

# (Quasi)additivity Properties of the Legendre-Fenchel Transform and its Inverse, with Applications in Probability

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The notion of the Hölder convolution is introduced. The main result is that, under general conditions on functions  $L_1, \dots, L_n$ , one has  $(L_1 \boxplus \dots \boxplus L_n)^{*^{-1}} = L_1^{*^{-1}} + \dots + L_n^{*^{-1}}$ , where  $\boxplus$  denotes the Hölder convolution and  $L^{*^{-1}}$  is the function inverse to the Legendre-Fenchel transform  $L^*$  of a given function  $L$ . General properties of the functions  $L^*$  and  $L^{*^{-1}}$  are discussed. Applications to probability theory are presented. In particular, an upper bound on the quantiles of the distribution of the sum of (possibly dependent) random variables is given.

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## 1. (Quasi)additivity properties of the Legendre-Fenchel transform and its inverse

For brevity, let  $T := (0, \infty)$ . Take any function  $L: T \rightarrow [-\infty, \infty]$ . To avoid unpleasant trivialities, assume that

$$L(T) \neq \{\infty\}; \tag{1}$$

that is, for some  $t \in T$  one has  $L(t) < \infty$ .

The Legendre-Fenchel transform  $L^*$  of  $L$  may be defined by the formula

$$L^*(x) := \sup_{t \in T} [tx - L(t)] \tag{2}$$

for all  $x \in \mathbb{R}$ , so that  $L^*(x)$  may take any value on the extended real line  $[-\infty, \infty]$ .

Next, introduce the function  $L^{*-1}$ , which is the generalized inverse of the Legendre–Fenchel transform  $L^*$ , by the formula

$$L^{*-1}(u) := \inf E_L(u), \quad \text{where } E_L(u) := \{x \in \mathbb{R} : L^*(x) \geq u\}, \quad (3)$$

for all  $u \in \mathbb{R}$  (recall that, according to the standard convention, for any subset  $E$  of  $\mathbb{R}$ ,  $\inf E = \infty$  if and only if  $E = \emptyset$ ).

Take now any natural  $n$  and any functions  $L_1, \dots, L_n$  mapping  $T$  into  $(-\infty, \infty]$ . Then introduce what we shall refer to as the *Hölder convolution*,  $L_1 \boxplus \dots \boxplus L_n$ , by the formula

$$(L_1 \boxplus \dots \boxplus L_n)(t) := \inf \left\{ \sum_{j=1}^n \alpha_j L_j \left( \frac{t}{\alpha_j} \right) : (\alpha_1, \dots, \alpha_n) \in \Sigma_n \right\}$$

for all  $t \in \mathbb{R}$ , where

$$\Sigma_n := \{(\alpha_1, \dots, \alpha_n) \in (0, \infty)^n : \sum_{j=1}^n \alpha_j = 1\}.$$

The reason for using the term “Hölder convolution” will be given later. Here one may note that, according to [11, Theorem 5.6], in the case when  $L_1, \dots, L_n$  are proper convex functions, the Hölder convolution  $L_1 \boxplus \dots \boxplus L_n$  coincides with what is called in [11] the convex hull of the set  $\{L_1, \dots, L_n\}$  of functions.

At this point, let us also note the following additivity property of the Hölder convolution with respect to the family of power functions (cf. [1]):

$$p_{r,a} \boxplus p_{r,b} = p_{r,a+b} \quad (4)$$

for all real  $r \geq 1$  and  $a, b \geq 0$ , where  $p_{r,a}(t) := (at)^r$  for all  $t \in T$ . An immediate application of (4) is the following proof of the Minkowski inequality, say for any random variables (r. & s)  $X$  and  $Y$  and any real  $r \geq 1$  (cf. [1]):

$$\begin{aligned} \|X + Y\|_r^r &\leq \mathbb{E}(|X| + |Y|)^r = \mathbb{E} p_{r,|X|+|Y|}(1) = \mathbb{E}(p_{r,|X|} \boxplus p_{r,|Y|})(1) \\ &= \mathbb{E} \inf_{0 < \beta < 1} \left[ \beta p_{r,|X|} \left( \frac{1}{\beta} \right) + (1 - \beta) p_{r,|Y|} \left( \frac{1}{1-\beta} \right) \right] \\ &\leq \inf_{0 < \beta < 1} \mathbb{E} \left[ \beta p_{r,|X|} \left( \frac{1}{\beta} \right) + (1 - \beta) p_{r,|Y|} \left( \frac{1}{1-\beta} \right) \right] \\ &= \inf_{0 < \beta < 1} \left[ \beta p_{r,\|X\|_r} \left( \frac{1}{\beta} \right) + (1 - \beta) p_{r,\|Y\|_r} \left( \frac{1}{1-\beta} \right) \right] \\ &= (p_{r,\|X\|_r} \boxplus p_{r,\|Y\|_r})(1) = p_{r,\|X\|_r + \|Y\|_r}(1) = (\|X\|_r + \|Y\|_r)^r. \end{aligned}$$

The following theorem, which expresses an additivity property of the inverse of the Legendre–Fenchel transform, is the main result in this paper.

