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Polynomial Inequalities on the $\pi/4$–Circle Sector

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A number of sharp inequalities are proved for the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the supremum norm on the sector $D\left(\frac{\pi}{4}\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$. Among the main results we can find sharp Bernstein and Markov inequalities and the calculation of the polarization constant and the unconditional constant of the canonical basis of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

Keywords: Bernstein and Markov inequalities, unconditional constants, polarizations constants, polynomial inequalities, homogeneous polynomials, extreme point.

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1. Preliminaries

The study of low dimensional spaces of polynomials can be an interesting source of examples and counterexamples related to more general questions. In this paper we mind 2-variable, real 2-homogeneous polynomials endowed with the supremum norm on the sector $D\left(\frac{\pi}{4}\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$. The space of such polynomials is represented by $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. This paper can be seen as a continuation of [15] and [20]. Other publications in the same spirit can be found in [11, 12, 21, 22, 24, 25].

If $P(x, y) = ax^2 + by^2 + cxy$, we will often represent P as the point (a, b, c) in $\mathbb{R}^3$. Hence, the norm of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is in fact the norm in $\mathbb{R}^3$ given by

$$\|(a, b, c)\|_{D\left(\frac{\pi}{4}\right)} = \sup\left\{|ax^2 + by^2 + cxy| : (x, y) \in D\left(\frac{\pi}{4}\right)\right\}.$$

In Section 3, the notation $\mathcal{L}^s\left({}^2D\left(\frac{\pi}{4}\right)\right)$ will be useful to represent the symmetric bilinear forms on $\mathbb{R}^2$ endowed with the supremum norm on $D\left(\frac{\pi}{4}\right)$.

In order to obtain sharp polynomial inequalities in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ we will use the so called Krein-Milman approach, which is based on the fact that norm attaining convex functions attain their norm at an extreme point of their domain. Hence, an explicit description of the norm $\|\cdot\|_{D\left(\frac{\pi}{4}\right)}$ and the extreme points of the unit ball $B_{D\left(\frac{\pi}{4}\right)}$, denoted by $\operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)$, will be required. Both are presented below:

Lemma 1.1 ([20, Theorem 3.1]). *If $P(x, y) = ax^2 + by^2 + cxy$, then*

$$\|P\|_{D\left(\frac{\pi}{4}\right)} = \begin{cases} \max\left\{|a|, \frac{1}{2}|a+b+c|, \frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\} & \text{if } c(a-b) \geq 0, \\ \max\{|a|, \frac{1}{2}|a+b+c|\} & \text{if } c(a-b) \leq 0, \end{cases}$$

Lemma 1.2 ([20, Theorem 4.4]). *The extreme points of the unit ball of $\mathcal{P}({}^2D(\frac{\pi}{4}))$ are given by*

$$\operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right) = \left\{\pm P_t, \pm Q_s, \pm(1, 1, 0) : \ -1 \leq t \leq 1 \text{ and } 1 \leq s \leq 5 + 4\sqrt{2}\right\},$$

where

$$P_t := (t, 4 + t + 4\sqrt{1+t}, -2 - 2t - 4\sqrt{1+t}),$$
$$Q_s := (1, s, -2\sqrt{2(1+s)}).$$

Let us describe now the three inequalities that will be studied in this paper. Section 2 is devoted to obtain a Bernstein type inequality for polynomials in

$\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Namely, for a fixed $(x,y) \in D\left(\frac{\pi}{4}\right)$, we find the best (smallest) constant $\Phi(x,y)$ in the inequality

$$\|\nabla P(x,y)\|_2 \leq \Phi(x,y)\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^2$. Similarly, we also obtain a Markov global estimate on the gradient of polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, or in other words, the smallest constant $M > 0$ in the inequality

$$\|\nabla P(x,y)\|_2 \leq M\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $(x,y) \in D\left(\frac{\pi}{4}\right)$. It is necessary to mention that the study of Bernstein and Markov type inequalities has a longstanding tradition. The interested reader can find further information on this classical topic in [2, 13, 14, 16, 18, 19, 23, 26, 28, 29, 30, 31].

In Section 3 we find the smallest constant $K > 0$ in the inequality

$$\|L\|_{D\left(\frac{\pi}{4}\right)} \leq K\|P\|_{D\left(\frac{\pi}{4}\right)},$$

where P is an arbitrary polynomial in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $L \in \mathcal{L}^s\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is the polar of P. Observe that here $\|L\|_{D\left(\frac{\pi}{4}\right)}$ stands for the sup norm of L over $D\left(\frac{\pi}{4}\right)^2$. Hence, what we do is to provide the polarization constant of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. The calculation of polarization constants in various polynomial spaces is largely motivated as the extensive, existing bibliography on the topic shows (see for instance [10, 13, 17, 27]).

Finally, in Section 4 we investigate the smallest constant $C > 0$ in the inequality

$$\||P|\|_{D\left(\frac{\pi}{4}\right)} \leq C\|P\|_{D\left(\frac{\pi}{4}\right)}, \tag{1}$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, where $|P|$ is the modulus of P, i.e., if $P(x,y) = ax^2 + by^2 + cxy$, then $|P|(x,y) = |a|x^2 + |b|y^2 + |c|xy$. The constant C turns out to be the unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. It is interesting to note that already in 1914, H. Bohr [4] studied this type of inequalities for infinite complex power series. Actually, the study of Bohr radii is nowadays a fruitful field (see for instance [1, 3, 6, 7, 8, 9]). Observe that the relationship between unconditional constants in polynomial spaces and inequalities of the type (1) was already noticed in [7].

2. Bernstein and Markov-type inequalities for polynomials on sectors

In this section we provide sharp estimates on the Euclidean length of the gradient ∇P of a polynomial P in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

Theorem 2.1. *For every $(x,y) \in D\left(\frac{\pi}{4}\right)$ and $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ we have*

$$\|\nabla P\|_2 \leq \Phi(x,y)\|P\|_{D\left(\frac{\pi}{4}\right)},$$

where

$$\Phi(x,y) = \begin{cases} 4\left[\left(13+8\sqrt{2}\right)x^2 + \left(69+48\sqrt{2}\right)y^2 - 2\left(28+20\sqrt{2}\right)xy\right] \\ \qquad \text{if } 0 \leq y \leq \frac{\sqrt{2}-1}{2}x \text{ or } \left(4\sqrt{2}-5\right)x \leq y \leq x, \\ \frac{x^4}{y^2} + 4(x^2+y^2) \\ \qquad \text{if } \frac{\sqrt{2}-1}{2}x \leq y \leq \left(\sqrt{2}-1\right)x, \\ \frac{\left(3x^2-2xy+3y^2\right)^2}{2(x-y)^2} \\ \qquad \text{if } \left(\sqrt{2}-1\right)x \leq y \leq \left(4\sqrt{2}-5\right)x. \end{cases}$$

Proof. In order to calculate $\Phi(x,y) := \sup\{\|\nabla P(x,y)\|_2 : \|P\|_{D\left(\frac{\pi}{4}\right)} \leq 1\}$, by the Krein-Milman approach, it is sufficient to calculate

$$\sup\{\|\nabla P(x,y)\|_2 : P \in \operatorname{ext}(B_{D\left(\frac{\pi}{4}\right)})\}.$$

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with $P_t(x,y) = tx^2 + \left(4+t+4\sqrt{1+t}\right)y^2 - 2\left(1+t+2\sqrt{1+t}\right)xy$, $t \in [-1,1]$. Then,

$$\begin{aligned} &\nabla P_t(x,y) \\ &= \left(2tx - 2\left(1+t+2\sqrt{1+t}\right)y, 2\left(4+t+4\sqrt{1+t}\right)y - 2\left(1+t+2\sqrt{1+t}\right)x\right), \end{aligned}$$

so that

$$\begin{aligned} \|\nabla P_t(x,y)\|_2^2 = 4t^2x^2 &+ 4\left(1+t+2\sqrt{1+t}\right)^2y^2 - 8t\left(1+t+2\sqrt{1+t}\right)xy \\ &+ 4\left(4+t+4\sqrt{1+t}\right)^2y^2 + 4\left(1+t+2\sqrt{1+t}\right)^2x^2 \\ &- 8\left(4+t+4\sqrt{1+t}\right)\left(1+t+2\sqrt{1+t}\right)xy. \end{aligned}$$

Make now the change $u = \sqrt{1+t} \in \left[0, \sqrt{2}\right]$, so that

$$\begin{aligned} &\|\nabla P_u(x,y)\|_2^2 \\ &= 8(x-y)^2u^4 + 16\left(x^2-4xy+3y^2\right)u^3 \\ &\qquad + 8\left(x^2-10xy+13y^2\right)u^2 + 32\left(3y^2-xy\right)u + 4\left(x^2+9y^2\right). \end{aligned}$$

Since

$$\begin{aligned} &\frac{\partial}{\partial u}\|\nabla P_u(x,y)\|_2^2 \\ &= 16\left(2\left(x-y\right)^2u^2 + \left(x^2-8xy+7y^2\right)u + 2y\left(3y-x\right)\right)\left(u+1\right), \end{aligned}$$

it follows that the critical points of $\|DP_u(x,y)\|_2^2$ are $u = \frac{2y}{x-y}$, $u = \frac{3y-x}{2(x-y)}$ and $u = -1$ if $x \neq y$ and $u = 4$ and $u = -1$ if $x = y$. Since we need to consider $0 \leq u \leq \sqrt{2}$, we can directly omit the case $x = y$.

