with [2, page 343]). Thus there exists a non-decreasing map $T : \text{supp}(f) \rightarrow \text{supp}(g)$ such that for every $x \in \text{supp}(f)$,

$$f(x) = g(T(x))T'(x),$$

see e.g. [3], [25]. Since T is non-decreasing and $\partial\varphi/\partial x, \partial\varphi/\partial y > 0$, the function $\Theta : \text{supp}(f) \rightarrow \varphi(\text{supp}(f), T(\text{supp}(f)))$ defined by $\Theta(x) = \varphi(x, T(x))$ is bijective. Hence the change of variable $z = \Theta(x)$ is admissible and one has,

$$\begin{aligned} \int h(z)dz &\geq \int_{\text{supp}(f)} h(\varphi(x, T(x))) \left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} T'(x) \right) dx \\ &\geq \int_{\text{supp}(f)} \Phi(f(x), g(T(x))T'(x)) dx \\ &= \int \Phi(f(x), f(x)) dx. \end{aligned}$$

Using homogeneity of Φ , one deduces that

$$\int h \geq \Phi(1, 1) \int f(x) dx = \Phi\left(\int f, \int g\right).$$

Step 2: (Tensorization). Let n be a positive integer and assume that Theorem 1.1 holds in $\mathbb{R}^n$. Let f, g, h, φ, Φ satisfying the assumptions of Theorem 1.1 in $\mathbb{R}^{n+1}$. Recall that the inequality

$$h(\varphi(x, y)) \prod_{k=1}^{n+1} \left(\frac{\partial\varphi_k}{\partial x_k} \rho_k + \frac{\partial\varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(f(x) \prod_{k=1}^{n+1} \rho_k, g(y) \prod_{k=1}^{n+1} \eta_k), \quad (2)$$

holds for every $x \in \text{supp}(f), y \in \text{supp}(g)$, and for every $\rho_1, \dots, \rho_{n+1}, \eta_1, \dots, \eta_{n+1} > 0$. Let us define, for $x_{n+1}, y_{n+1}, z_{n+1} \in \mathbb{R}$,

$$\begin{aligned} F(x_{n+1}) &= \int_{\mathbb{R}^n} f(x, x_{n+1}) dx, & G(y_{n+1}) &= \int_{\mathbb{R}^n} g(x, y_{n+1}) dx, \\ H(z_{n+1}) &= \int_{\mathbb{R}^n} h(x, z_{n+1}) dx. \end{aligned}$$

Since $\int f > 0, \int g > 0$, the support of F and the support of G are nonempty. Let $x_{n+1} \in \text{supp}(F), y_{n+1} \in \text{supp}(G)$, and let $\rho_{n+1}, \eta_{n+1} > 0$. Let us define, for $x, y, z \in \mathbb{R}^n$,

$$\begin{aligned} f_{x_{n+1}}(x) &= f(x, x_{n+1})\rho_{n+1}, & g_{y_{n+1}}(y) &= g(y, y_{n+1})\eta_{n+1}, \\ \bar{\varphi}(x, y) &= (\varphi_1(x_1, y_1), \dots, \varphi_n(x_n, y_n)), \\ h_{\varphi_{n+1}}(z) &= h(z, \varphi_{n+1}(x_{n+1}, y_{n+1})) \left(\frac{\partial\varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial\varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right). \end{aligned}$$

Let $x \in \text{supp}(f_{x_{n+1}}), y \in \text{supp}(g_{y_{n+1}})$, and let $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$. One has

$$\begin{aligned} & h_{\varphi_{n+1}}(\overline{\varphi}(x, y)) \Pi_{k=1}^n \left(\frac{\partial \overline{\varphi}_k}{\partial x_k} \rho_k + \frac{\partial \overline{\varphi}_k}{\partial y_k} \eta_k \right) \\ &= h(\varphi(x, x_{n+1}, y, y_{n+1})) \Pi_{k=1}^{n+1} \left(\frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \\ &\geq \Phi(f(x, x_{n+1}) \Pi_{k=1}^{n+1} \rho_k, g(y, y_{n+1}) \Pi_{k=1}^{n+1} \eta_k) \\ &= \Phi(f_{x_{n+1}}(x) \Pi_{k=1}^n \rho_k, g_{y_{n+1}}(y) \Pi_{k=1}^n \eta_k), \end{aligned}$$

where the inequality follows from inequality (2). Hence, applying Theorem 1.1 in dimension n , one has

$$\int_{\mathbb{R}^n} h_{\varphi_{n+1}}(x) dx \geq \Phi \left(\int_{\mathbb{R}^n} f_{x_{n+1}}(x) dx, \int_{\mathbb{R}^n} g_{y_{n+1}}(x) dx \right).$$