**Theorem 1.1.** *Suppose that the condition (1) holds with  $L_j$  in place of  $L$ , for each  $j \in \{1, \dots, n\}$ . Then*

$$(L_1 \boxplus \dots \boxplus L_n)^{*-1} = L_1^{*-1} + \dots + L_n^{*-1}. \quad (5)$$

Note that the right-hand side of equality (5) is correctly defined, in view of property (e) in Proposition A.1.

Theorem 1.1 is based on the following proposition, which appears to be of independent interest as well.

**Proposition 1.2.** *For any real  $x_1, \dots, x_n$*

$$\begin{aligned} \min (L_1^*(x_1), \dots, L_n^*(x_n)) &\leq (L_1 \square \cdots \square L_n)^*(x_1 + \cdots + x_n) \\ &\leq \max (L_1^*(x_1), \dots, L_n^*(x_n)). \end{aligned} \quad (6)$$

To quickly appreciate the relevance of Proposition 1.2 regarding Theorem 1.1, one can first make the easy observation that the function  $L^*$  is always nondecreasing (cf. Proposition A.1(c)). Consider now the easier case when the functions  $L_1^*, \dots, L_n^*$  and  $(L_1 \square \cdots \square L_n)^*$  are continuous and strictly increasing. Suppose then that for some real  $u$  and  $x_1, \dots, x_n$  and for all  $j \in \{1, \dots, n\}$  one has  $L_j^*(x_j) = u$ , so that  $x_j = L_j^{*-1}(u)$ . Then Proposition 1.2 yields  $(L_1 \square \cdots \square L_n)^*(x_1 + \cdots + x_n) = u$  and hence  $(L_1 \square \cdots \square L_n)^{*-1}(u) = x_1 + \cdots + x_n = \sum_1^n L_j^{*-1}(u)$ ; cf. (5). Of course, here there will be some technical difficulties to overcome, since in general we do not assume the additional conditions that the functions  $L_1^*, \dots, L_n^*$  and  $(L_1 \square \cdots \square L_n)^*$  are continuous and strictly increasing and all the equations  $L_j^*(x_j) = u$  for  $x_j$  have solutions for all real  $u$ . However, as shown in the proof of Theorem 1.1 below, these difficulties are not excessive.

**Proof of Proposition 1.2.** Take indeed any real  $x_1, \dots, x_n$ .

Consider the bijective correspondence

$$\begin{aligned} (0, \infty) \times \Sigma_n &\ni (t, \alpha_1, \dots, \alpha_n) \\ \longleftrightarrow \mathbf{t} &:= (t_1, \dots, t_n) := \left( \frac{t}{\alpha_1}, \dots, \frac{t}{\alpha_n} \right) \in (0, \infty)^n, \end{aligned} \quad (7)$$

under which one has  $\alpha_j = \alpha_j(\mathbf{t}) := \frac{1/t_j}{1/t_1 + \cdots + 1/t_n}$  for all  $j \in \{1, \dots, n\}$  and  $t = \frac{1}{1/t_1 + \cdots + 1/t_n}$ . It follows that for

$$L := L_1 \square \cdots \square L_n \quad (8)$$

one has

$$\begin{aligned} L^*\left(\sum_1^n x_j\right) &= \sup_{t>0} [t \sum_1^n x_j - L(t)] \\ &= \sup \left\{ t \sum_1^n x_j - \sum_1^n \alpha_j L_j\left(\frac{t}{\alpha_j}\right) : (t, \alpha_1, \dots, \alpha_n) \in (0, \infty) \times \Sigma_n \right\} \\ &= \sup \left\{ \sum_1^n \alpha_j(\mathbf{t}) [t_j x_j - L_j(t_j)] : (t_1, \dots, t_n) \in (0, \infty)^n \right\} \\ &\leq \sup \left\{ \max_{1 \leq j \leq n} [t_j x_j - L_j(t_j)] : (t_1, \dots, t_n) \in (0, \infty)^n \right\} \\ &= \max_{1 \leq j \leq n} \sup_{t_j>0} [t_j x_j - L_j(t_j)] = \max_{1 \leq j \leq n} L_j^*(x_j), \end{aligned}$$

which proves the second inequality in (6). Somewhat similarly, the expression in the third line of the above multi-line display is no less than

$$\begin{aligned}\mathcal{S} &:= \sup \left\{ \min_{1 \leq j \leq n} [t_j x_j - L_j(t_j)] : (t_1, \dots, t_n) \in (0, \infty)^n \right\} \\ &\geq \min_{1 \leq j \leq n} \sup_{t_j > 0} [t_j x_j - L_j(t_j)] = \min_{1 \leq j \leq n} L_j^*(x_j).\end{aligned}\tag{9}$$

To verify the inequality in (9) (which is in fact an equality), for each  $j \in \{1, \dots, n\}$  take any real  $v_j < \sup_{t_j > 0} [t_j x_j - L_j(t_j)]$ ; then there is some real  $t_j > 0$  such that  $t_j x_j - L_j(t_j) > v_j$ , whence

$$\min_{1 \leq j \leq n} [t_j x_j - L_j(t_j)] > \min_{1 \leq j \leq n} v_j,$$

so that  $\mathcal{S} > \min_{1 \leq j \leq n} v_j$ , for any real numbers  $v_j < \sup_{t_j > 0} [t_j x_j - L_j(t_j)]$ . Thus, the first inequality in (6) is proved as well.  $\square$