Therefore, we can write

$$\frac{\partial}{\partial u}\|\nabla P_u(x,y)\|_2^2 = 32(x-y)^2\left(u - \frac{2y}{x-y}\right)\left(u - \frac{3y-x}{2(x-y)}\right)(u+1).$$

Let $u_1 = \frac{2y}{x-y}$ and $u_2 = \frac{3y-x}{2(x-y)}$ (again, since we need to consider $0 \leq u \leq \sqrt{2}$, we can omit the solution $u = -1$). Also, we have the extra conditions $u_1 \in [0, \sqrt{2}]$ whenever $0 \leq y \leq \left(\sqrt{2}-1\right)x$ and $u_2 \in [0, \sqrt{2}]$ whenever $\frac{1}{3}x \leq y \leq \left(4\sqrt{2}-5\right)x$. Considering all these facts, we need to compare the quantities

$$\begin{aligned} C_1(x,y) &:= \|\nabla P_{u_1}(x,y)\|_2^2 = \|\nabla P_{t_1}\|_2^2 \\ &= 4\frac{x^6 - 4x^5y + 7x^4y^2 - 8x^3y^3 + 7x^2y^4 - 4xy^5 + y^6}{(x-y)^4} \\ &= 4\left(x^2 + y^2\right), \end{aligned}$$

for $0 \leq y \leq \left(\sqrt{2}-1\right)x$ and $t_1 = \frac{3y^2+2xy-x^2}{(x-y)^2}$,

$$\begin{aligned} C_2(x,y) &:= \|\nabla P_{u_2}(x,y)\|_2^2 = \|\nabla P_{t_2}\|_2^2 \\ &= \frac{9x^6 - 30x^5y + 55x^4y^2 - 68x^3y^3 + 55x^2y^4 - 30xy^5 + 9y^6}{2(x-y)^4} \\ &= \frac{\left(3x^2 - 2xy + 3y^2\right)^2}{2(x-y)^2}, \end{aligned}$$

for $\frac{1}{3}x \leq y \leq \left(4\sqrt{2}-5\right)x$ and $t_2 = \frac{5y^2+2xy-3x^2}{4(x-y)^2}$,

$$C_3(x,y) := \|\nabla P_{t_3=-1}\|_2^2 = 4\left(x^2 + 9y^2\right),$$

and

$$\begin{aligned} C_4(x,y) &:= \|\nabla P_{t_4=1}\|_2^2 \\ &= 4\big[\big(13+8\sqrt{2}\big)x^2 + \big(69+48\sqrt{2}\big)y^2 - 2\big(28+20\sqrt{2}\big)xy\big]. \end{aligned}$$

Let us focus now on $Q_s = \left(1, s, -2\sqrt{2(1+s)}\right)$, $1 \leq s \leq 5+4\sqrt{2}$. Then, we have

$$\|\nabla Q_s(x,y)\|_2^2 = 4x^2 + 4s^2y^2 + 8(1+s)(x^2+y^2) - 8(1+s)\sqrt{2(1+s)}xy.$$

Making the change $v = \sqrt{2(1+s)} \in \left[2, 2+2\sqrt{2}\right]$, we need to study the function

$$\|\nabla Q_v(x,y)\|_2^2 = v^2\left(y^2v^2 - 4xyv + 4x^2\right) + 4\left(x^2+y^2\right).$$

If $x = y = 0$ we have $\|\nabla Q_v(x,y)\|_2^2 = 0$, so we will assume both $x \neq 0$ and $y \neq 0$. The critical points of $\|\nabla Q_v(x,y)\|_2^2$ are $v = \frac{x}{y}, v = \frac{2x}{y}$ and $v = 0$ (but $0 \notin [2, 2+2\sqrt{2}]$). Observe that $v_1 = \frac{x}{y} \in \left[2, 2+2\sqrt{2}\right]$ whenever $\frac{\sqrt{2}-1}{2}x \leq y \leq \frac{1}{2}x$ and $v_2 = \frac{2x}{y} \in \left[2, 2+2\sqrt{2}\right]$ whenever $y \geq \left(\sqrt{2}-1\right)x$. Thus, we also need to compare the quantities

$$C_5(x,y) := \|\nabla Q_{v_1}(x,y)\|_2^2 = \|\nabla Q_{s_1}(x,y)\|_2^2 = \frac{x^4}{y^2} + 4\left(x^2+y^2\right),$$

for $\frac{\sqrt{2}-1}{2}x \leq y \leq \frac{1}{2}x$ and $s_1 = \frac{x^2-2y^2}{2y^2}$,

$$C_6(x,y) := \|\nabla Q_{v_2}(x,y)\|_2^2 = \|\nabla Q_{s_2}(x,y)\|_2^2 = 4\left(x^2+y^2\right),$$

for $\left(\sqrt{2}-1\right)x \leq y \leq x$ and $s_2 = \frac{2x^2-y^2}{y^2}$, and also

$$C_7(x,y) := \|\nabla Q_{s_3=1}\|_2^2 = 4\left(x^2+y^2\right) + 16(x-y)^2,$$

and

$$\begin{aligned} C_8(x,y) &:= \|\nabla Q_{s_4=5+4\sqrt{2}}\|_2^2 \\ &= \left(12+8\sqrt{2}\right)\left[4x^2 + \left(12+8\sqrt{2}\right)y^2 - \left(8+8\sqrt{2}\right)xy\right] + 4\left(x^2+y^2\right) \\ &= 4\left[\left(13+8\sqrt{2}\right)x^2 + \left(69+48\sqrt{2}\right)y^2 - 2\left(28+20\sqrt{2}\right)xy\right]. \end{aligned}$$

Note that (the reader can take a look at Figures 2.1, 2.2 and 2.3)

$$C_1(x,y), C_6(x,y) \leq C_7(x,y) \leq \begin{cases} C_4(x,y) & \text{if } 0 \leq y \leq \frac{2-\sqrt{2}}{2}x \text{ or } \frac{1}{2}x \leq y \leq x, \\ C_5(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x \leq y \leq \frac{1}{2}x, \end{cases}$$

$$C_3(x,y) \leq \begin{cases} C_2(x,y) & \text{if } \frac{1}{3}x \leq y \leq \left(4\sqrt{2}-5\right)x, \\ C_4(x,y) & \text{if } 0 \leq y \leq \frac{1}{3}x \text{ or } (4\sqrt{2}-5)x \leq y \leq x, \end{cases}$$

$$C_8(x,y) = C_4(x,y).$$

Hence, for $(x,y) \in D\left(\frac{\pi}{4}\right)$,

$$\begin{aligned} \Phi(x,y) = &\ \sup\left\{\|\nabla P(x,y)\|_2 \,:\, P \in \operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)\right\} \\ = &\begin{cases} C_4(x,y) & \text{if } 0 \leq y \leq \frac{\sqrt{2}-1}{2}x \text{ or } (4\sqrt{2}-5)x \leq y \leq x, \\ C_5(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x \leq y \leq (\sqrt{2}-1)x, \\ C_2(x,y) & \text{if } (\sqrt{2}-1)x \leq y \leq (4\sqrt{2}-5)x. \end{cases} \end{aligned}$$

In order to illustrate the previous step, the reader can take a look at Figure 2.4. $\square$

Corollary 2.2. *If $P \in \mathcal{P}\left(D\left(\frac{\pi}{4}\right)\right)$, then*

$$\sup\left\{\|\nabla P(x,y)\|_2 : (x,y) \in D\left(\frac{\pi}{4}\right)\right\} \leq 4(13+8\sqrt{2})\|P\|_{D\left(\frac{\pi}{4}\right)},$$

with equality for the polynomials $P_1(x,y) = \pm\left(x^2 + (5+4\sqrt{2})y^2 - 2(2+2\sqrt{2})xy\right)$.

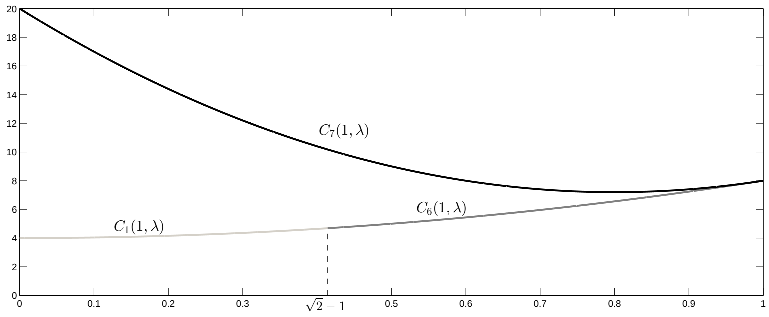

Figure 2.1: Graphs of the mappings $C_1(1,\lambda)$, $C_6(1,\lambda)$, $C_7(1,\lambda)$.

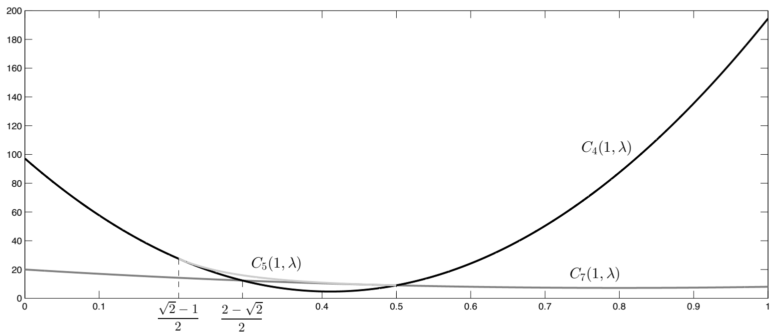

Figure 2.2: Graphs of the mappings $C_4(1,\lambda)$, $C_5(1,\lambda)$, $C_7(1,\lambda)$.

