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On Rectangular Constant in Normed Linear Spaces

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We study the properties of rectangular constant $\mu(\mathbb{X})$ in a normed linear space $\mathbb{X}$. We prove that $\mu(\mathbb{X}) = 3$ if and only if the unit sphere contains a straight line segment of length 2. In fact, we prove that the rectangular modulus attains its upper bound if and only if the unit sphere contains a straight line segment of length 2. We prove that if the dimension of the space $\mathbb{X}$ is finite then $\mu(\mathbb{X})$ is attained. We also find a necessary and sufficient condition for a normed linear space to be an inner product space in terms of conditions involving rectangular constant.

Keywords: Birkhoff-James Orthogonality, rectangular constant

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1. Introduction

Let $(\mathbb{X}, \|\ \ \|)$ be a normed linear space, unless otherwise mentioned we assume the space to be real. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ be the unit ball and unit sphere respectively of the normed linear space $\mathbb{X}$. There are various notions of orthogonality in a normed linear space. Among them one of the most important and widely used notion is orthogonality in the sense of Birkhoff-James [2, 7]:
For any two elements $x, y \in \mathbb{X}$, x is said to be orthogonal to y in the sense of Birkhoff-James, written as $x \perp_B y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in K(= \mathbb{R}\ or\ \mathbb{C})$.
In the year of 1969, Joly [8] introduced the notion of rectangular constant of a

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real normed linear space of dimension ≥ 2, which is defined as follows:

$$\mu(\mathbb{X}) = \sup_{x \perp_B y} \left\{ \frac{\|x\| + \|y\|}{\|x + y\|} \right\}.$$

The notion of rectangular constant plays a very important role in the geometry of normed linear spaces and it has been studied by Joly [8], del Rio and Benitez [3], Desbiens [4] and Baronti and Casini [1]. In 1999 the concept of the rectangular constant of a normed linear space was generalized by Serb [9] to introduce rectangular modulus, $\mu_{\mathbb{X}}(\lambda)$, as a function $\mu_{\mathbb{X}} : (0, \infty) \longrightarrow \mathbb{R}$ defined as follows

$$\mu_{\mathbb{X}}(\lambda) = \sup \left\{ \max\{ \frac{\lambda^2 + t}{\|\lambda u + tv\|}, \frac{1 + \lambda^2 t}{\|u + \lambda tv\|} \} : t > 0, u, v \in S_{\mathbb{X}}, u \perp_B v \right\}.$$

For $\lambda = 1$, $\mu_{\mathbb{X}}(1)$ is the rectangular constant of the normed linear space $\mathbb{X}$. He also defined $*$-rectangular modulus $\mu^*_{\mathbb{X}}(\lambda)$ as

$$\mu^*_{\mathbb{X}}(\lambda) = \sup \left\{ \frac{\lambda^2 + t}{\|\lambda u + tv\|} : t > 0, u, v \in S_{\mathbb{X}}, u \perp_B v \right\}$$

and proved that

$$\mu_{\mathbb{X}}(\lambda) = \max\{\mu^*_{\mathbb{X}}(\lambda), \lambda \mu^*_{\mathbb{X}}(1/\lambda)\}.$$

Serb [9] proved that

$$\sqrt{1 + \lambda^2} \leq \mu_{\mathbb{X}}(\lambda) \leq \max\{\lambda + 2, 1 + 2\lambda\} \quad \text{for all } \lambda > 0.$$