**Proof of Theorem 1.1.** Take any  $u \in \mathbb{R}$ . Recall the definition of the set  $E_L(u)$  in (3) and let, for brevity,

$$E := E_L(u), \quad z := \inf E, \quad E_j := E_{L_j}(u), \quad z_j := \inf E_j, \tag{10}$$

where  $L$  is as in (8) and  $j \in \{1, \dots, n\}$ . In view of part (b) of Proposition A.1 and Proposition 1.2, each of the sets  $E_1, \dots, E_n, E$  is nonempty, so that  $z_1 < \infty, \dots, z_n < \infty, z < \infty$ . In particular, the sum  $z_1 + \dots + z_n$  makes sense. Since the functions  $L_1^*, \dots, L_n^*, L^*$  are nondecreasing, each of the sets  $E_1, \dots, E_n, E$  is an interval in  $\mathbb{R}$  whose right endpoint is  $\infty$ , and the complements  $E_1^c, \dots, E_n^c, E^c$  of these sets are intervals whose left endpoint is  $-\infty$ .

On the other hand, by Proposition 1.2, the Minkowski sum  $E_1 + \dots + E_n$  of the sets  $E_1, \dots, E_n$  is a subset of  $E$ , which yields  $z_1 + \dots + z_n \geq z$ . Quite similarly,  $E_1^c + \dots + E_n^c \subseteq E^c$ , which yields  $z_1 + \dots + z_n \leq z$ , since  $\sup E_j^c = \inf E_j = z_j$  for all  $j$  and  $\sup E^c = \inf E = z$ . We conclude that  $z_1 + \dots + z_n = z$ . It remains to recall the definitions in (10) and (3).  $\square$

**Corollary 1.3.** *In the conditions of Theorem 1.1, suppose also that, for each  $j \in \{1, \dots, n\}$ ,  $L_j(t)/t$  is nondecreasing in  $t > 0$  (in particular, this will be the case when  $L_j$  is convex with  $L_j(0+) \leq 0$ ). Then*

$$(L_1 + \dots + L_n)^{*-1} \leq L_1^{*-1} + \dots + L_n^{*-1}.$$

This follows immediately from Theorem 1.1; indeed, if  $L_j(t)/t$  is nondecreasing in  $t > 0$  for each  $j \in \{1, \dots, n\}$ , then  $L_1 + \dots + L_n \leq L_1 \boxed{\text{H}} \dots \boxed{\text{H}} L_n$ .

Here one may note that the conditions of Corollary 1.3 will be satisfied if for all  $j \in \{1, \dots, n\}$  and  $t \in T$  one has  $L_j(t) = \ln \mathbb{E} e^{tX_j}$ , where  $X_1, \dots, X_n$  are any random variables (r.v.'s) such that  $\mathbb{E} e^{t_j X_j} < \infty$  for each  $j \in \{1, \dots, n\}$  and some  $t_j \in T$ .

Instead of the definition (3) of the (smallest possible) version,  $L^{*-1}$ , of the generalized inverse of the  $L^*$ , one can consider the following largest possible version of it, given by the formula

$$\widetilde{L^{*-1}}(u) := \inf\{x \in \mathbb{R} : L^*(x) > u\} \quad (11)$$

for all  $u \in \mathbb{R}$ . The functions  $L^{*-1}$  and  $\widetilde{L^{*-1}}$  are closely related:

**Proposition 1.4.** *For any  $u \in \mathbb{R}$*

$$\widetilde{L^{*-1}}(u) = L^{*-1}(u+) \quad \text{and} \quad L^{*-1}(u) = \widetilde{L^{*-1}}(u-).$$

**Proof of Proposition 1.4.** Introduce the sets

$$E_u := \{x \in \mathbb{R} : L^*(x) \geq u\} \quad \text{and} \quad \tilde{E}_u := \{x \in \mathbb{R} : L^*(x) > u\};$$

here and subsequently in this proof,  $u$  stands for an arbitrary real number. Then  $\tilde{E}_u = \bigcup_{v>u} E_v$  and  $E_u = \bigcap_{w<u} \tilde{E}_w$ . Hence,

$$\widetilde{L^{*-1}}(u) = \inf \tilde{E}_u = \inf_{v>u} \inf E_v = \inf_{v>u} L^{*-1}(v) = L^{*-1}(u+); \quad (12)$$

the second equality here follows because  $\inf \bigcup_{S \in \mathcal{S}} S = \inf_{S \in \mathcal{S}} \inf S$  for any set  $\mathcal{S}$  of subsets of  $\mathbb{R}$ , whereas the last equality in (12) is due to the function  $L^{*-1}$  being nondecreasing (see Proposition A.1(f)). Somewhat similarly,

$$L^{*-1}(u) = \inf E_u = \sup \inf_{w<u} \tilde{E}_w = \sup_{w<u} \widetilde{L^{*-1}}(w) = \widetilde{L^{*-1}}(u-);$$

the second equality here follows because, by Proposition A.1(c), the sets  $\tilde{E}_w$  are intervals in  $\mathbb{R}$  with the right endpoint equal  $\infty$ , and  $\inf \bigcap_{I \in \mathcal{I}} I = \sup_{I \in \mathcal{I}} \inf I$  for any set  $\mathcal{I}$  of such intervals.  $\square$