3. Polarization constants for polynomials on sectors

In this section we find the exact value of the polarization constant of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. In order to do that, we prove a Bernstein type inequality for polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Observe that if $P\in\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $(x,y)\in D\left(\frac{\pi}{4}\right)$ then the differential $DP(x,y)$ of P at (x,y) can be viewed as a linear form. What we shall do is to find the best estimate for $\|DP(x,y)\|_{D\left(\frac{\pi}{4}\right)}$ (the sup norm of $DP(x,y)$ over the sector $D\left(\frac{\pi}{4}\right)$) in terms of (x,y) and $\|P\|_{D\left(\frac{\pi}{4}\right)}$. First, we state a lemma that will be useful in the future:

Lemma 3.1. *Let $a,b\in\mathbb{R}$. Then,*

$$\begin{aligned}
&\sup_{\theta\in\left[0,\frac{\pi}{4}\right]}|a\cos\theta+b\sin\theta|\\
&=\begin{cases}\max\left\{|a|,\frac{\sqrt{2}}{2}|a+b|\right\} & \text{if } \frac{b}{a}>1 \text{ or } \frac{b}{a}<0,\\ \sqrt{a^2+b^2} & \text{otherwise.}\end{cases}\\
&=\begin{cases}\sqrt{a^2+b^2} & \text{if } 0<\frac{b}{a}<1,\\ \frac{\sqrt{2}}{2}|a+b| & \text{if } \left(1-\sqrt{2}\right)b<a<b \text{ or } b<a<\left(1-\sqrt{2}\right)b,\\ |a| & \text{if } -\left(1+\sqrt{2}\right)a<b<0 \text{ or } 0<b<-\left(1+\sqrt{2}\right)a.\end{cases}
\end{aligned}$$

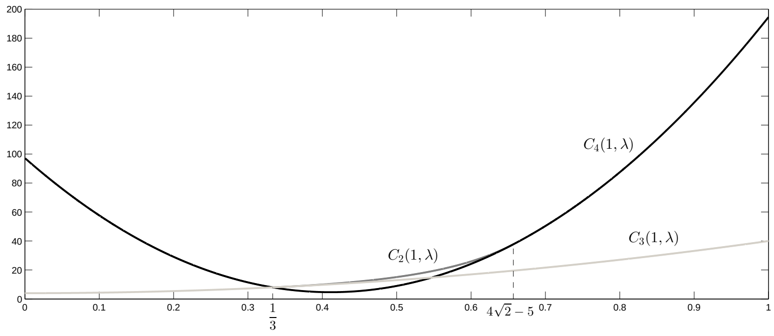

Figure 2.3: Graphs of the mappings $C_2(1,\lambda)$, $C_3(1,\lambda)$, $C_4(1,\lambda)$.

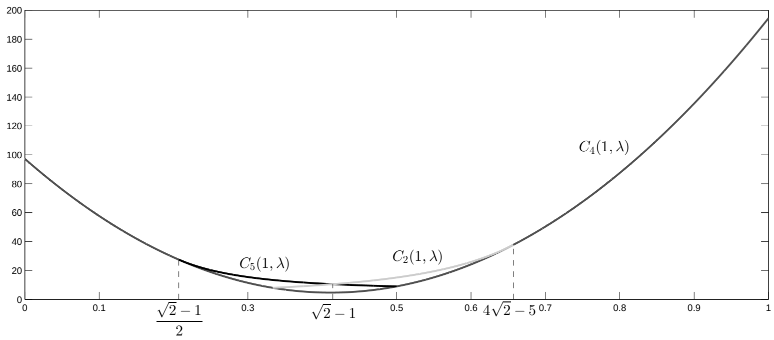

Figure 2.4: Graphs of the mappings $C_2(1,\lambda)$, $C_4(1,\lambda)$, $C_5(1,\lambda)$.

Theorem 3.2. *For every* $(x,y) \in D(\frac{\pi}{4})$ *and* $P \in \mathcal{P}(^2D(\frac{\pi}{4}))$ *we have that*

$$\|DP(x,y)\|_{D(\frac{\pi}{4})} \leq \Psi(x,y)\|P\|_{D(\frac{\pi}{4})}, \tag{2}$$

where

$$\Psi(x,y) = \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x - \left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < (\sqrt{2}-1)x, \\ 2\left(x + \frac{y^2}{x-y}\right) & \text{if } (\sqrt{2}-1)x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x \end{cases}$$

Moreover, inequality (2) *is optimal for each* $(x,y) \in D(\frac{\pi}{4})$.

Proof. In order to calculate $\Psi(x,y) := \sup\{\|DP(x,y)\|_{D(\frac{\pi}{4})} : \|P\|_{D(\frac{\pi}{4})}) \leq 1\}$, by the Krein-Milman approach, it suffices to calculate

$$\sup\{\|DP(x,y)\|_{D(\frac{\pi}{4})} : P \in \text{ext}(B_{D(\frac{\pi}{4})})\}.$$

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with

$$P_t(x,y) = tx^2 + \left(4+t+4\sqrt{1+t}\right)y^2 - \left(2+2t+4\sqrt{1+t}\right)xy.$$

So we may write

$$\nabla P_t(x,y) = \big(2tx - \big(2+2t+4\sqrt{1+t}\big)y,\ 2\big(4+t+4\sqrt{1+t}\big)y - \big(2+2t+4\sqrt{1+t}\big)x\big),$$

from which

$$\begin{aligned}
&\|DP_t(x,y)\|_{D(\frac{\pi}{4})} \\
&= \sup_{0\le\theta\le\frac{\pi}{4}} \big|2\big[tx - \big(1+t+2\sqrt{1+t}\big)y\big]\cos\theta \\
&\qquad + 2\big[\big(4+t+4\sqrt{1+t}\big)y - \big(1+t+2\sqrt{1+t}\big)x\big]\sin\theta\big| \\
&= 2x \sup_{0\le\theta\le\frac{\pi}{4}} |f_\lambda(t,\theta)|,
\end{aligned}$$

$$\text{for } f_\lambda(t,\theta) = \big[t - \big(1+t+2\sqrt{1+t}\big)\lambda\big]\cos\theta + \big[\big(4+t+4\sqrt{1+t}\big)\lambda - \big(1+t+2\sqrt{1+t}\big)\big]\sin\theta,$$

where $\lambda = \frac{y}{x}$, $x \neq 0$ (the case $x = 0$ is trivial, since the only point in $D(\frac{\pi}{4})$ where $x = 0$ is $(0,0)$, in which case $P_t(0,0) = \|DP_t(0,0)\|_{D(\frac{\pi}{4})} = 0$).

We need to calculate

$$\sup_{-1\le t\le 1} \|DP_t(x,y)\|_{D(\frac{\pi}{4})} = 2x \sup_{\substack{0\le\theta\le\frac{\pi}{4}\\ -1\le t\le 1}} |f_\lambda(t,\theta)|.$$

Let us define $C_1 = [-1,1]\times[0,\frac{\pi}{4}]$. We will analyze 5 cases.

Case 1: $(t,\theta)\in(-1,1)\times(0,\frac{\pi}{4})$. We are interested just in critical points. Hence,

$$\begin{aligned}
\frac{\partial f_\lambda}{\partial t}(t,\theta) &= \left[\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)\right]\sin\theta \\
&\quad + \left[1 - \left(1+\frac{1}{\sqrt{1+t}}\right)\lambda\right]\cos\theta = 0,
\end{aligned} \tag{3}$$

$$\begin{aligned}
\frac{\partial f_\lambda}{\partial \theta}(t,\theta) &= \big[\big(1+t+2\sqrt{1+t}\big)\lambda - t\big]\sin\theta \\
&\quad + \big[\big(4+t+4\sqrt{1+t}\big)\lambda - \big(1+t+2\sqrt{1+t}\big)\big]\cos\theta = 0
\end{aligned} \tag{4}$$

Equation (4) tells us that

$$\sin\theta = \frac{\big(4+t+4\sqrt{1+t}\big)\lambda - \big(1+t+2\sqrt{1+t}\big)}{t - \big(1+t+2\sqrt{1+t}\big)\lambda}\cos\theta. \tag{5}$$

If we now plug (5) in equation (3), we obtain

$$\begin{aligned}
0 = &\left\{\left[1 - \left(1+\frac{1}{\sqrt{1+t}}\right)\lambda\right] + \left[\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)\right]\right. \\
&\left.\times \frac{\big(4+t+4\sqrt{1+t}\big)\lambda - \big(1+t+2\sqrt{1+t}\big)}{t - \big(1+t+2\sqrt{1+t}\big)\lambda}\right\}\cos\theta.
\end{aligned}$$