It is well known that for any real normed linear space $\mathbb{X}$, $\sqrt{2} \leq \mu(\mathbb{X}) \leq 3$. Joly [8] proved that if dimension of $\mathbb{X}$ is greater than 2 then $\mu(\mathbb{X}) = \sqrt{2}$ if and only if $\mathbb{X}$ is an inner product space. Later del Rio and Benitez [3] proved it for dimension 2. However there is no characterisation for the attainment of upper bound of the rectangular constant. In this paper we prove that $\mu(\mathbb{X}) = 3$ if and only if the unit sphere contains a straight line segment of length 2. In fact, we prove that the rectangular modulus attains its upper bound if and only if the unit sphere contains a straight line segment of length 2. We further prove that if the dimension of the space $\mathbb{X}$ is finite then $\mu(\mathbb{X})$ is attained. We also prove that a normed linear space is an inner product space if and only if $\sup\{\frac{1+|t|}{\|y+tx\|} : x, y \in S_{\mathbb{X}} \text{ with } x \perp_B y\} \leq \sqrt{2}$ for all t satisfying $|t| \in (3 - 2\sqrt{2}, \sqrt{2} + 1)$.

2. Main Results

Let

$$\mu(x, y) = \frac{\|x\| + \|y\|}{\|x + y\|}, \quad x, y \in \mathbb{X} - \{0\}, x \perp_B y.$$

Then

$$\mu(\mathbb{X}) = \sup\{\mu(x, y) : x, y \in \mathbb{X} - \{0\}, x \perp_B y\}.$$

As the rectangular constant is defined by taking the supremum of $\mu(x,y)$ over the set $\{(x,y) \in (\mathbb{X}-\{0\}) \times (\mathbb{X}-\{0\}) : x \perp_B y\}$ so it is natural to ask whether it is attained i.e., whether there exists some $x_0, y_0 \in \mathbb{X}-\{0\}, x_0 \perp_B y_0$ such that $\mu(\mathbb{X}) = \mu(x_0, y_0)$.
We here show that the rectangular constant is attained in a finite dimensional normed linear space.

Theorem 2.1. *Rectangular constant $\mu(\mathbb{X})$ is attained in any finite dimensional normed linear space $\mathbb{X}$.*

Proof. It is easy to check that the rectangular constant $\mu(\mathbb{X})$ can be written as

$$\mu(\mathbb{X}) = \sup\left\{\frac{1+|t|}{\|y+tx\|} : x, y \in S_{\mathbb{X}}, x \perp_B y, t \in \mathbb{R}\right\}.$$

Let $M = \{(x,y) \in S_{\mathbb{X}} \times S_{\mathbb{X}} : x \perp_B y\}$. Clearly M is compact as $\mathbb{X}$ is finite dimensional. Now there exists a sequence $\{((x_n, y_n), t_n)\} \in M \times \mathbb{R}$ such that

$$\mu(\mathbb{X}) = \lim \frac{1+|t_n|}{\|y_n + t_n x_n\|}.$$

Then clearly $\{t_n\}$ is bounded, otherwise $\mu(\mathbb{X}) = \lim \frac{1+|t_n|}{\|y_n+t_n x_n\|} = 1$, which contradicts that $\mu(\mathbb{X}) \geq \sqrt{2}$. Now as M is compact and $\{t_n\}$ is bounded, we can see that there exists $((x,y),t) \in M \times \mathbb{R}$ such that

$$\mu(\mathbb{X}) = \frac{1+|t|}{\|y+tx\|}.$$

that is rectangular constant is attained. This completes the proof. $\square$

Following the same method we can show that for each λ, the $*$-rectangular modulus $\mu^*_{\mathbb{X}}(\lambda)$ is attained in any finite dimensional normed linear space.

Remark 2.2. Although we could not give an example to show that the rectangular constant may not be attained in an infinite dimensional normed linear space, existence of such a space seems more probable to us.

We next prove that the rectangular constant $\mu(\mathbb{X}) = 3$ if and only if the unit sphere contains a straight line segment of length 2. In fact, we prove that the rectangular modulus attains its upper bound if and only if the unit sphere contains a straight line segment of length 2. For this we first prove the following theorem.

Theorem 2.3. *Let $\mathbb{X}$ be a finite dimensional normed linear space. Let l be a real number such that*

$$\|u-v\| < l \text{ for all } u, v \in S_{\mathbb{X}} \text{ with } \|tu+(1-t)v\| = 1 \text{ for all } t \in [0,1]$$

Then for any $x, y \in S_{\mathbb{X}}$ with $x \perp_B y$ we have $\|x+\lambda y\| > 1$ for all λ with $|\lambda| \geq l$.

