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# Coproximinality in Spaces of Bochner Integrable Functions

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*Dedicated to the memory of my friend and collaborator Sudipta Dutta.*

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In this short note we use a new way of applying von Neumann's selection theorem for obtaining best coapproximation in spaces of measurable functions. For a coproximinal closed subspace $Y$ of a Banach space $X$, we show that if $Y$ is constrained in a weakly compactly generated dual space, then the space $L^1(\mu, Y)$ of $Y$-valued Bochner integrable functions is coproximinal in $L^1(\mu, X)$. This extends a result of Haddadi et al., proved when $Y$ is reflexive.

*Keywords:* Coproximinality, constrained subspaces, weakly compactly generated spaces, spaces of Bochner integrable functions, von Neumann's selection theorem

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## 1. Introduction

Let $X$ be a Banach space. We recall that a closed subspace $Y \subset X$ is said to be coproximinal, if for any $x \in X$, there is a $y_0 \in Y$ such that for all $y \in Y$, $\|y - y_0\| \leq \|y - x\|$ for all $y \in Y$. Such an element $y_0$ is called a best coapproximation. It is easy to see that if $P : X \to Y$ is a linear, surjective projection of norm one, then $Y$ is coproximinal in $X$. Such a subspace is called a constrained subspace. Most often coproximinality assumption on $Y$ is sufficient to solve problems in optimization theory, relative to $Y$. See [6] and [7].

Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability space. Let $L^1(\mu, X)$, denote the space of $X$-valued Bochner integrable functions. It was proved in [3], using Dunford's theorem on relatively weakly compact sets (see [2]), that, $L^1(\mu, Y)$ is coproximinal in $L^1(\mu, X)$, if $Y$ is a reflexive coproximinal subspace of $X$. In this paper we obtain the same conclusion when $Y$ is the constrained subspace of a weakly compactly generated dual space. As reflexive spaces are weakly compactly gen-

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erated, this is an extension of the earlier result with a completely different argument. We use von Neumann's measurable selection theorem, ([9]) to achieve this. Using the conventional approach, we note that if $L^1(\mu, Y)$ is a separable space and $Y \subset X$ is coproximinal, then $L^1(\mu, Y)$ is coproximinal in $L^1(\mu, X)$.

## 2. Main Result

We first note that preserving coproximinality when the measure $\mu$ is purely atomic, is simple to handle and follows from the following more general lemma, which is easy to prove.

**Lemma 2.1.** *Let $\{Y_i\}_{i\in I}$ be a family of closed coproximinal subspaces of the family of Banach spaces, $\{X_i\}_{i\in I}$. Then the $\ell^1$-direct sum $\bigoplus_1\{Y_i : i \in I\}$ is a coproximinal subspace of $\bigoplus_1\{X_i : i \in I\}$.*

*Thus we assume from now on that the measure space is purely nonatomic.*

In the case of spaces of Bochner integrable functions, as obtaining a measurable best coapproximation involves solving uncountably many norm inequalities, in the proof of the following theorem we use properties of weakly compactly generated dual spaces and the rest of the hypothesis to reduce the problem to solving countably many inequalities and apply von Neumann's selection theorem to get the desired conclusion.

**Theorem 2.2.** *Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability space. Let $Y \subset X$ be a coproximinal subspace that is isometric to a constrained subspace of a weakly compactly generated dual space. Then $L^1(\mu, Y)$ is a coproximinal subspace of $L^1(\mu, X)$.*

**Proof.** Let $f \in L^1(\mu, X)$. We will first show that for any finite collection $g_1, ..., g_n \in L^1(\mu, Y)$

there exists a $g_0 \in L^1(\mu, Y)$ such that $\|g_i - g_0\| \le \|g_i - f\|$ for $1 \le i \le n$.

Since each $g_i$ is a.e. separably valued we may assume w.l.o.g. that there is a separable subspace $Y' \subset Y$ such that $g_i$'s are a.e. valued in $Y'$. Since $Y$ is weakly compactly generated by Lemma 4 from [1], there is a separable constrained subspace $Z$ of $Y$ with $Y' \subset Z \subset Y$. It is easy to see that $Z$ is a coproximinal subspace of $X$.

As the measure space is complete and since we are dealing with only finitely many functions, by removing a common null set if necessary we may and do assume that all the functions are everywhere valued. Also as these functions are separably valued, we may also assume that the $\sigma$-field is countably generated. Define $F : \Omega \to 2^Z$ by $F(t) = \{z \in Z : \|z - g_i(t)\| \le \|f(t) - g_i(t)\| \ for \ 1 \le i \le n\}$. Since $Z$ is a coproximinal subspace of $X$, these sets are non-empty. Now to apply Von Neumann's theorem (see [9] Corollary 5.5.8) to get a measurable selector for $F$, we show that the graph $\{(t, F(t)) : t \in \Omega\}$ is a measurable set. As the $g_i$'s and $f$ are measurable and the norm function is continuous, we get that the graph is measurable. Let $g_0 : \Omega \to Z$ be a measurable function such that $g_0(t) \in F(t)$ for all $t \in \Omega$.

Now as $Z$ separable $g_0$ is strongly measurable and by integrating the point-wise inequalities we have that $g_0 \in L^1(\mu, Y)$ and $\|g_0 - g_i\| \le \|f - g_i\|$ for $1 \le i \le n$.