Using that $0 < \theta < \frac{\pi}{4}$, we can conclude

$$0 = \left[1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda\right] + \left[\left(1 + \frac{2}{\sqrt{1+t}}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right)\right] \times \frac{\left(4 + t + 4\sqrt{1+t}\right)\lambda - \left(1 + t + 2\sqrt{1+t}\right)}{t - \left(1 + t + 2\sqrt{1+t}\right)\lambda}$$

and thus

$$\begin{aligned}
0 = {} & \left[1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda\right] \cdot \left[t - \left(1 + t + 2\sqrt{1+t}\right)\lambda\right] \\
& + \left[\left(1 + \frac{2}{\sqrt{1+t}}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right)\right] \\
& \cdot \left[\left(4 + t + 4\sqrt{1+t}\right)\lambda - \left(1 + t + 2\sqrt{1+t}\right)\right] \\
= {} & t - \left(1 + t + 2\sqrt{1+t}\right)\lambda - t\lambda + \left(1 + t + 2\sqrt{1+t}\right)\lambda^2 - \frac{\lambda t}{\sqrt{1+t}} \\
& + \frac{\lambda^2}{\sqrt{1+t}}\left(1 + t + 2\sqrt{1+t}\right) + \left(1 + \frac{2}{\sqrt{1+t}}\right)\left(4 + t + 4\sqrt{1+t}\right)\lambda^2 \\
& - \left(1 + \frac{2}{\sqrt{1+t}}\right)\left(1 + t + 2\sqrt{1+t}\right)\lambda \\
& - \left(1 + \frac{1}{\sqrt{1+t}}\right)\left(4 + t + 4\sqrt{1+t}\right)\lambda \\
& + \left(1 + \frac{1}{\sqrt{1+t}}\right)\left(1 + t + 2\sqrt{1+t}\right) \\
= {} & t\left(1 - 2\lambda + 2\lambda^2 - 2\lambda + 1\right) + \left(-2\lambda + 2\lambda^2 + 4\lambda^2 - 2\lambda - 4\lambda + 2\right)\sqrt{1+t} \\
& + \frac{t}{\sqrt{1+t}}\left(-\lambda + \lambda^2 + 2\lambda^2 - 2\lambda - \lambda + 1\right) \\
& + \frac{1}{\sqrt{1+t}}\left(\lambda^2 + 8\lambda^2 - 2\lambda - 4\lambda + 1\right) \\
& + \left(-\lambda + \lambda^2 + 2\lambda^2 + 4\lambda^2 - \lambda - 4\lambda + 1 + 2 + 8\lambda^2 - 8\lambda\right) \\
= {} & 2t(\lambda - 1)^2 + 6\sqrt{1+t}(\lambda - 1)\left(\lambda - \frac{1}{3}\right) + 3\frac{t}{\sqrt{1+t}}(\lambda - 1)\left(\lambda - \frac{1}{3}\right) \\
& + \frac{1}{\sqrt{1+t}}(3\lambda - 1)^2 + 15\left(\lambda - \frac{1}{3}\right)\left(\lambda - \frac{3}{5}\right).
\end{aligned}$$

Working with this last expression, we get

$$\begin{aligned}
0 = {} & 2t\sqrt{1+t}(\lambda - 1)^2 + 6(1+t)(\lambda - 1)\left(\lambda - \frac{1}{3}\right) + 3t(\lambda - 1)\left(\lambda - \frac{1}{3}\right) \\
& + (3\lambda - 1)^2 + 15\sqrt{1+t}\left(\lambda - \frac{1}{3}\right)\left(\lambda - \frac{3}{5}\right)
\end{aligned}$$

and hence, rearranging terms,

$$\begin{aligned} &\sqrt{1+t}\left[15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right)+2t(\lambda-1)^2\right] \\ =& -9t(\lambda-1)\left(\lambda-\frac{1}{3}\right)-15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right). \end{aligned} \tag{6}$$

If $\lambda = 1$, we obtain

$$\sqrt{1+t}+1=0$$

and so, in particular, we have $\lambda \neq 1$. Equation (6) has two solutions,

$$t_1(\lambda)=\frac{-1+2\lambda+3\lambda^2}{(\lambda-1)^2} \quad \text{and} \quad t_2(\lambda)=\frac{5\lambda^2+2\lambda-3}{4(\lambda-1)^2}.$$

Using equation (3), we may see

$$\tan\theta=\frac{\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda-1}{\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda-\left(1+\frac{1}{\sqrt{1+t}}\right)}.$$

In particular, evaluating in $t_1(\lambda)$ we obtain

$$\tan\theta_1=\frac{\left(1+\frac{1-\lambda}{2\lambda}\right)\lambda-1}{\left(1+\frac{1-\lambda}{\lambda}\right)\lambda-\left(1+\frac{1-\lambda}{2\lambda}\right)}=\lambda,$$

in which case we have

$$D_{1,1}(\lambda):=|f_\lambda(t_1,\theta_1)|=\left|-\sqrt{1+\lambda^2}\right|=\sqrt{1+\lambda^2}.$$

Regarding $t_2\left(\lambda\right)$, we obtain

$$\tan\theta_2=\frac{\left(1+\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)\lambda-1}{\left(1+2\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)\lambda-\left(1+\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)}.$$

Since $\theta_2 \in \left(0,\frac{\pi}{4}\right)$, we need to guarantee $0<\tan\theta_2<1$, and for this we need $0<\lambda<\frac{1}{5}$. Therefore

$$\tan\theta_2=\frac{5\lambda-1}{7\lambda-3}$$

and in this case,

$$\begin{aligned}
&D_{1,2}(\lambda) := |f_\lambda(t_2,\theta_2)| \\
&= \left| \left[\frac{5\lambda^2+2\lambda-3}{4(\lambda-1)^2} - \left(\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2} + \frac{3\lambda-1}{\lambda-1} \right) \lambda \right] \frac{3-7\lambda}{\sqrt{74\lambda^2-52\lambda+10}} \right. \\
&\qquad + \left[\left(3 + \frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2} + \frac{6\lambda-2}{\lambda-1} \right) \lambda \right. \\
&\qquad \left. \left. - \left(\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2} + \frac{3\lambda-1}{\lambda-1} \right) \right] \frac{1-5\lambda}{\sqrt{74\lambda^2-52\lambda+10}} \right| \\
&= \left| -\frac{78\lambda^4-208\lambda^3+196\lambda^2-80\lambda+14}{4(\lambda-1)^2\sqrt{74\lambda^2-52\lambda+10}} \right| \\
&= \left| -\frac{39\lambda^2-26\lambda+7}{2\sqrt{74\lambda^2-52\lambda+10}} \right| \\
&= \frac{39\lambda^2-26\lambda+7}{2\sqrt{74\lambda^2-52\lambda+10}}.
\end{aligned}$$

Case 2: $\theta = 0, -1 \le t \le 1$. We have

$$f_\lambda(t,0) = t - \big(1 + t + 2\sqrt{1+t}\big)\lambda.$$

Then,

$$\begin{aligned}
f_\lambda(-1,0) &= -1, \\
f_\lambda(1,0) &= 1 - 2\big(1+\sqrt{2}\big)\lambda,
\end{aligned}$$

and hence

$$|f_\lambda(1,0)| = \begin{cases} 1 - 2(1+\sqrt{2})\lambda & \text{if } 0 \le \lambda < \frac{\sqrt{2}-1}{2}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{\sqrt{2}-1}{2} \le \lambda \le 1. \end{cases}$$

Working now on $(-1,1)$, since

$$f'_\lambda(t,0) = 1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda,$$

the critical point of $f_\lambda(t,0)$ is

$$t = \frac{\lambda^2}{(1-\lambda)^2} - 1.$$

Recall that we need to make sure that $-1 < t < 1$. Therefore, in this case we also need to ask

$$\lambda < \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2}.$$

Plugging the critical point of $f_\lambda(t,0)$ into $f_\lambda(t,0)$, we obtain

$$f_\lambda\left(\frac{\lambda^2}{(\lambda-1)^2}-1,0\right)=\frac{\lambda^2}{(\lambda-1)^2}-1-\left[\frac{\lambda^2}{(\lambda-1)^2}+\frac{2\lambda}{1-\lambda}\right]\lambda=\frac{\lambda^2}{\lambda-1}-1,$$

and hence

$$\left|f_\lambda\left(\frac{\lambda^2}{(\lambda-1)^2}-1,0\right)\right|=1+\frac{\lambda^2}{1-\lambda}.$$

- Assume first $0\le\lambda<\frac{\sqrt{2}-1}{2}$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)|=\max\left\{1,\,1-2\left(1+\sqrt{2}\right)\lambda,\,1+\frac{\lambda^2}{1-\lambda}\right\}=1+\frac{\lambda^2}{1-\lambda}.$$

- Assume now $\frac{\sqrt{2}-1}{2}\le\lambda<2-\sqrt{2}$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)|=\max\left\{1,\,2\left(1+\sqrt{2}\right)\lambda-1,\,1+\frac{\lambda^2}{1-\lambda}\right\}=1+\frac{\lambda^2}{1-\lambda}.$$

- Assume finally $2-\sqrt{2}\le\lambda\le 1$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)|=\max\left\{1,\,2\big(1+\sqrt{2}\big)\lambda-1\right\}=2\big(1+\sqrt{2}\big)\lambda-1.$$

So, in conclusion,

$$\begin{aligned}\sup_{-1\le t\le 1}|f_\lambda(t,0)|&=\begin{cases}1+\frac{\lambda^2}{1-\lambda} & \text{if } 0\le\lambda<2-\sqrt{2},\\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } 2-\sqrt{2}\le\lambda\le 1,\end{cases}\\ &=:\begin{cases}D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2},\\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1.\end{cases}\end{aligned}$$

(3) $\theta=\frac{\pi}{4}$ and $-1\le t\le 1$.