Proof. If possible let there be $x, y \in S_{\mathbb{X}}$ with $x \perp_B y$ such that $\|x + \lambda_0 y\| = 1$ for some λ_0 with $|\lambda_0| \geq l$. Now for $t \in (0,1)$ we have

$$\begin{aligned} 1 &= t\|x\| + (1-t)\|x + \lambda_0 y\| \\ &\geq \|tx + (1-t)(x + \lambda_0 y)\| \\ &= \|x + \lambda_0(1-t)y\| \\ &\geq 1 \end{aligned}$$

$\Rightarrow \|tx + (1-t)(x + \lambda_0 y)\| = 1$ for all $t \in (0,1)$.
So $\|x - (x + \lambda_0 y)\| = \|\lambda_0 y\| = |\lambda_0| \geq l$, this is a contradiction. Hence $\|x + \lambda y\| > 1$ for all λ with $|\lambda| \geq l$. This completes the proof. □

The converse part of the above theorem also holds as follows from the next theorem. For this we prove a lemma first which we will use repeatedly.

Lemma 2.4. *Let $\mathbb{X}$ be a normed linear space. If $x, y \in S_{\mathbb{X}}$ such that $\|tx + (1-t)y\| = 1$ for all $t \in (0,1)$, then $x \perp_B (y - x)$.*

Proof. For this it is sufficient to show that $\|x + k(y - x)\| \geq 1$ for all k such that $|k| < 1$.
If $0 < k < 1$ then $\|x + k(y - x)\| = \|(1-k)x + ky\| = 1$ and if $-1 < k < 0$, then $\|x + k(y - x)\| = \|(1-k)x + ky\| \geq | \|(1-k)x\| - \|ky\| | = 1$. □

Theorem 2.5. *Let $\mathbb{X}$ be a finite dimensional normed linear space. Let l be a real number such that for any $x, y \in S_{\mathbb{X}}$ with $x \perp_B y$ we have $\|x + \lambda y\| > 1$ for all λ with $|\lambda| \geq l$. Then*

$$\|u - v\| < l \textit{ for all } u, v \in S_{\mathbb{X}} \textit{ with } \|tu + (1-t)v\| = 1 \textit{ for all } t \in [0,1]$$

Proof. If possible suppose $u, v \in S_{\mathbb{X}}$ are such that $\|tu + (1-t)v\| = 1$ for all $t \in (0,1)$ and $\|u - v\| \geq l$.
Then by Lemma 2.4, we have $u \perp_B (v - u)$ and so by homogenity of Birkhoff-James orthogonality $u \perp_B \frac{v-u}{\|v-u\|}$. Let $x = u, y = \frac{v-u}{\|v-u\|}$ and $\lambda = \|u - v\|$.
Then $| \lambda | \geq l$ and $\|x + \lambda y\| = \|u + v - u\| = 1$, which is a contradiction to our assumption. This completes the proof. □

We next prove the theorem which shows that the $*$-rectangular modulus attains its upper bound if and only if the unit sphere contains a straight line segment of length 2.

Theorem 2.6. *Let $\mathbb{X}$ be a finite dimensional normed linear space. Then $*$-rectangular modulus $\mu^*_{\mathbb{X}}(\lambda) = \lambda + 2$ if and only if there exists two points $u', v' \in S_{\mathbb{X}}$ such that $\|tu' + (1-t)v'\| = 1$ for all $t \in (0,1)$ and $\|u' - v'\| = 2$.*