This yields that for every $x_{n+1} \in \text{supp}(F), y_{n+1} \in \text{supp}(G)$, and for every $\rho_{n+1}, \eta_{n+1} > 0$,

$$H(\varphi_{n+1}(x_{n+1}, y_{n+1})) \left(\frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right) \geq \Phi(F(x_{n+1}), G(y_{n+1})).$$

Hence, applying Theorem 1.1 in dimension 1, one has

$$\int_{\mathbb{R}} H(x) dx \geq \Phi \left(\int_{\mathbb{R}} F(x) dx, \int_{\mathbb{R}} G(x) dx \right).$$

This yields the desired inequality. $\square$

3. Applications to the log-Brunn-Minkowski inequality

In this section, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang [7].

Recall that a *convex body* in $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. Böröczky et al. conjectured the following inequality.

Conjecture 3.1 (log-Brunn-Minkowski inequality). *Let K, L be symmetric convex bodies in $\mathbb{R}^n$ and let $\lambda \in [0, 1]$. Then,*

$$|(1 - \lambda) \cdot K \oplus_0 \lambda \cdot L| \geq |K|^{1-\lambda} |L|^\lambda.$$

Here,

$$\begin{aligned} & (1 - \lambda) \cdot K \oplus_0 \lambda \cdot L \\ &= \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u)^{1-\lambda} h_L(u)^\lambda, \text{ for all } u \in S^{n-1}\}, \end{aligned}$$

where S^{n-1} denotes the n -dimensional Euclidean unit sphere, h_K denotes the support function of K , defined by $h_K(u) = \max_{x \in K} \langle x, u \rangle$, and $|\cdot|$ stands for Lebesgue measure.

Böröczky et al. [7] proved that Conjecture 3.1 holds in the plane. Using Corollary 1.5 with $\mathbf{p} = (0, \dots, 0)$, Saroglou [21] proved that Conjecture 3.1 holds for unconditional convex bodies in $\mathbb{R}^n$ (a set $K \subset \mathbb{R}^n$ is *unconditional* if for every $(x_1, \dots, x_n) \in K$ and for every $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, one has $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \in K$).

Recall that a measure μ is s -concave, $s \in [-\infty, +\infty]$, if the inequality

$$\mu((1-\lambda)A + \lambda B) \geq M_s^\lambda(\mu(A), \mu(B))$$

holds for all compact sets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$ and for every $\lambda \in [0, 1]$ (see [5], [6]). The 0-concave measures are also called *log-concave measures*, and the $-\infty$ -concave measures are also called *convex measures*. A function $f: \mathbb{R}^n \rightarrow [0, +\infty)$ is α -concave, $\alpha \in [-\infty, +\infty]$, if the inequality

$$f((1-\lambda)x + \lambda y) \geq M_\alpha^\lambda(f(x), f(y))$$

holds for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and for every $\lambda \in [0, 1]$.

Saroglou [22] recently proved that if the log-Brunn-Minkowski inequality holds, then the inequality

$$\mu((1-\lambda) \cdot K \oplus_0 \lambda \cdot L) \geq \mu(K)^{1-\lambda} \mu(L)^\lambda$$

holds for every symmetric log-concave measure μ , for all symmetric convex bodies K, L in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$.

An extension of the log-Brunn-Minkowski inequality for convex measures was proposed by the author in [19], and reads as follows:

Conjecture 3.2. *Let $p \in [0, 1]$. Let μ be a symmetric measure in $\mathbb{R}^n$ that has an α -concave density function, with $\alpha \geq -\frac{p}{n}$. Then for every symmetric convex body K, L in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$,*

$$\mu((1-\lambda) \cdot K \oplus_p \lambda \cdot L) \geq M_{\left(\frac{n}{p} + \frac{1}{\alpha}\right)^{-1}}^\lambda(\mu(K), \mu(L)). \quad (3)$$

Here,

$$\begin{aligned} & (1-\lambda) \cdot K \oplus_p \lambda \cdot L \\ &= \{x \in \mathbb{R}^n : \langle x, u \rangle \leq M_p^\lambda(h_K(u), h_L(u)), \text{ for all } u \in S^{n-1}\}. \end{aligned}$$

In Conjecture 3.2, if α or p is equal to 0, then $(n/p + 1/\alpha)^{-1}$ is defined by continuity and is equal to 0. Notice that Conjecture 3.1 is a particular case of Conjecture 3.2 when taking μ to be Lebesgue measure and $p = 0$.