It follows by Proposition 1.4 and Proposition A.1(e) that Theorem 1.1 and Corollary 1.3 hold with  $\widetilde{L_j^{*-1}}$  in place of  $L_j^{*-1}$ . It further follows that these results will hold for any “weighted” generalized inverses of the form  $(1-\alpha)L_j^{*-1} + \alpha \widetilde{L_j^{*-1}}$ , for any fixed  $\alpha \in [0, 1]$ . However, in applications in probability as the one to be presented in Corollary 2.2, one should prefer to use  $L_j^{*-1}$ , the smallest member of this family of generalized inverses – because, at least formally, this choice maximizes the left-hand side of inequality (20) and thus provides the strongest version of that inequality.

On the other hand, it is straightforward to modify the proof of Theorem 1.1 so that to obtain its counterpart for the largest generalized inverse directly, rather than via Proposition 1.4. Also, Proposition 1.4 allows one to obtain an alternative proof of Theorem 1.1 (as originally stated, for the smallest generalized inverse), based on the following proposition (for the largest generalized inverse).

**Proposition 1.5.** *For any  $u \in \mathbb{R}$*

$$\widetilde{L^{*-1}}(u) = \hat{L}(u) := \inf_{t>0} \frac{u + L(t)}{t}. \quad (13)$$

Indeed, for  $L$  as in (8) and for any  $u \in \mathbb{R}$ , twice using Proposition 1.5 (and also the bijective correspondence in (7)) one has

$$\begin{aligned} \widetilde{L^{*-1}}(u) &= \hat{L}(u) = \inf_{t>0} \frac{u+L(t)}{t} \\ &= \inf \left\{ \frac{1}{t} \left[ u + \sum_1^n \alpha_j L_j \left( \frac{t}{\alpha_j} \right) \right] : (t, \alpha_1, \dots, \alpha_n) \in (0, \infty) \times \Sigma_n \right\} \\ &= \inf \left\{ \sum_1^n \frac{u+L_j(t_j)}{t_j} : (t_1, \dots, t_n) \in (0, \infty)^n \right\} \\ &= \sum_1^n \inf_{t_j>0} \frac{u+L_j(t_j)}{t_j} = \sum_1^n \widetilde{L_j^{*-1}}(u), \end{aligned} \quad (14)$$

so that one has (5) with  $\widetilde{L_j^{*-1}}$  instead of  $L_j^{*-1}$ .

**Proof of Proposition 1.5.** For any real  $u$  and  $x$

$$\begin{aligned} L^*(x) > u &\iff (\exists t > 0 \quad tx - L(t) > u) \\ &\iff (\exists t > 0 \quad x > \frac{u+L(t)}{t}) \iff x > \inf_{t>0} \frac{u+L(t)}{t}; \end{aligned}$$

thus,  $L^*(x) > u \iff x > \hat{L}(u) = \inf_{t>0} \frac{u+L(t)}{t}$ , see (13).  $\square$

Identity (13) was given by Rio [7, page 159] and [9, (4)] in the case when the function  $L$  is convex, nondecreasing, left-continuous, and with  $L(0) = 0$ ; also, it was tacitly assumed in the proof there that the infimum in (13) is attained and the function  $L^*$  is strictly increasing.

## 2. Applications to probability

Take any real-valued r.v.  $X$  and consider the corresponding Cramér–Chernoff function  $L_X$  defined by the formula

$$L_X(t) := \ln \mathbb{E} e^{tX} \quad (15)$$

for all  $t \in (0, \infty)$ ; thus,  $L_X$  is the logarithm of the moment generating function of  $X$ . Note that  $L_X(t) \in (-\infty, \infty]$  for all  $t \in (0, \infty)$ . Note also that the condition (1) is satisfied with  $L_X$  in place of  $L$  if and only if

$$\mathbb{E} e^{tX} < \infty \quad \text{for some } t \in T, \quad (16)$$

which will be henceforth assumed; then, by previous discussion,  $L_X^*(x) > -\infty$  and  $L_X^{*-1}(u) < \infty$  for all real  $x$  and  $u$ .

By Markov's inequality,

$$\mathbf{P}(X \geq x) \leq \exp\{-L_X^*(x)\} \quad (17)$$

for all  $x \in \mathbb{R}$ , with the convention  $\exp\{-\infty\} := 0$ .

Take now any real  $u$  and then any  $z \in (y, \infty)$ , where  $y := L^{*-1}(u)$  and  $L := L_X$ . Then there is  $x_z \in (-\infty, z)$  such that  $L^*(x_z) \geq u$ . Since  $L^*$  is nondecreasing, it follows that  $L^*(z) \geq u$ , for all  $z \in (y, \infty)$ . So,  $\mathbf{P}(X > y) = \lim_{z \downarrow y} \mathbf{P}(X > z) \leq \limsup_{z \downarrow y} \exp\{-L^*(z)\} \leq e^{-u}$ . Therefore,

$$\mathbf{P}(X > L_X^{*-1}(u)) \leq e^{-u} \quad (18)$$

for all real  $u$ . Thus,  $L_X^{*-1}(-\ln(1-q))$  may be considered as an upper bound on the  $q$ -quantile of the distribution of  $X$ , for any  $q \in (0, 1)$ .