We have

$$\begin{aligned}&f_\lambda\left(t,\frac{\pi}{4}\right)\\ &=\frac{\sqrt{2}}{2}\left[t-\big(1+t+2\sqrt{1+t}\big)\lambda+\big(4+t+4\sqrt{1+t}\big)\lambda-\big(1+t+2\sqrt{1+t}\big)\right]\\ &=\frac{\sqrt{2}}{2}\left[\big(3+2\sqrt{1+t}\big)\lambda-\big(1+2\sqrt{1+t}\big)\right].\end{aligned}$$

Again, we have

$$\begin{aligned} f_\lambda\left(-1,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left(3\lambda-1\right), \\ f_\lambda\left(1,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[(3+2\sqrt{2})\lambda-\left(1+2\sqrt{2}\right)\right], \\ f'_\lambda\left(t,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\frac{\lambda}{\sqrt{1+t}}-\frac{1}{\sqrt{1+t}}\right], \end{aligned}$$

and $f'_\lambda(t,\frac{\pi}{4})=0$ implies $\lambda=1$ (in which case $f_\lambda(t,\frac{\pi}{4})=\sqrt{2}$ for every t).

- Assume first $0\le\lambda<\frac{1}{3}$. Then,

$$\begin{aligned} \sup_{-1\le t\le 1}\left|f_\lambda\left(t,\frac{\pi}{4}\right)\right| &= \frac{\sqrt{2}}{2}\max\left\{\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda,\, 1-3\lambda\right\} \\ &= \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right]. \end{aligned}$$

- Assume now $\frac{1}{3}\le\lambda<4\sqrt{2}-5$. Then,

$$\begin{aligned} &\sup_{-1\le t\le 1}\left|f_\lambda\left(t,\frac{\pi}{4}\right)\right| \\ &= \frac{\sqrt{2}}{2}\max\left\{\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda,\, 3\lambda-1\right\} \\ &= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } \frac{1}{3}\le\lambda<\frac{2\sqrt{2}+1}{7}, \\ \frac{\sqrt{2}}{2}(3\lambda-1) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda<4\sqrt{2}-5. \end{cases} \end{aligned}$$

- Assume finally $4\sqrt{2}-5\le\lambda\le 1$. Then,

$$\begin{aligned} &\sup_{-1\le t\le 1}\left|f_\lambda\left(t,\frac{\pi}{4}\right)\right| \\ &= \frac{\sqrt{2}}{2}\max\left\{3\lambda-1,\, \left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right\} = \frac{\sqrt{2}}{2}(3\lambda-1). \end{aligned}$$

Hence, we can say that

$$\begin{aligned} \sup_{-1\le t\le 1}\left|f_\lambda\left(t,\frac{\pi}{4}\right)\right| &= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{2\sqrt{2}+1}{7}, \\ \frac{\sqrt{2}}{2}\left(3\lambda-1\right) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda\le 1. \end{cases} \\ &=: \begin{cases} D_{3,1}(\lambda) & \text{if } 0\le\lambda<\frac{2\sqrt{2}+1}{7}, \\ D_{3,2}(\lambda) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda\le 1. \end{cases} \end{aligned}$$

Case 4: $t = -1$, $0 \le \theta \le \frac{\pi}{4}$. Applying Lemma 3.1, we obtain

$$\sup_{0\le\theta\le\frac{\pi}{4}} f_\lambda(-1,\theta) = \begin{cases} 1 & \text{if } 0 \le \lambda < \frac{1+\sqrt{2}}{3}, \\ \frac{\sqrt{2}}{2}(3\lambda - 1) & \text{if } \frac{1+\sqrt{2}}{3} \le \lambda \le 1. \end{cases}$$
$$=: \begin{cases} D_{4,1}(\lambda) & \text{if } 0 \le \lambda < \frac{1+\sqrt{2}}{3}, \\ D_{4,2}(\lambda) & \text{if } \frac{1+\sqrt{2}}{3} \le \lambda \le 1. \end{cases}$$

Case 5: $t = 1$, $0 \le \theta \le \frac{\pi}{4}$. We use again Lemma 3.1, with $a = 1 - \left(2 + 2\sqrt{2}\right)\lambda$ and $b = \left(5 + 4\sqrt{2}\right)\lambda - \left(2 + 2\sqrt{2}\right)$. Through standard calculations, we see that $\frac{b}{a} < 0$ if and only if $\lambda \in \left[0, \frac{\sqrt{2}-1}{2}\right) \cup \left(\frac{6-2\sqrt{2}}{7}, 1\right]$ and $\frac{b}{a} > 1$ if and only if $\frac{\sqrt{2}-1}{2} < \lambda < \frac{3+4\sqrt{2}}{23}$. Therefore,

$$\sup_{0\le\theta\le\frac{\pi}{4}} |f_\lambda(1,\theta)|$$
$$= \begin{cases} \max\left\{\left|1 - \left(2+2\sqrt{2}\right)\lambda\right|, \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda - \left(1+2\sqrt{2}\right)\right|\right\} \\ \qquad \text{if } 0 \le \lambda < \frac{3+4\sqrt{2}}{23}, \\ \sqrt{\left(1 - \left(2+2\sqrt{2}\right)\lambda\right)^2 + \left(\left(5+4\sqrt{2}\right)\lambda - \left(2+2\sqrt{2}\right)\right)^2} \\ \qquad \text{if } \frac{3+4\sqrt{2}}{23} \le \lambda < \frac{6-2\sqrt{2}}{7}, \\ \max\left\{\left|1 - \left(2+2\sqrt{2}\right)\lambda\right|, \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda - \left(1+2\sqrt{2}\right)\right|\right\} \\ \qquad \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda \le 1. \end{cases}$$

Since $0 \le \lambda < \sqrt{2}-1$ implies $\left|1 - \left(2+2\sqrt{2}\right)\lambda\right| < \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda - \left(1+2\sqrt{2}\right)\right|$, it follows that

$$\sup_{0\le\theta\le\frac{\pi}{4}} |f_\lambda(1,\theta)|$$
$$= \begin{cases} \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda - \left(1+2\sqrt{2}\right)\right| & \text{if } 0 \le \lambda < \frac{3+4\sqrt{2}}{23}, \\ \sqrt{48\sqrt{2}\lambda^2 - 56\lambda + 69\lambda^2 - 40\sqrt{2}\lambda + 8\sqrt{2} + 13} & \text{if } \frac{3+4\sqrt{2}}{23} \le \lambda < \frac{6-2\sqrt{2}}{7}, \\ \left|1 - \left(2+2\sqrt{2}\right)\lambda\right| & \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda \le 1 \end{cases}$$
$$= \begin{cases} \frac{\sqrt{2}}{2}\left[1 + 2\sqrt{2} - \left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \le \lambda < \frac{3+4\sqrt{2}}{23}, \\ \sqrt{48\sqrt{2}\lambda^2 - 56\lambda + 69\lambda^2 - 40\sqrt{2}\lambda + 8\sqrt{2} + 13} & \text{if } \frac{3+4\sqrt{2}}{23} \le \lambda < \frac{6-2\sqrt{2}}{7}, \\ \left(2+2\sqrt{2}\right)\lambda - 1 & \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda \le 1. \end{cases}$$
$$=: \begin{cases} D_{5,1}(\lambda) & \text{if } 0 \le \lambda < \frac{3+4\sqrt{2}}{23}, \\ D_{5,2}(\lambda) & \text{if } \frac{3+4\sqrt{2}}{23} \le \lambda < \frac{6-2\sqrt{2}}{7}, \\ D_{5,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda \le 1. \end{cases}$$

Since (see Figures 3.1 and 3.2)

$$D_{1,1}(\lambda) \leq \begin{cases} D_{2,1}(\lambda) & \text{if } 0 \leq \lambda < 2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2} \leq \lambda \leq 1, \end{cases}$$

$$D_{1,2}(\lambda) \leq D_{3,1}(\lambda) \ \text{ for } 0 < \lambda < \tfrac{1}{5},$$

we can rule out Case 1. Since

$$D_{3,1}(\lambda) = D_{5,1}(\lambda) \ \text{ for } 0 \leq \lambda \leq \tfrac{3+4\sqrt{2}}{23},$$

$$D_{3,2}(\lambda) = D_{4,2}(\lambda) \ \text{ for } \tfrac{1+\sqrt{2}}{3} \leq \lambda \leq 1,$$

we can directly rule out Case 3. Since (see Figures 3.1 and 3.3)

$$D_{4,1}(\lambda) = 1 \leq \begin{cases} D_{2,1}(\lambda) & \text{if } 0 \leq \lambda < 2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2} \leq \lambda < \tfrac{1+\sqrt{2}}{3}, \end{cases}$$

$$D_{4,2}(\lambda) \leq D_{2,2} \ \text{ for } \tfrac{1+\sqrt{2}}{3} \leq \lambda \leq 1,$$

we can rule out Case 4. Finally, since (see Figure 3.4)

$$D_{5,2}(\lambda) \leq D_{2,1}(\lambda) \ \text{ for } \tfrac{3+4\sqrt{2}}{23} \leq \lambda < \tfrac{6-2\sqrt{2}}{7},$$

$$D_{5,3}(\lambda) = D_{2,2}(\lambda) \ \text{ for } 2-\sqrt{2} \leq \lambda \leq 1,$$

we can rule out the expressions $D_{5,2}(\lambda)$ and $D_{5,3}(\lambda)$ of Case 5.