Proof. We prove the necessary part by the method of contradiction. We first assume that $\|u' - v'\| < 2$ whenever $u', v' \in S_{\mathbb{X}}$ satisfies $\|tu' + (1-t)v'\| = 1$ for all $t \in (0,1)$. It is easy to check that the $*$-rectangular modulus $\mu^*_{\mathbb{X}}(\lambda)$ can be written as

$$\mu^*_{\mathbb{X}}(\lambda) = \sup\left\{\frac{\lambda + t}{\|u + tv\|} : u\perp_B v, u, v \in S_{\mathbb{X}}, t > 0\right\}.$$

If $0 < t \leq 1$ then $\frac{\lambda+t}{\|u+tv\|} \leq 1 + \lambda$ and if $1 < t < 2$ then $\frac{\lambda+t}{\|u+tv\|} < \lambda + 2$.
If $2 \leq t < 2 + \frac{1}{\lambda}$, then $\|u + tv\| \geq t - 1$ and $\|u + tv\| > 1$, by Theorem 2.3.
So $\|u + tv\| > 1.\frac{\lambda}{\lambda+2} + (t-1)\frac{2}{\lambda+2} \geq \frac{\lambda+t}{\lambda+2}$, which implies $\frac{\lambda+t}{\|u+tv\|} < \lambda + 2$.
Finally let $t \geq 2+\frac{1}{\lambda}$. Then $\|u+tv\| \geq t-1 \geq \frac{\lambda+t}{\lambda+1}$, which implies $\frac{\lambda+t}{\|u+tv\|} \leq \lambda+1$.
Since $*$-rectangular modulus is attained in any finite dimensional normed linear space, $\sup\{\frac{\lambda+t}{\|u+tv\|} : u\perp_B v, u, v \in S_{\mathbb{X}}, t > 0\} < 2 + \lambda$.
Considering other cases we can similarly show that $\mu^*_X(\lambda) < 2 + \lambda$.

Conversely, suppose that there exist two points $u, v \in S_{\mathbb{X}}$ such that $\|tu + (1-t)v\| = 1$ for all $t \in (0,1)$ and $\|u - v\| = 2$. Now by Lemma 2.4, $u\perp_B(v-u)$ and so $u\perp_B\frac{v-u}{\|v-u\|}$, as before. Then

$$\frac{\lambda + \|v - u\|}{\|u + \|v - u\|\frac{v-u}{\|v-u\|}\|} = \lambda + \|v - u\| = \lambda + 2.$$

Hence $\mu^*_{\mathbb{X}}(\lambda) = \sup\{\frac{\lambda+t}{\|u+tv\|} : u\perp_B v, u, v \in S_{\mathbb{X}}\} = 2 + \lambda$.
This completes the proof. □

Corollary 2.7. *Let $\mathbb{X}$ be a finite dimensional normed linear space. Then rectangular modulus $\mu_{\mathbb{X}}(\lambda) = \max\{2 + \lambda, 1 + 2\lambda\}$ if and only if there exists two points $u, v \in S_{\mathbb{X}}$ such that $\|tu+(1-t)v\| = 1$ for all $t \in (0,1)$ and $\|u-v\| = 2$.*

Proof. Since $\mu_{\mathbb{X}}(\lambda) = \max\{\mu^*_{\mathbb{X}}(\lambda), \lambda\mu^*_{\mathbb{X}}(\frac{1}{\lambda})\}$, the result follows from the last theorem and the fact that $\mu^*_{\mathbb{X}}(\lambda) = \lambda + 2$ if and only if $\lambda\ \mu^*_{\mathbb{X}}(\frac{1}{\lambda}) = 1 + 2\lambda$. □

Corollary 2.8. *Let $\mathbb{X}$ be a finite dimensional normed linear space. Then rectangular constant $\mu(\mathbb{X}) = 3$ if and only if there exists two points $u, v \in S_{\mathbb{X}}$ such that $\|tu + (1-t)v\| = 1$ for all $t \in (0,1)$ and $\|u - v\| = 2$.*

Geometric interpretation. The above result shows that the rectangular constant of the space is equal to 3 if and only if the unit sphere of the space contains a straight line segment of length 2.