We next claim that there exists a dual space $W^*$ containing $L^1(\mu, Y)$ and a linear surjective projection $P : W^* \to L^1(\mu, Y)$ of norm one. Assuming the claim, let $f \in L^1(\mu, X)$. Consider the closed balls $\{B(g, \|g-f\|) : g \in L^1(\mu, Y)\}$ in $W^*$. By what we have proved, any finitely many of them have non-empty intersection in $L^1(\mu, Y)$ and hence in $W^*$. By weak$^*$-compactness there exists a $w$ such that $\|w - g\| \le \|g - f\|$ for all $g \in L^1(\mu, Y)$. Now

$$\|P(w) - g\| = \|P(w - g)\| \le \|w - g\| \le \|g - f\|$$

for all $g \in L^1(\mu, Y)$. This completes the proof.

To see the claim, we first note that as the question under consideration is invariant under an onto isometry, we may work with another probability space $(\Omega', \mathcal{A}', \lambda)$, such that $L^1(\mu, X)$ is isometric to $L^1(\lambda, X)$ and $L^1(\mu, Y)$ gets mapped onto $L^1(\lambda, Y)$. By transferring the measure $\mu$ to a nonatomic Borel probability measure on the Stone space $K$ of $L^\infty(\mu)$, via the Gelfand map, and denoting it by $\lambda$, one has that $L^1(\mu)$ is isometric to $L^1(\lambda)$. See [8]. This isometry is easily seen to extend to an isometry of $L^1(\mu, X)$ onto $L^1(\lambda, X)$ and also maps $L^1(\mu, Y)$ onto $L^1(\lambda, Y)$. Let $S^*$ be a weakly compactly generated dual space and let $Q : S^* \to Y$ be a linear projection of norm one. Let $C(K, S)$ be the space of $S$-valued continuous functions equipped with the supremum norm. $C(K, S)^*$ can be identified with $M(K, S^*)$, the space of $S^*$-valued measures, equipped with the total variation norm. Since $S^*$ is weakly compactly generated, it has the Radon-Nikodým property (see [2]). For any $F \in M(K, S^*)$, let $F_a$ denote its absolutely continuous part with respect to $\lambda$. Define $R : C(K, S)^* \to L^1(\lambda, Y)$ by $R(F) = Q \circ \frac{dF_a}{d\lambda}$. It is fairly easy to check that this is a linear surjective projection of norm one. $\square$

In the next result we establish coproximinality, assuming $L^1(\mu, Y)$ is separable.

**Proposition 2.3.** *Let $Y \subset X$ be a separable coproximinal subspace. Let $(\Omega, \mathcal{A}, \mu)$ be a complete countably generated probability space. Then $L^1(\mu, Y)$ is coproximinal in $L^1(\mu, X)$.*

**Proof.** By our assumptions $L^1(\mu, Y)$ is a separable Banach space. Let $\{g_n\}_{n \ge 1}$ be a dense sequence in $L^1(\mu, Y)$. Let $f \in L^1(\mu, X)$. Since there are only countably many functions under consideration we assume as before that they are point wise defined. Let $F : \Omega \to 2^Y$ be defined by $F(t) = \{y \in Y : \|y - g_n(t)\| \le \|y - f(t)\| \ for\ all\ n\}$.

Since $Y$ is coproximinal in $X$, these sets are not empty. As there are only countably many functions involved as before we get that the graph of $F$ is measurable . Thus again by the Von Neumann's selection theorem and the separability of $Y$, we get a selector $g \in L^1(\mu, Y)$ for $F$. Again by integration, we get $\|g - g_n\| \leq \|g - f\|$ for all $n$. Hence by the denseness of the $g_n$'s we get that $g$ is a best coapproximation . $\square$

Fixing the domain set $\Omega$ and the measure $\mu$, let $\mathcal{B} \subset \mathcal{A}$ be a sub-$\sigma$-field. We denote by $L^1(\mathcal{B}, X)$, the space of Bochner integrable functions that are integrable w.r.t., the sub-$\sigma$-field $\mathcal{B}$. Using the well-known conditional expectation projection, $E : L^1(\mathcal{A}) \to L^1(\mathcal{B})$, in the scalar-valued case (via the Radon-Nikodým theorem), we recall from [2] that the vector-valued conditional expectation projection $E' : L^1(\mathcal{A}, X) \to L^1(\mathcal{B}, X)$ is defined by its values on a simple function $s = \sum_1^n x_i \chi_{A_i}$ by $E'(s) = \sum_1^n x_i E(\chi_{A_i})$, for $x_1, ..., x_n \in X$ and $A_1, ..., A_n \in \mathcal{A}$. This is also a surjective projection of norm one.

**Proposition 2.4.** *Let $Y \subset X$ be a closed subspace such that $L^1(\mathcal{A}, Y)$ is coproximinal in $L^1(\mathcal{A}, X)$. Then $L^1(\mathcal{B}, Y)$ is coproximinal in $L^1(\mathcal{A}, X)$.*

**Proof.** Let $f \in L^1(\mathcal{A}, X)$, let $g_0 \in L^1(\mathcal{A}, Y)$ be such that $\|g_0 - g\| \leq \|f - g\|$ for all $g \in L^1(\mathcal{A}, Y)$. Now $E'(g_0) \in L^1(\mathcal{B}, Y)$ and for any $g \in L^1(\mathcal{B}, Y)$, $\|E'(g_0) - g\| = \|E'(g_0 - g)\| \leq \|g_0 - g\| \leq \|f - g\|$. Thus the conclusion follows. $\square$

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