Comparing (17) and (18), one may ask as to whether one can write

$$\mathbf{P}(X \geq L_X^{*-1}(u)) \leq e^{-u} \quad (19)$$

for all real  $u$ . A complete answer to this question is given by

**Proposition 2.1.** *Take any  $u \in \mathbb{R}$ . Then inequality (19) fails to hold if and only if all of the following conditions take place:*

- (i)  $x_{\max} := \sup \text{supp } X < \infty$  (here, as usual,  $\text{supp } X$  denotes the support set of the distribution of  $X$ );
- (ii)  $p_{\max} := \mathbf{P}(X = x_{\max}) > 0$ ;
- (iii)  $u > -\ln p_{\max}$ .

A proof of Proposition 2.1 is given at the end of Appendix A.

Note that condition (i) of Proposition 2.1 is actually implied by its condition (ii), since the r.v.  $X$  was assumed to be real-valued; however, it will be convenient to present condition (i) explicitly. It follows from Proposition 2.1 that inequality (19) holds for all  $u \in \mathbb{R}$  whenever  $x_{\max}$  is not an atom of the distribution of the r.v.  $X$ ; in particular, that will be the case if  $\text{supp } X$  is not bounded from above.

**Corollary 2.2.** *Let  $X_1, \dots, X_n$  be any r.v.'s such that the condition (16) is satisfied for all  $j \in \{1, \dots, n\}$  with  $X_j$  in place of  $X$ . Then for all  $u \in \mathbb{R}$*

$$\mathbf{P}(X_1 + \dots + X_n > L_{X_1}^{*-1}(u) + \dots + L_{X_n}^{*-1}(u)) \leq e^{-u}. \quad (20)$$

This follows immediately from (18) and Theorem 1.1. Indeed, by Hölder's inequality,  $\mathbf{E} \exp\{t \sum_{j=1}^n X_j\} \leq \prod_{j=1}^n (\mathbf{E} e^{tX_j/\alpha_j})^{\alpha_j}$  for all  $(\alpha_1, \dots, \alpha_n) \in \Sigma_n$ , so that

$$L_{X_1 + \dots + X_n} \leq L_{X_1} \boxed{\mathbf{H}} \cdots \boxed{\mathbf{H}} L_{X_n} \quad (21)$$

and hence, by (2) and (3),  $(L_{X_1+\dots+X_n})^{*-1} \leq (L_{X_1} \boxplus \dots \boxplus L_{X_n})^{*-1}$ . Now Theorem 1.1 yields

$$(L_{X_1+\dots+X_n})^{*-1}(u) \leq L_{X_1}^{*-1}(u) + \dots + L_{X_n}^{*-1}(u) \quad (22)$$

for all  $u \in \mathbb{R}$ . It remains to use (18) (with  $X = X_1 + \dots + X_n$ ).

One might note here that, in the case when the r.v.'s  $X_1, \dots, X_n$  are independent, one has  $L_{X_1+\dots+X_n} = L_{X_1} + \dots + L_{X_n}$  and thus inequality (22) can also be obtained from Corollary 1.3 in this case.

The use of Hölder's inequality to obtain (21) was the reason for this author to adopt the term “Hölder convolution”, used in this paper for the operation  $\boxplus$ . This operation was implicitly used (for  $n = 2$ ) by Rio in his paper [6], which originally inspired the present study. However, reasoning somewhat similar in spirit to that in the proof of Theorem 1.1 was used earlier in the proofs in [4] and [5, Corollary 1]; cf. also [2, Propositions 3.1 and 3.8].

In the case when  $n = 2$  and  $X_1$  and  $X_2$  are centered non-degenerate r.v.'s with moment generating functions (m.g.f.'s) finite in a neighborhood of 0, Rio [6, Lemma 2.1] proved (22) for  $u > 0$ , which of course implies (20) in this case. The assumptions that  $X_1$  and  $X_2$  be centered and non-degenerate r.v.'s with m.g.f.'s finite in a left neighborhood of 0 were removed in [8], and the proof was significantly shortened; in fact, display (14) follows largely the lines of reasoning presented in [8]. Note also that [6, Lemma 2.1] was used in [10].