By using Corollary 1.6, we will prove that Conjecture 3.1 implies Conjecture 3.2, when $\alpha \leq 1$, generalizing Saroglou's result discussed earlier.

Theorem 3.3. *If the log-Brunn-Minkowski inequality holds, then the inequality*

$$\mu((1-\lambda) \cdot K \oplus_p \lambda \cdot L) \geq M_{\left(\frac{n}{p} + \frac{1}{\alpha}\right)}^\lambda(\mu(K), \mu(L))$$

holds for every $p \in [0, 1]$, for every symmetric measure μ in $\mathbb{R}^n$ that has an α -concave density function, with $1 \geq \alpha \geq -\frac{p}{n}$, for every symmetric convex body K, L in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$.

Proof. Let K_0, K_1 be symmetric convex bodies in $\mathbb{R}^n$ and let $\lambda \in (0, 1)$. Let us denote $K_\lambda = (1-\lambda) \cdot K_0 \oplus_p \lambda \cdot K_1$ and let us denote by ψ the density function of μ . Let us define, for $t > 0$, $h(t) = |K_\lambda \cap \{\psi \geq t\}|$, $f(t) = |K_0 \cap \{\psi \geq t\}|$ and $g(t) = |K_1 \cap \{\psi \geq t\}|$. Notice that

$$\begin{aligned} \mu(K_\lambda) &= \int_{K_\lambda} \psi(x) dx = \int_{K_\lambda} \int_0^{\psi(x)} dt dx \\ &= \int_0^{+\infty} |K_\lambda \cap \{\psi \geq t\}| dt = \int_0^{+\infty} h(t) dt. \end{aligned}$$

Similarly, one has

$$\mu(K_0) = \int_0^{+\infty} f(t) dt, \quad \mu(K_1) = \int_0^{+\infty} g(t) dt.$$

Let $t, s > 0$ such that the sets $\{\psi \geq t\}$ and $\{\psi \geq s\}$ are nonempty. Let us denote $L_0 = \{\psi \geq t\}$, $L_1 = \{\psi \geq s\}$ and $L_\lambda = \{\psi \geq M_\alpha^\lambda(t, s)\}$. If $x \in L_0$ and $y \in L_1$, then $\psi((1-\lambda)x + \lambda y) \geq M_\alpha^\lambda(\psi(x), \psi(y)) \geq M_\alpha^\lambda(t, s)$. Hence,

$$L_\lambda \supset (1-\lambda)L_0 + \lambda L_1 \supset (1-\lambda) \cdot L_0 \oplus_p \lambda \cdot L_1,$$

the last inclusion following from the fact that $p \leq 1$. We deduce that

$$\begin{aligned} K_\lambda \cap L_\lambda &\supset ((1-\lambda) \cdot K_0 \oplus_p \lambda \cdot K_1) \cap ((1-\lambda) \cdot L_0 \oplus_p \lambda \cdot L_1) \\ &\supset (1-\lambda) \cdot (K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1). \end{aligned}$$

Hence,

$$\begin{aligned} h(M_\alpha^\lambda(t, s)) &= |K_\lambda \cap L_\lambda| \geq |(1-\lambda) \cdot (K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1)| \\ &\geq M_{\frac{n}{p}}^\lambda(f(t), g(s)), \end{aligned}$$

the last inequality is valid for $p \geq 0$ and follows from the log-Brunn-Minkowski inequality by using homogeneity of Lebesgue measure (see [7, beginning of Section 3]). Thus we may apply Corollary 1.6 to conclude that

$$\begin{aligned} \mu(K_\lambda) &= \int_0^{+\infty} h \geq M_{\left(\frac{n}{p} + \frac{1}{\alpha}\right)}^\lambda \left(\int_0^{+\infty} f, \int_0^{+\infty} g \right) \\ &= M_{\left(\frac{n}{p} + \frac{1}{\alpha}\right)}^\lambda(\mu(K_0), \mu(K_1)). \end{aligned} \quad \square$$

Since the log-Brunn-Minkowski inequality holds true in the plane, we deduce that Conjecture 3.2 holds true in the plane (with the restriction $\alpha \leq 1$). Notice that Conjecture 3.2 holds true in the unconditional case as a consequence of Corollary 1.5 (see [19]).

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