Thus, putting all the above cases together, we may reach the conclusion

$$\sup_{(t,\theta)\in C_1} |f_\lambda(t,\theta)|$$

$$= \begin{cases} D_{5,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}, \\ D_{2,1}(\lambda) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14} \leq \lambda < 2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2} \leq \lambda \leq 1, \end{cases}$$

$$= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \leq \lambda < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}, \\ 1 + \frac{\lambda^2}{1-\lambda} & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14} \leq \lambda < 2-\sqrt{2}, \\ \left(2+2\sqrt{2}\right)\lambda - 1 & \text{if } 2-\sqrt{2} \leq \lambda \leq 1, \end{cases}$$

and hence

$$\sup_{-1\leq t\leq 1} \|DP_t(x,y)\|_{D(\frac{\pi}{4})} = 2x \sup_{(t,\theta)\in C_1} |f_\lambda(t,\theta)|$$

$$= \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x - \left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}x, \\ 2\left(x + \frac{y^2}{x-y}\right) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x, \end{cases}$$

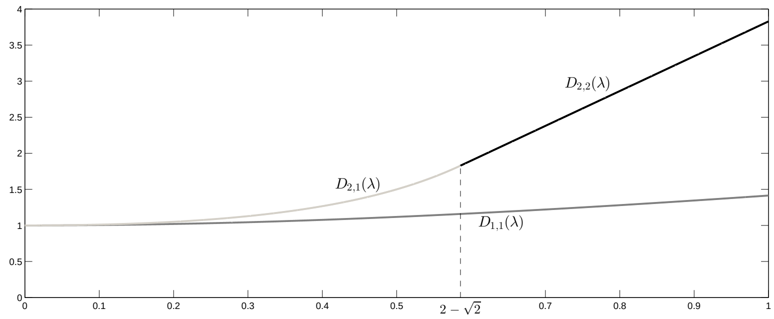

Figure 3.1: Graphs of the mappings $D_{1,1}(\lambda)$, $D_{2,1}(\lambda)$ and $D_{2,2}(\lambda)$.

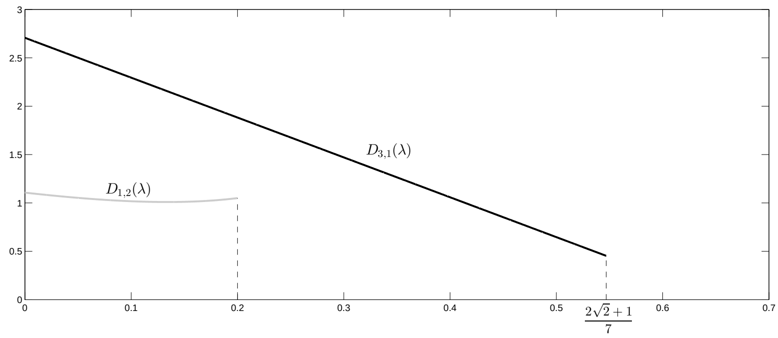

Figure 3.2: Graphs of the mappings $D_{1,2}(\lambda)$ and $D_{3,1}(\lambda)$.

assuming in every moment $x \neq 0$ (in order to illustrate the previous step, the reader can take a look at Figure 3.5).

Let us deal now with the polynomials

$$Q_s(x, y) = x^2 + sy^2 - 2\sqrt{2(1+s)}xy, \quad 1 \leq s \leq 5 + 4\sqrt{2}.$$

Then,

$$\nabla Q_s(x, y) = \big(2x - 2\sqrt{2(1+s)}y,\, 2sy - 2\sqrt{2(1+s)}x\big),$$

$$\begin{aligned} &\|DQ_s(x, y)\|_{D(\frac{\pi}{4})} \\ = &\sup_{0\leq\theta\leq\frac{\pi}{4}} \big|2x\big[\big(1 - \sqrt{2(1+s)}\lambda\big)\cos\theta + \big(s\lambda - \sqrt{2(1+s)}\big)\sin\theta\big]\big|, \end{aligned}$$

and thus

$$\sup_{1\leq s\leq 5+4\sqrt{2}} \|DQ_s(x, y)\|_{D(\frac{\pi}{4})} = 2x \sup_{(s,\theta)\in C_2} |g_\lambda(s, \theta)|,$$

with

$$g_\lambda(s, \theta) = \big(1 - \sqrt{2(1+s)}\lambda\big)\cos\theta + \big(s\lambda - \sqrt{2(1+s)}\big)\sin\theta$$

and $C_2 = [1, 5 + 4\sqrt{2}] \times [0, \frac{\pi}{4}]$. Again, we have several cases:

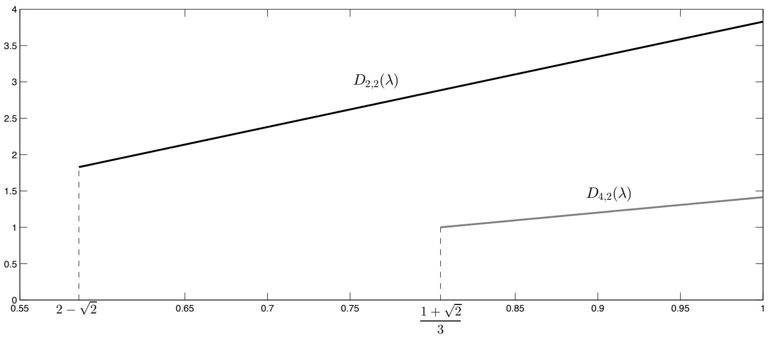

Figure 3.3: Graphs of the mappings $D_{2,2}(\lambda)$ and $D_{4,2}(\lambda)$.

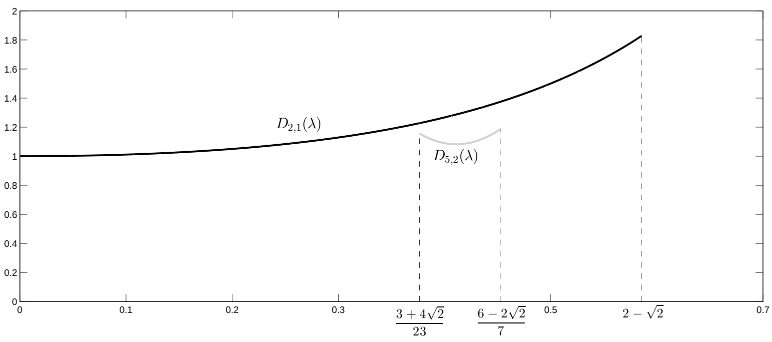

Figure 3.4: Graphs of the mappings $D_{2,1}(\lambda)$ and $D_{5,2}(\lambda)$.

Case 6: $(s,\theta) \in (1, 5+4\sqrt{2}) \times (0, \frac{\pi}{4})$. Let us first calculate the critical points of g_λ over C_2.

$$\frac{\partial g_\lambda}{\partial s}(s_0,\theta_0) = \frac{-\lambda}{\sqrt{2(1+s_0)}}\cos\theta_0 + \left(\lambda - \frac{1}{\sqrt{2(1+s_0)}}\right)\sin\theta_0,$$

$$\frac{\partial g_\lambda}{\partial \theta}(s_0,\theta_0) = \left(s_0\lambda - \sqrt{2(1+s_0)}\right)\cos\theta_0 - \left(1-\sqrt{2(1+s_0)}\lambda\right)\sin\theta_0,$$

so, if $Dg_\lambda(s_0,\theta_0) = 0$, using the first expression, we obtain $\tan\theta_0 = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda - 1}$, and, using the second one, we obtain $\tan\theta_0 = \frac{s_0\lambda - \sqrt{2(1+s_0)}}{1-\sqrt{2(1+s_0)}\lambda}$. Hence, we may say

$$\frac{s_0\lambda - \sqrt{2(1+s_0)}}{1-\sqrt{2(1+s_0)}\lambda} = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda - 1}$$

and thus

$$s_0 = \frac{2-\lambda^2}{\lambda^2}.$$

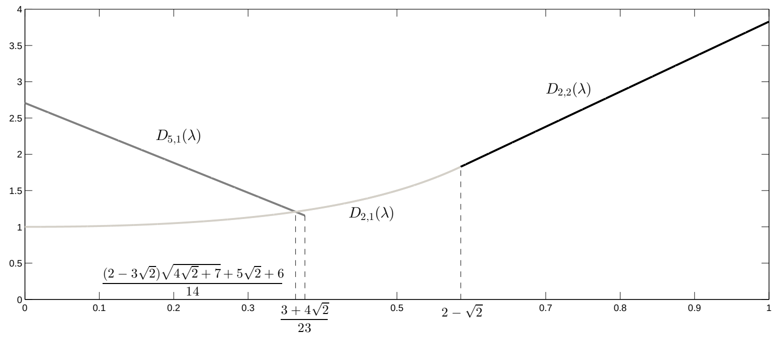

Figure 3.5: Graphs of the mappings $D_{2,1}(\lambda)$, $D_{2,2}(\lambda)$ and $D_{5,1}(\lambda)$.