We next prove that if $\mathbb{X}$ is a two dimensional polyhedral space (i.e., a space with finitely many extreme points) then in finding $\mu(\mathbb{X})$, it is sufficient to consider the supremum over all orthogonal $x, y \in \mathbb{X} - \{0\}$, where x is an extreme point of $S_{\mathbb{X}}$.

Theorem 2.9. *Let $\mathbb{X}$ be a two-dimensional real polyhedral space. Then for any $x, y \in \mathbb{X} - \{0\}$ with $x\perp_B y$, $\mu(x, y) = \frac{\|x\|+\|y\|}{\|x+y\|} \leq \frac{\|u\|+\|v\|}{\|u+v\|}$ where $u \in S_{\mathbb{X}}$ is an extreme point and $u\perp_B v$.*

Proof. We first prove the following two lemmas:

Lemma 2.10. *Let $\mathbb{X}$ be a finite dimensional polyhedral space. Let $x_1, x_2 \in S_{\mathbb{X}}$ are two adjacent extreme points and let $x = tx_1 + (1-t)x_2 \in S_{\mathbb{X}}$, for some $t \in (0,1)$. Then $x \perp_B y \Rightarrow x_1 \perp_B y$ and $x_2 \perp_B y$.*

Proof. Assume $x \perp_B y$. By Theorem 2.1 of James [6], we can find a linear functional f on $\mathbb{X}$ such that $\|f\| = f(x) = 1$ and $f(y) = 0$.
Now $|f(x_1)| \leq \|x_1\| = 1$ and $|f(x_2)| \leq \|x_2\| = 1$.
If $|f(x_1)| < 1$ then

$$\begin{aligned} 1 &= \|tx_1 + (1-t)x_2\| \\ &= t\|x_1\| + (1-t)\|x_2\| \\ &> t|f(x_1)| + (1-t)|f(x_2)| \\ &\geq |f(tx_1 + (1-t)x_2)| \\ &= |f(x)|, \quad \text{which is a contradiction.} \end{aligned}$$

So $|f(x_1)| = 1$. Similarly $|f(x_2)| = 1$. As $f(y) = 0$ so by the same Theorem 2.1 of James [6], we get $x_1 \perp_B y$ and $x_2 \perp_B y$.
Thus $x \perp_B y \Rightarrow x_1 \perp_B y$ and $x_2 \perp_B y$. This completes the proof of first lemma. □

Lemma 2.11. *Let $x, y \in \mathbb{X}$ be such that $x \perp_B y$. Then $\|x + \lambda_1 y\| \geq \|x + \lambda_2 y\|$, if $\lambda_1 > \lambda_2 > 0$ or $\lambda_1 < \lambda_2 < 0$.*

Proof. Let $\lambda_1 > \lambda_2 > 0$, then we can write $\lambda_1 = \lambda_2 + c$, for some $c > 0$. To show

$$\|x + \lambda_1 y\| = \|x + \lambda_2 y + cy\| \geq \|x + \lambda_2 y\|$$

If $\|x + \lambda_2 y\| = \|x\|$, then

$$\|x + \lambda_1 y\| \geq \|x\| = \|x + \lambda_2 y\|$$

and we are done.
Let $\|x + \lambda_2 y\| > \|x\|$.
We know, for the two elements $(x + \lambda_2 y)$ and y, either $\|x + \lambda_2 y + \lambda y\| \geq \|x + \lambda_2 y\|$ for all $\lambda \geq 0$ or $\|x + \lambda_2 y + \lambda y\| \geq \|x + \lambda_2 y\|$ for all $\lambda \leq 0$.
Put $\lambda = -\lambda_2 < 0$, then $\|x\| = \|x + \lambda_2 y - \lambda_2 y\| < \|x + \lambda_2 y\|$
So $\|x + \lambda_2 y + cy\| \geq \|x + \lambda_2 y\|$ i.e. $\|x + \lambda_1 y\| \geq \|x + \lambda_2 y\|$.
Similarly we can prove the result for $\lambda_1 < \lambda_2 < 0$. This completes the proof of the lemma. □