## A. Supplements and auxiliaries

First here, let us list some general properties of the functions  $L^*$  and  $L^{*-1}$ :

### Proposition A.1.

- (a)  $L^*(x) > -\infty$  for all  $x \in \mathbb{R}$ .
- (b)  $L^*(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .
- (c) The function  $L^*$  is nondecreasing, convex, lower semi-continuous on  $\mathbb{R}$ , and hence continuous on its effective domain [11]  $\text{dom}(L^*) := \{x \in \mathbb{R} : L^*(x) < \infty\}$ . Let

$$x_\infty := \sup \text{dom}(L^*), \quad (23)$$

using the standard convention  $\sup \emptyset := -\infty$ ; so,

$$(i) \text{ dom}(L^*) = (-\infty, x_\infty) \quad \text{or} \quad (ii) \text{ dom}(L^*) = (-\infty, x_\infty]. \quad (24)$$

- (d) The monotonicity of  $L^*$  makes possible the definitions

$$u_{-\infty} := \lim_{x \rightarrow -\infty} L^*(x) \quad \text{and} \quad u_\infty := \lim_{x \uparrow x_\infty} L^*(x) \quad (25)$$

if  $x_\infty > -\infty$ ; if  $x_\infty = -\infty$  (i.e., if  $\text{dom}(L^*) = \emptyset$ ), let  $u_\infty := \infty$ ; thus, in any case  $-\infty \leq u_{-\infty} \leq u_\infty \leq \infty$ . Moreover,

$$u_\infty = \begin{cases} \infty & \text{if } \text{dom}(L^*) = (-\infty, x_\infty), \\ L^*(x_\infty) & \text{if } \text{dom}(L^*) = (-\infty, x_\infty]. \end{cases} \quad (26)$$

- (e)  $L^{*-1}(u) \leq \widetilde{L^{*-1}}(u) < \infty$  for all  $u \in \mathbb{R}$ .  
 (f) The functions  $L^{*-1}$  and  $\widetilde{L^{*-1}}$  are nondecreasing on  $\mathbb{R}$  and strictly increasing on the interval  $(u_{-\infty}, u_\infty)$  (which may be empty).

**Proof of Proposition A.1.** Properties (a) and (b) in Proposition A.1 follow by (1).

Property (c) is due to the fact that, by the definition (2),  $L^*$  is the pointwise supremum of a family of increasing affine functions on  $\mathbb{R}$ .

The inequality  $u_{-\infty} \leq u_\infty$  in part (d) follows from the monotonicity of  $L^*$ , stated in part (c), the first line in (26) follows by the lower semi-continuity of  $L^*$  and (b), and the second line in (26) follows by (25) and the continuity of  $L^*$  on  $\text{dom}(L^*)$ .

The first inequality of property (e) follows straight from the definitions (3) and (11) of  $L^{*-1}$  and  $\widetilde{L^{*-1}}$ , and the second inequality follows from property (b).

Finally, concerning property (f): that the functions  $L^{*-1}$  and  $\widetilde{L^{*-1}}$  are nondecreasing on  $\mathbb{R}$  follows trivially from the definitions, and the strict increase of these functions on  $(u_{-\infty}, u_\infty)$  follows because  $L^*$  is nondecreasing on  $\mathbb{R}$  and continuous on  $(-\infty, x_\infty)$ , and  $L^*$  maps  $(-\infty, x_\infty)$  onto an interval with the endpoints  $u_{-\infty}$  and  $u_\infty$ .  $\square$

The following proposition deals with the important special case when  $L$  is convex and, at least partially, strictly convex.

**Proposition A.2.** Suppose that the function  $L$  is convex on  $(0, \infty)$  and strictly convex (and hence real-valued) on an interval  $(t_0, t_1)$  such that  $0 < t_0 < t_1 < \infty$ . For any  $t \in (t_0, t_1)$ , let  $L'(t)$  and  $L'(t-)$  denote, respectively, the right and left derivatives of  $L$  at  $t$ . Let also  $L'(t_0+) := \lim_{t \downarrow t_0} L'(t)$ . Then the interval  $(L'(t_0+), L'(t_1-))$  is nonempty and the function  $L^*$  is strictly increasing on  $(L'(t_0+), L'(t_1-))$ . Moreover,  $(L'(t_0+), L'(t_1-)) \subseteq \text{dom}(L^*)$  and  $L^*$  is strictly increasing on the entire interval  $(L'(t_0+), \infty) \cap \text{dom}(L^*)$ .

**Proof of Proposition A.2.** The strict convexity of  $L$  implies that  $L'$  is strictly increasing and right-continuous on the nonempty interval  $(t_0, t_1)$ . So, the in-

terval  $(L'(t_0+), L'(t_1-))$  is nonempty and for any  $x$  in this interval one has

$$\begin{aligned} t_x &:= L'^{-1}(x) := \inf\{t \in (t_0, t_1) : L'(t) \geq x\} \\ &= \min\{t \in (t_0, t_1) : L'(t) \geq x\} \in (t_0, t_1). \end{aligned}$$

Clearly,  $t_x$  is nondecreasing in  $x \in (L'(t_0+), L'(t_1-))$ , and

$$L'(t_x-) \leq x \leq L'(t_x) \quad (27)$$

for all  $x \in (L'(t_0+), L'(t_1-))$ .

Take any  $x \in (L'(t_0+), L'(t_1-))$ . The right derivative of  $tx - L(t)$  in  $t \in (t_0, t_1)$  is  $x - L'(t)$ , which is greater than 0 for  $t \in (0, t_x)$  and no greater than 0 for  $t \in [t_x, \infty)$ . So,  $tx - L(t)$  is increasing in  $t \in (t_0, t_x]$  and non-increasing in  $t \in [t_x, t_1)$ . Since  $tx - L(t)$  is concave in  $t \in (0, \infty)$ , it follows that  $tx - L(t)$  is increasing in  $t \in (0, t_x]$  and non-increasing in  $t \in [t_x, \infty)$ . Thus,

$$L^*(x) = t_x x - L(t_x), \quad (28)$$

for all  $x \in (L'(t_0+), L'(t_1-))$ .