Then, $\tan\theta_0 = \lambda$ and also, if we want to guarantee that $1 < s_0 < 5 + 4\sqrt{2}$, we need $\sqrt{2} - 1 < \lambda < 1$.
In that case, $\sin\theta_0 = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $\cos\theta_0 = \frac{1}{\sqrt{1+\lambda^2}}$, and then

$$g_\lambda(s_0, \theta_0) = \frac{-1}{\sqrt{1+\lambda^2}} + \frac{-\lambda^2}{\sqrt{1+\lambda^2}} = -\sqrt{1+\lambda^2},$$

so

$$D_6(\lambda) := |g_\lambda(s_0, \theta_0)| = \sqrt{1+\lambda^2}.$$

Case 7: $s = 1$, $0 \le \theta \le \frac{\pi}{4}$. Apply Lemma 3.1 with $a = 1 - 2\lambda$ and $b = \lambda - 2$. Using $0 \le \lambda \le 1$, observe that we always have $b < 0$ and $b \le a$. Also, $a < \left(1 - \sqrt{2}\right) b$ if and only if $\lambda > \frac{5-3\sqrt{2}}{7}$.
Putting everything together, we can say that

$$\begin{aligned}\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(1,\theta)| &= \begin{cases} 1 - 2\lambda & \text{if } 0 \le \lambda < \frac{5-3\sqrt{2}}{7}, \\ \frac{\sqrt{2}}{2}(1+\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \le \lambda \le 1, \end{cases} \\ &=: \begin{cases} D_{7,1}(\lambda) & \text{if } 0 \le \lambda < \frac{5-3\sqrt{2}}{7}, \\ D_{7,2}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \le \lambda \le 1. \end{cases}\end{aligned}$$

Case 8: $s = 5 + 4\sqrt{2}$, $0 \le \theta \le \frac{\pi}{4}$. Apply again Lemma 3.1, this time to $a = 1 - 2\left(1+\sqrt{2}\right)\lambda$ and $b = \left(5+4\sqrt{2}\right)\lambda - 2\left(1+\sqrt{2}\right)$. As usual, we notice that $a < 0$ if and only if $\lambda > \frac{\sqrt{2}-1}{2}$, $b < 0$ if and only if $\lambda < \frac{6-2\sqrt{2}}{7}$ and $a < b$ if and only if $\lambda > \frac{3+4\sqrt{2}}{23}$. All together, we can say that, for $\frac{3+4\sqrt{2}}{23} < \lambda < \frac{6-2\sqrt{2}}{7}$, we have

$$\begin{aligned}\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(5+4\sqrt{2},\theta)| &= \sqrt{a^2+b^2} \\ &= \sqrt{13 + 8\sqrt{2} - \left(56+40\sqrt{2}\right)\lambda + \left(69+48\sqrt{2}\right)\lambda^2}.\end{aligned}$$

Also, notice that, for any $\lambda \in [0,1]$, we are going to have $b < -\left(1+\sqrt{2}\right) a$ and $a < \left(1-\sqrt{2}\right) b$. Hence,

$$\begin{aligned}
&\sup_{0\leq\theta\leq\frac{\pi}{4}} |g_\lambda(5+4\sqrt{2},\theta)| \\
&= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \leq \lambda < \frac{3+4\sqrt{2}}{23}, \\ \sqrt{13+8\sqrt{2}-\left(56+40\sqrt{2}\right)\lambda+\left(69+48\sqrt{2}\right)\lambda^2} & \text{if } \frac{3+4\sqrt{2}}{23} \leq \lambda < \frac{6-2\sqrt{2}}{7}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{6-2\sqrt{2}}{7} \leq \lambda \leq 1, \end{cases} \\
&=: \begin{cases} D_{8,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{3+4\sqrt{2}}{23}, \\ D_{8,2}(\lambda) & \text{if } \frac{3+4\sqrt{2}}{23} \leq \lambda < \frac{6-2\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7} \leq \lambda \leq 1. \end{cases}
\end{aligned}$$

Case 9: $\theta = 0$, $1 \leq s \leq 5+4\sqrt{2}$. We have

$$\begin{gathered}
g_\lambda(s,0) = 1-\sqrt{2(1+s)}\lambda, \\
g_\lambda(1,0) = 1-2\lambda, \\
g_\lambda(5+4\sqrt{2},0) = 1-2\left(1+\sqrt{2}\right)\lambda, \\
g'_\lambda(s,0) = -\frac{\lambda}{\sqrt{2(1+s)}} \neq 0 \quad \text{for } \lambda \neq 0.
\end{gathered}$$

Then,

$$\begin{aligned}
\sup_{1\leq s\leq 5+4\sqrt{2}} |g_\lambda(s,0)| &= \max\left\{|1-2\lambda|,\, |1-2(1+\sqrt{2})\lambda|\right\} \\
&= \begin{cases} 1-2\lambda & \text{if } 0 \leq \lambda < \frac{2-\sqrt{2}}{2}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{2-\sqrt{2}}{2} \leq \lambda \leq 1, \end{cases} \\
&=: \begin{cases} D_{9,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2-\sqrt{2}}{2}, \\ D_{9,2}(\lambda) & \text{if } \frac{2-\sqrt{2}}{2} \leq \lambda \leq 1. \end{cases}
\end{aligned}$$

Case 10: $\theta = \frac{\pi}{4}$, $1 \leq s \leq 5+4\sqrt{2}$. We have

$$g_\lambda\left(s,\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left[1+s\lambda-\sqrt{2(1+s)}(1+\lambda)\right].$$

Then

$$\begin{gathered}
g_\lambda\left(1,\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}(1+\lambda), \\
g_\lambda\left(5+4\sqrt{2},\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left[(3+2\sqrt{2})\lambda-\left(1+2\sqrt{2}\right)\right], \\
g'_\lambda\left(s_0,\frac{\pi}{4}\right) = 0 \ \text{ if and only if } s_0 = \frac{(1+\lambda)^2}{2\lambda^2} - 1
\end{gathered}$$

and since we need to ensure that $1 < s_0 < 5 + 4\sqrt{2}$, we need $\frac{2\sqrt{2}-1}{7} < \lambda < 1$. In that case,

$$g_\lambda\left(s_0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda}.$$

Hence,

$$\begin{aligned}\sup_{1\le s\le 5+4\sqrt{2}} \left|g_\lambda\left(s.\frac{\pi}{4}\right)\right| &= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \le \lambda < \frac{2\sqrt{2}-1}{7}, \\ \frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} & \text{if } \frac{2\sqrt{2}-1}{7} \le \lambda \le 1, \end{cases} \\ &=: \begin{cases} D_{10,1}(\lambda) & \text{if } 0 \le \lambda < \frac{2\sqrt{2}-1}{7}, \\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \le \lambda \le 1. \end{cases}\end{aligned}$$

Since (the reader can take a look at Figure 3.6)

$$D_6(\lambda) \le \begin{cases} D_{8,2}(\lambda) & \text{if } \sqrt{2}-1 < \lambda < \frac{6-2\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda < 1, \end{cases}$$

we can rule out Case 6. Since (see Figures 3.7 and 3.8)

$$D_{7,1}(\lambda) \le D_{10,1}(\lambda) \text{ for } 0 \le \lambda < \tfrac{5-3\sqrt{2}}{7},$$

$$D_{7,2}(\lambda) \le \begin{cases} D_{10,1}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \le \lambda < \frac{2\sqrt{2}-1}{7}, \\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \le \lambda \le 1, \end{cases}$$

we can rule out Case 7. Since

$$D_{8,1}(\lambda) = D_{10,1}(\lambda) \text{ for } 0 \le \lambda < \frac{2\sqrt{2}-1}{7},$$

we can rule out the expression $D_{8,1}(\lambda)$ of Case 8. Since

$$\begin{aligned} D_{9,1}(\lambda) &= D_{7,1}(\lambda) \text{ for } 0 \le \lambda < \tfrac{5-3\sqrt{2}}{7}, \\ D_{9,2}(\lambda) &= D_{8,3}(\lambda) \text{ for } \tfrac{6-2\sqrt{2}}{7} \le \lambda \le 1, \end{aligned}$$

we can directly rule out Case 9. Furthermore, since (see Figure 3.9)

$$D_{8,2}(\lambda) \le D_{10,2}(\lambda) \text{ for } \tfrac{3+4\sqrt{2}}{23} \le \lambda < \tfrac{6-2\sqrt{2}}{7},$$

$$D_{8,3}(\lambda) \le D_{10,2}(\lambda) \text{ for } \tfrac{6-2\sqrt{2}}{7} \le \lambda \le \tfrac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7},$$

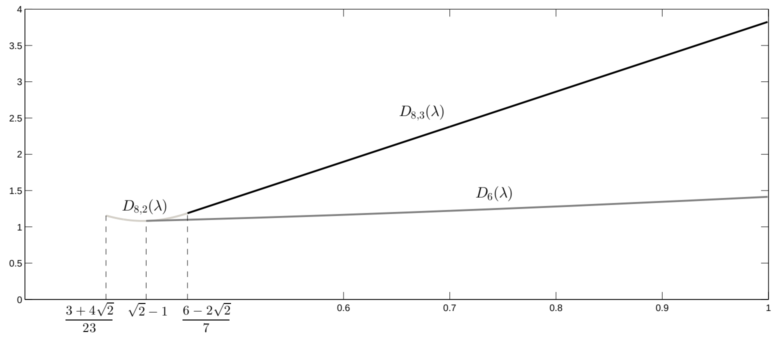

Figure 3.6: Graphs of the mappings $D_6(\lambda)$, $D_{8,2}(\lambda)$ and $D_{8,3}(\lambda)$.

we can conclude that

$$\begin{aligned}
&\sup_{(s,\theta)\in C_2} |g_\lambda(s,\theta)| \\
&= \begin{cases} D_{10,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} \leq \lambda \leq 1. \end{cases} \\
&= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\ \frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} \leq \lambda \leq 1, \end{cases}
\end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{1\leq s\leq 5+4\sqrt{2}} \|DQ_s(x,y)\|_{D(\frac{\pi}{4})} \\
&= \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}x \leq y \leq x. \end{cases}
\end{aligned}$$

Finally, if we compare the results obtained with P_t and Q_s, since $\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} \geq 1+\frac{\lambda^2}{1-\lambda}$ whenever $\lambda \leq \sqrt{2}-1$, we obtain

$$\begin{aligned}
&\Phi(x,y) \\
&= \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < \left(\sqrt{2}-1\right)x, \\ 2\left(x+\frac{y^2}{x-y}\right) & \text{if } \left(\sqrt{2}-1\right)x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y-2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x. \end{cases} \qquad \square
\end{aligned}$$

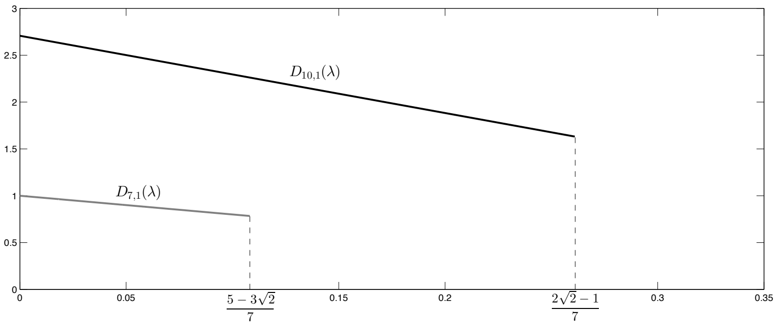

Figure 3.7: Graphs of the mappings $D_{7,1}(\lambda)$ and $D_{10,1}(\lambda)$.