Proof of the theorem continued: Let $x', y' \in \mathbb{X} - \{0\}$ and $x' \perp_B y'$. Then $\mu(x', y') = \frac{\|x'\| + \|y'\|}{\|x' + y'\|} = \frac{1 + |\lambda|}{\|x + \lambda y\|}$, for some $x, y \in S_{\mathbb{X}}$ with $x \perp_B y$.
If $x \in S_{\mathbb{X}}$ is an extreme point, then we are done.
Suppose $x \in S_{\mathbb{X}}$ is not an extreme point. Then $x = tx_1 + (1-t)x_2$, for some

$t \in (0,1)$ and for two adjacent extreme points $x_1, x_2 \in S_{\mathbb{X}}$.
Let $|\lambda| \leq \|x_1 - x_2\|$
By Lemma 2.4, $x_1 \perp_B (x_2 - x_1)$. Now

$$\frac{1+|\lambda|}{\|x+\lambda y\|} \leq 1 + |\lambda| \leq 1 + \|x_2 - x_1\| \leq \frac{\|x_1\| + \|x_2 - x_1\|}{\|x_1 + x_2 - x_1\|}.$$

Let $|\lambda| > \|x_1 - x_2\|$. As $x \perp_B y$ so $x_1 \perp_B y$ and $x_2 \perp_B y$, by Lemma 2.10. Then $x_1 = x + \lambda_1 y$ and $x_2 = x + \lambda_2 y$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Claim: λ_1, λ_2 are of opposite sign and $|\lambda_1|, |\lambda_2| < \|x_1 - x_2\|$.
If possible, let $\lambda_1 > 0, \lambda_2 > 0$. Then

$$\begin{aligned} x_1 = x + \lambda_1 y &= tx_1 + (1-t)x_2 + \lambda_1 y \\ \Rightarrow (1-t)x_1 &= (1-t)x_2 + \lambda_1 y \\ \Rightarrow x_1 - x_2 &= \frac{\lambda_1 y}{1-t} \qquad (i) \\ \Rightarrow \|x_1 - x_2\| &= \frac{|\lambda_1|}{1-t} > |\lambda_1| \end{aligned}$$

Proceeding similarly from $x_2 = x + \lambda_2 y$, we get (ii) $x_2 - x_1 = \frac{\lambda_2 y}{t}$.
Then $|\lambda_2| < \|x_1 - x_2\|$
From (i) and (ii) we get $\frac{\lambda_1}{t-1} = \frac{\lambda_2}{t}$ and so $t = \frac{\lambda_2}{\lambda_2 - \lambda_1}$
If $\lambda_1 > \lambda_2$, then $t < 0$ and if $\lambda_2 > \lambda_1$, then $t > 1$, contradiction in both cases.
Similar case for $\lambda_1 < 0$, $\lambda_2 < 0$.

Let us first assume that $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda > 0$.
As $|\lambda_2| < \|x_1 - x_2\| < |\lambda|$, we have $0 < \lambda + \lambda_2 < \lambda$.
Then by Lemma 2.11, $\|x_2 + \lambda y\| = \|x + \lambda_2 y + \lambda y\| \leq \|x + \lambda y\|$.
Let us next assume that $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda < 0$, then $\lambda < \lambda + \lambda_1 < 0$.
So $\|x_1 + \lambda y\| = \|x + \lambda_1 y + \lambda y\| = \|x + (\lambda_1 + \lambda)y\| \leq \|x + \lambda y\|$.
Similarly, if $\lambda_1 < 0$, $\lambda_2 > 0$, then,
$\|x_1 + \lambda y\| \leq \|x + \lambda y\|$, when $\lambda > 0$
$\|x_2 + \lambda y\| \leq \|x + \lambda y\|$, when $\lambda < 0$.
Now $\mu(x', y') = \frac{1+|\lambda|}{\|x+\lambda y\|} \leq \frac{1+|\lambda|}{\|x_i+\lambda y\|}$, $i \in \{1,2\}$
$\Rightarrow \mu(x', y') \leq \frac{\|x_i\|+\|\lambda y\|}{\|x_i+\lambda y\|}$, $i \in \{1,2\}$.
This completes the proof. □

We next give an example showing that v in Theorem 2.9 can not be taken as either an extreme point or a scalar multiple of an extreme point on the unit sphere.