Take now any  $x$  and  $y$  such that  $L'(t_0+) < x < y < L'(t_1-)$ . For brevity, let here  $s := t_x$  and  $t := t_y$ , so that one has  $0 < t_0 < s \leq t < t_1$ . If  $s = t$  then  $L^*(x) = sx - L(s) = tx - L(t) < ty - L(t) = L^*(y)$ , which yields  $L^*(x) < L^*(y)$ . In the remaining case, when  $s < t$ , recall (27) and write

$$\begin{aligned} L^*(y) - L^*(x) &= [ty - L(t)] - [sx - L(s)] \\ &\geq [tL'(t-) - L(t)] - [sL'(s) - L(s)] \\ &= s[L'(t-) - L'(s)] + \int_s^t [L'(t-) - L'(\tau)] d\tau > 0, \end{aligned}$$

since  $s > 0$  and  $L'$  is strictly increasing on  $(t_0, t_1)$ . Thus, in either case  $L^*(x) < L^*(y)$ .

This shows that indeed  $L^*$  is strictly increasing on the open interval  $(L'(t_0+), L'(t_1-))$ , which in turn immediately implies that  $(L'(t_0+), L'(t_1-)) \subseteq \text{dom}(L^*)$ . The final sentence of Proposition A.2 follows because of this general fact (implicitly used to obtain (28)): if a function  $f$  is convex on  $(0, \infty)$ ,  $0 \leq a < b \leq c < \infty$ ,  $(a, c] \subseteq \text{dom } f$ , and  $f$  is strictly increasing on  $(a, b)$ , then  $f$  is so on the entire interval  $(a, c]$ .  $\square$

The following lemma seems to be of general interest; it will also be used in the proof of Proposition 2.1.

**Lemma A.3.** *For brevity, let here  $L := L_X$  and recall the definitions of  $x_\infty$  and  $u_\infty$  in (23) and (25). Then*

$$x_\infty = x_{\max}. \quad (29)$$

Moreover, if condition (i) of Proposition 2.1 holds, then

$$u_\infty = L^*(x_{\max}) = -\ln p_{\max}; \quad (30)$$

if, in addition, condition (ii) of Proposition 2.1 holds as well, then

$$L^{*-1}(u) = \begin{cases} x_{\max} & \text{if } u \in (-\ln p_{\max}, \infty), \\ x_{\max} & \text{if } u = -\ln p_{\max} \text{ and } p_{\max} < 1, \\ -\infty & \text{if } u = -\ln p_{\max} \text{ and } p_{\max} = 1. \end{cases} \quad (31)$$

**Proof of Lemma A.3.** If for some real  $y$  one has  $y \in (x_{\max}, \infty)$ , then  $x_{\max} \in \mathbb{R}$  and  $\mathbf{P}(X \leq x_{\max}) = 1$ , whence  $L(t) \leq tx_{\max}$  for all  $t > 0$  and  $L^*(y) \geq \sup_{t>0} [ty - tx_{\max}] = \infty$ . On the other hand, if  $y \in (-\infty, x_{\max})$  then  $p_y := \mathbf{P}(X > y) > 0$  and hence  $L(t) \geq ty + \ln p_y$  for all  $t > 0$  and  $L^*(y) \leq -\ln p_y < \infty$ . Thus,  $L^*(y) = \infty$  for all  $y \in (x_{\max}, \infty)$  and  $L^*(y) < \infty$  for all  $y \in (-\infty, x_{\max})$ . Now (29) follows.

To prove (30), suppose that indeed  $x_{\max} < \infty$ ; then, by the definition of  $x_{\max}$  in condition (i) of Proposition 2.1,  $x_{\max} \in \mathbb{R}$ . The first equality in (30) now follows by (26) and (29); indeed, (26) shows that  $u_\infty = L^*(x_\infty)$  whenever  $x_\infty \in \mathbb{R}$ . Concerning the second equality in (30), one has  $\mathbf{P}(X \leq x_{\max}) = 1$  and hence  $\mathbf{E} \exp\{t(X - x_{\max})\}$  is non-increasing in  $t$ . Using this monotonicity and dominated convergence, one sees that

$$\begin{aligned} \exp\{-L^*(x_{\max})\} &= \inf_{t>0} \mathbf{E} \exp\{t(X - x_{\max})\} = \lim_{t \rightarrow \infty} \mathbf{E} \exp\{t(X - x_{\max})\} \\ &= \mathbf{E} \mathbf{I}\{X = x_{\max}\} = p_{\max}; \end{aligned}$$

here  $\exp\{-L^*(x_{\max})\}$  is naturally understood as 0 when  $L^*(x_{\max}) = \infty$ . Thus, (30) is completely proved.

It remains to prove (31), assuming that  $x_{\max} < \infty$  and  $p_{\max} > 0$ , so that, by (30),  $u_\infty < \infty$ . Now (26) and (24) yield  $\text{dom}(L^*) = (-\infty, x_\infty]$ , whence, by (29),

$$\text{dom}(L^*) = (-\infty, x_{\max}]. \quad (32)$$

Take any  $u \in \mathbb{R}$ .