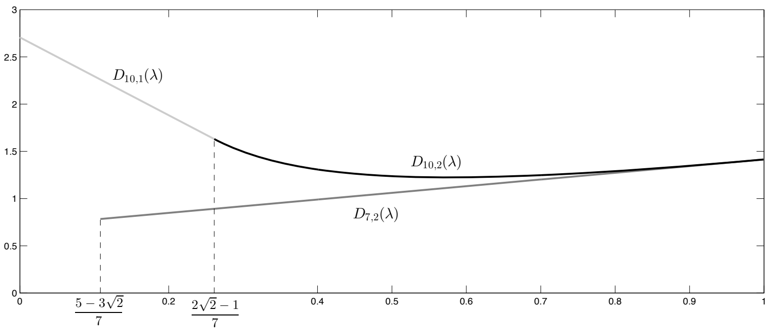

Figure 3.8: Graphs of the mappings $D_{7,2}(\lambda)$, $D_{10,1}(\lambda)$ and $D_{10,2}(\lambda)$.

We can see that $\Phi(x,y) \leq 4+\sqrt{2}$, for all $(x,y) \in D\left(\frac{\pi}{4}\right)$. Furthermore, the maximum is attained by the polynomials

$$P_1(x,y) = x^2 + \big(5+4\sqrt{2}\big)y^2 - \big(4+4\sqrt{2}\big)xy = Q_{5+4\sqrt{2}}(x,y).$$

Corollary 3.3. *Let $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and assume $L \in \mathcal{L}^s\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is the polar of P. Then*

$$\|L\|_{D\left(\frac{\pi}{4}\right)} \leq \left(2+\frac{\sqrt{2}}{2}\right)\|P\|_{D\left(\frac{\pi}{4}\right)}.$$

Moreover, equality is achieved for $P_1(x,y) = Q_{5+4\sqrt{2}}(x,y) = x^2+\left(5+4\sqrt{2}\right)y^2-\left(4+4\sqrt{2}\right)xy$. Hence, the polarization constant of the polynomial space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is $2+\frac{\sqrt{2}}{2}$.

4. Unconditional constants for polynomials on sectors

Here, we obtain a sharp estimate on the norm of the modulus of a polynomial in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ in terms of it norm. That sharp estimate turns out to be the unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

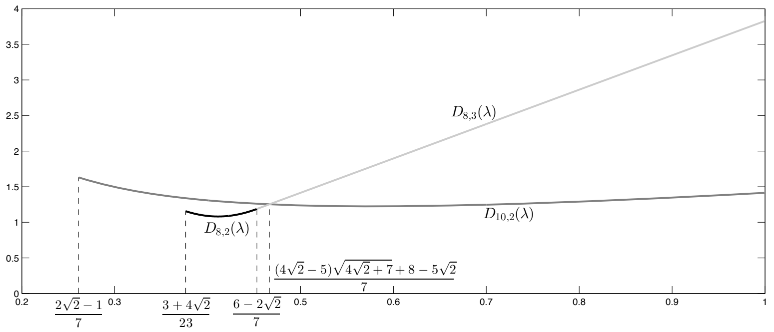

Figure 3.9: Graphs of the mappings $D_{8,2}(\lambda)$, $D_{8,3}(\lambda)$ and $D_{10,2}(\lambda)$.

Theorem 4.1. *The unconditional constant of the canonical basis of* $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ *is* $5+4\sqrt{2}$. *In other words, the inequality*

$$|||P|||_{D\left(\frac{\pi}{4}\right)} \leq (5+4\sqrt{2})\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. *Furthermore, the previous inequality is sharp and equality is attained for the polynomials* $\pm P_1(x,y) = \pm Q_{5+4\sqrt{2}}(x,y) = \pm\left[x^2+(5+4\sqrt{2})y^2-(4+4\sqrt{2})xy\right]$.

Proof. We just need to calculate

$$\sup\left\{|||P|||_{D\left(\frac{\pi}{4}\right)} : \ P \in \operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)\right\}.$$

In order to calculate the above supremum we use the extreme polynomials described in Lemma 1.2. If we consider first the polynomials P_t, then $|P_t| = \left(|t|, 4+t+4\sqrt{1+t}, 2+2t+4\sqrt{1+t}\right)$. Now, using Lemma 1.1 we have

$$\begin{aligned}
&\sup_{-1\leq t\leq 1} |||P_t|||_{D\left(\frac{\pi}{4}\right)} \\
&= \sup_{-1\leq t\leq 1} \max\left\{|t|, \frac{1}{2}\left(|t|+4+t+4\sqrt{1+t}+2+2t+4\sqrt{1+t}\right)\right\} \\
&= \sup_{-1\leq t\leq 1} \frac{1}{2}\left(|t|+6+3t+8\sqrt{1+t}\right) = 5+4\sqrt{2}.
\end{aligned}$$

Notice that the above supremum is attained at $t=1$. On the other hand, if we consider the polynomials Q_s, we have $|Q_s| = \left(1, s, 2\sqrt{2(1+s)}\right)$. Now, using Lemma 1.1 we have

$$\begin{aligned}
&\sup_{1\leq s\leq 5+4\sqrt{2}} |||Q_s|||_{D\left(\frac{\pi}{4}\right)} \\
&= \sup_{1\leq s\leq 5+4\sqrt{2}} \max\left\{1, \frac{1}{2}\left(1+s+2\sqrt{2(1+s)}\right)\right\} \\
&= \sup_{1\leq s\leq 5+4\sqrt{2}} \frac{1}{2}\left(1+s+2\sqrt{2(1+s)}\right) = 5+4\sqrt{2}.
\end{aligned}$$

Observe that the last supremum is now attained at $s = 5 + 4\sqrt{2}$. $\square$

5. Conclusions

Comparing the results obtained in [11] and [25] for polynomials on the simplex Δ, in [12] for polynomials on the unit square $\square$, in [15] for polynomials on the sector $D\left(\frac{\pi}{2}\right)$ and the results obtained in the previous sections, we have the following:

	$\mathcal{P}(^2\Delta)$	$\mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$	$\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)$	$\mathcal{P}(^2\square)$
Markov constants	$2\sqrt{10}$	$2\sqrt{5}$	$4(13+8\sqrt{2})$	$\sqrt{13}$
Polarization constants	3	2	$2+\frac{\sqrt{2}}{2}$	$\frac{3}{2}$
Unconditional Constants	2	3	$5+4\sqrt{2}$	5

Furthermore, all the constants appearing in the previous table are sharp. Actually, the extreme polynomials where the constants are attained are the following:

(1) $\pm(x^2 + y^2 - 6xy)$ for the simplex.
(2) $\pm(x^2 + y^2 - 4xy)$ for the sector $D\left(\frac{\pi}{2}\right)$.
(3) $\pm\left(x^2 + (5+4\sqrt{2})y^2 - (4+4\sqrt{2})xy\right)$ for the sector $D\left(\frac{\pi}{4}\right)$.
(4) $\pm(x^2 + y^2 - 3xy)$ for the unit square.

Compare the previous table with similar results that hold for 2-homogeneous polynomials on the Banach spaces ℓ_1^2, ℓ_2^2 and ℓ_∞^2:

	$\mathcal{P}(^2\ell_1^2)$	$\mathcal{P}(^2\ell_2^2)$	$\mathcal{P}(^2\ell_\infty^2)$
Markov constants	4	2	$2\sqrt{2}$
Polarization constants	2	1	2
Unconditional Constants	$\frac{1+\sqrt{2}}{2}$	$\sqrt{2}$	$1+\sqrt{2}$

Observe that the Markov constants of the spaces $\mathcal{P}(^2\ell_1^2)$ and $\mathcal{P}(^2\ell_\infty^2)$ can be calculated taking into consideration the description of the geometry of those spaces given in [5]. Also, the Markov constant of $\mathcal{P}(^2\ell_2^2)$ is twice its polarization constant, or in other words, 2.

On the other hand, the constants appearing in the second line of the previous table are well-known results (see for instance [27]).

Finally, the unconditional constants corresponding to the third line of the previous table were calculated in Theorem 3.5, Theorem 3.19 and Theorem 3.6 of [11].

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