Example 2.12. Consider the normed linear space $(\mathbb{R}^2, \|\ \|_\infty)$, which is a two-dimensional polyhedral space. The only extreme points of the unit sphere are $\pm(1,1), \pm(-1,1)$. We first note that $(1,1) \perp_B k(1,-1)$ for any real k and

$(1,1)\perp_B(-2,0)$. We also see that $\frac{\|(1,1)\|+\|k(1,-1)\|}{\|(1,1)+k(1,-1)\|} = 1 < \frac{\|(1,1)\|+\|(-2,0)\|}{\|(1,1)+(-2,0)\|} = 3$. Similarly considering other extreme points of the unit sphere and their scalar multiples, we conclude that v in the previous theorem can not be taken as an extreme point in this case.

We next prove the theorem which characterises an inner product space. Joly [8] and del Rio and Benitez [3] proved that a finite dimensional normed linear space is an inner product space if and only if $\sup\{\frac{1+|\lambda|}{\|y+\lambda x\|} : x\perp_B y, x, y \in S_{\mathbb{X}}, \lambda \in \mathbb{R}\} = \sqrt{2}$. Here we prove that a finite dimensional normed linear space is an inner product space if and only if $\sup\{\frac{1+|\lambda|}{\|y+\lambda x\|} : x\perp_B y, x, y \in S_{\mathbb{X}}, \mid \lambda \mid \in (3-2\sqrt{2}, \sqrt{2}+1)\} \leq \sqrt{2}$.

Theorem 2.13. *A finite dimensional normed linear space $\mathbb{X}$ is an inner-product space if and only if $x, y \in S_{\mathbb{X}}$ with $x\perp_B y$ we have $\frac{1+|\lambda|}{\|y+\lambda x\|} \leq \sqrt{2}$ for all λ satisfying $|\lambda| \in (3-2\sqrt{2}, \sqrt{2}+1)$.*

Proof. The necessary part of the theorem follows easily. For the sufficient part we proceed as follows.
As $x\perp_B y$, we have $\|y+\lambda x\| \geq |\lambda|$.
$\Rightarrow \frac{1+|\lambda|}{\|y+\lambda x\|} \leq \frac{1+|\lambda|}{|\lambda|} \leq \sqrt{2}$ for all λ satisfying $|\lambda| \geq \sqrt{2}+1$.
Again, $\|y+\lambda x\| \geq |1-|\lambda||$.
Let $\|y+\lambda x\| \geq 1-|\lambda|$
$\Rightarrow \frac{1+|\lambda|}{\|y+\lambda x\|} \leq \frac{1+|\lambda|}{1-|\lambda|} \leq \sqrt{2}$ for all λ satisfying $|\lambda| \leq 3-2\sqrt{2}$.
Let $\|y+\lambda x\| \geq |\lambda|-1$
$\Rightarrow \frac{1+|\lambda|}{\|y+\lambda x\|} \leq \frac{1+|\lambda|}{|\lambda|-1} \leq \sqrt{2}$ for all λ satisfying $|\lambda| \geq 3+2\sqrt{2} > \sqrt{2}+1$.
So if $\frac{1+|\lambda|}{\|y+\lambda x\|} \leq \sqrt{2}$ for all λ with $|\lambda| \in (3-2\sqrt{2}, \sqrt{2}+1)$, then $\frac{1+|\lambda|}{\|y+\lambda x\|} \leq \sqrt{2}$ for all $\lambda \in \mathbb{R}$. So $\mathbb{X}$ is an inner product space. This completes the proof. □

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