If  $u \in (-\ln p_{\max}, \infty)$  then, by (30) and (32),  $L^*(x_{\max}) < u < L^*(x)$  for all  $x \in (x_{\max}, \infty)$  and hence  $L^{*-1}(u) = x_{\max}$ .

Consider next the case when  $u = -\ln p_{\max}$  and  $0 < p_{\max} < 1$ . Then, in view of (15) and (16),  $L''(t)(\mathbf{E} e^{tX})^2 = \mathbf{E} X^2 e^{tX} \mathbf{E} e^{tX} - (\mathbf{E} X e^{tX})^2 > 0$  (by the Cauchy-Schwarz inequality) for all  $t$  in a right neighborhood of 0, so that  $L$  is strictly convex in that neighborhood. Hence, by Proposition A.2 and (32), the function  $L^*$  is strictly increasing on a nonempty interval of the form  $(x, x_{\max}]$ . So, since the function  $L^*$  is nondecreasing on  $\mathbb{R}$ , it is strictly less than  $L^*(x_{\max})$  on the interval  $(-\infty, x_{\max})$ . Hence, by (3) and (30), indeed  $L^{*-1}(u) = x_{\max}$  if  $u = -\ln p_{\max}$  and  $p_{\max} < 1$ .

Finally, if  $p_{\max} = 1$  (i.e.,  $\mathbf{P}(X = x_{\max}) = 1$ ), then  $L(t) = tx_{\max}$  for all  $t \in (0, \infty)$ ,  $L^*(x)$  equals 0 for  $x \in (-\infty, x_{\max}]$  and  $\infty$  for  $x \in (x_{\max}, \infty)$ , so that  $L^{*-1}(u)$  equals  $-\infty$  for  $u \in (-\infty, 0]$  and  $x_{\max}$  for  $u \in (0, \infty)$ . In particular,  $L^{*-1}(u) = -\infty$  if  $u = 0 = -\ln p_{\max}$ .  $\square$

**Proof of Proposition 2.1.** Let again, for brevity,  $L := L_X$ .

To prove the “only if” part of Proposition 2.1, take any  $u \in \mathbb{R}$  violating (19). Then, by (18),

$$\mathbf{P}(X = L^{*-1}(u)) > 0.$$

Therefore and because the r.v.  $X$  was assumed to be real-valued, necessarily

$$L^{*-1}(u) > -\infty, \tag{33}$$

which, by the definition of  $u_{-\infty}$  in (25), in turn implies  $u > u_{-\infty}$ , so that the interval  $(u_{-\infty}, u]$  is nonempty.

If now  $u < u_{\infty}$  then the interval  $(u_{-\infty}, u]$  is contained in the interval  $(u_{-\infty}, u_{\infty})$ , on which, by Proposition A.1(f), the function  $L^{*-1}$  is strictly increasing. So, for any  $v \in (u_{-\infty}, u)$  one has  $L^{*-1}(u) > L^{*-1}(v)$  and hence

$$\mathbf{P}(X \geq L^{*-1}(u)) \leq \mathbf{P}(X > L^{*-1}(v)) \leq e^{-v},$$

by (18) with  $v$  in place of  $u$ ; letting now  $v \uparrow u$ , one obtains (18), which contradicts the assumption on  $u$ .

Thus, it is necessary that  $u \in [u_{\infty}, \infty)$ . Then, of course,  $u_{\infty} < \infty$ , which, in view of (26), means that case (ii) in (24) takes place, whence necessarily  $x_{\infty} \in \mathbb{R}$ . So, by (29), condition (i) of Proposition 2.1 holds. Further, by (30),  $u_{\infty} = -\ln p_{\max}$  and hence  $u \in [-\ln p_{\max}, \infty)$ .

If  $u = -\ln p_{\max}$  and  $p_{\max} < 1$  then, by (31),  $L^{*-1}(u) = x_{\max}$  and hence  $\mathbf{P}(X \geq L^{*-1}(u)) = \mathbf{P}(X \geq x_{\max}) = \mathbf{P}(X = x_{\max}) = p_{\max} = e^{-u}$ , so that (19) holds. If  $u = -\ln p_{\max}$  and  $p_{\max} = 1$  then, by (31),  $L^{*-1}(u) = -\infty$ , which contradicts (33).

We conclude that any  $u \in \mathbb{R}$  violating (19) must satisfy condition (iii) of Proposition 2.1 as well. Thus, the “only if” part is proved.

The “if” part of Proposition 2.1 follows immediately by Lemma A.3. Indeed, if  $p_{\max} := \mathbf{P}(X = x_{\max}) > 0$  and  $u > -\ln p_{\max}$ , then the right-hand side of (19) is strictly less  $p_{\max} = \mathbf{P}(X = x_{\max})$ , whereas, by (31), the left-hand side of (19) equals  $\mathbf{P}(X \geq x_{\max})$ , which is no less than  $\mathbf{P}(X = x_{\max})$ . Proposition 2.1 is now completely proved.  $\square$

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