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Best Approximative Properties of Exposed Faces of $C(Q)$

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If C is an exposed face of the unit sphere of the Banach space $C(Q)$, Q compact Hausdorff space, then the closed convex set C is strongly proximinal and the metric projection onto C is Hausdorff metric continuous.

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1. Introduction

Throughout X denotes a real Banach space, X^*, the dual of X, $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$, the unit sphere of X.

For any x in X and a subset C of X, $d(x, C)$ denotes the distance of x from C. Let

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

If $P_C(x)$ is a non-empty set for each x in X, we say the subset C is proximinal in X. For any $\delta > 0$ we set

$$P_C(x, \delta) = \{z \in C : \|x - z\| < d(x, C) + \delta\}.$$

We say a proximinal set C of a normed linear space X is *strongly proximinal* if for each x in X and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{d(z, P_C(x)) : z \in P_C(x, \delta)\} < \epsilon.$$

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Let X be a normed linear space and $\{f_1, f_2...f_n\} \subseteq X^*$.We now define subsets $J_X(f_1, f_2...f_i)$ for $1 \leq i \leq n$ inductively as follows.

$$J_X(f_1) = \{x \in S_X : f_1(x) = \|f_1\|\}$$

and having defined $J_X(f_1, f_2...f_{i-1})$, we inductively define

$$\begin{aligned} & J_X(f_1, f_2...f_i) \\ = \ & \{x \in J_X(f_1, f_2...f_{i-1}) : f_i(x) = \sup\{f_i(y) : y \in J_X(f_1, f_2...f_{i-1})\}\} \end{aligned} \tag{1}$$

for $2 \leq i \leq n$. Note that $J_X(f_1) \neq \phi \Leftrightarrow f_1 \in NA(X)$. The sets $J_X(f_1, f_2...f_i)$ can be empty and if non-empty, they are closed convex subsets of B_X.

If f is in X^*, we say f is norm attaining functional if the set $J_X(f) = \{x \in S_X : f(x) = \|f\|\}$ is non-empty. We denote by $NA(X)$, the set of all norm attaining functionals on X. If $H = \ker f$, then it is well known that [8]

$$f \in NA(X) \Leftrightarrow J_X(f) \neq \emptyset \Leftrightarrow H \text{ is proximinal in X.}$$

If f is in $NA(X)$, then the closed, convex subset $J_X(f)$ of S_X is called an exposed face of the unit ball of X. Let X and Y be normed linear spaces. We say F is a set valued map from X into Y if F is a map from X into the class of all non-empty subsets of Y and in this case, we write $F : X \longrightarrow Y$ is a set valued map. Let x_0 be in X. The set valued map F is said to be

1. **lower semi-continuous** at x_0 if, given $\epsilon > 0$ and z in $F(x_0)$, there exists $\delta = \delta(\epsilon, z) > 0$ such that the set $B(z, \epsilon) \cap F(x)$ is non-empty, for any x in $B(x_0, \delta)$. If δ can be chosen to be independent of z in $F(x_0)$ in the above definition, that is, given $\epsilon > 0$, there exists $\delta > 0$ such that the set $B(z, \epsilon) \cap F(x)$ is non-empty, for any x in $B(x_0, \delta)$ and any z in $F(x_0)$, then F is said to be **lower Hausdorff semi-continuous** at x_0 .
2. **upper semi-continuous** at x_0 if given any open neighbourhood U of zero in X, there exists $\delta > 0$ such that

   $$F(x) \subseteq F(x_0) + U$$

   for each x in $B(x_0, \delta)$. Replacing the arbitrary open set U by an open ball in the above, yields the notion of upper Hausdorff semi-continuity. More precisely, the map F is **upper Hausdorff semi-continuous** at x_0, if given $\epsilon > 0$ there exists $\delta > 0$ such that

   $$F(x) \subseteq F(x_0) + \epsilon S_X$$

   for each x in $B(x_0, \delta)$.

The set valued map F is lower (upper, lower Hausdorff, upper Hausdorff) semi-continuous on X if it is lower (upper, lower Hausdorff, upper Hausdorff) semi-continuous at each point of X and is called **Hausdorff metric continuous** if it is both lower and upper Hausdorff semi-continuous.

Definition 1.1. A selection for the set valued map F is a map $f : X \longrightarrow Y$ such that $f(x)$ is in $F(x)$, for each x in X. A selection of the set valued map F which is continuous on X, is called a continuous selection of the map F.

It is easily verified that the metric projection onto a strongly proximinal set is upper Hausdorff metric continuous.

2. The Space $C(Q)$

Let X denote $C_{\mathbb{R}}(Q)$, the space of all real valued continuous functions defined on a compact Hausdorff space Q with sup norm. Then the dual of X is the space of regular Borel measures on Q and for μ in X^*, the Jordan decomposition [7] of μ is given by

$$\mu = \mu^+ - \mu^-$$

where $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$. We recall that the support of a measure μ in X^* is the smallest closed subset F of Q such that $|\mu|(E) = 0$ for any E contained in $Q \setminus F$. We denote the support of a measure μ by $S(\mu)$. It is known and easily shown [1, 2] that

$$J_X(\mu) \neq \emptyset \Leftrightarrow S(\mu^+) \cap S(\mu^-) = \emptyset$$

and in this case

$$J_X(\mu) = \bigcap_{\sigma \in \{-1,1\}} \{f \in S_X : f \equiv \sigma \text{ on } S(\mu^\sigma)\}.$$

Now we consider a linearly independent set $\{\mu_1, \mu_2, ..., \mu_n\}$ in S_{X^*}, the unit sphere of the dual space X^* and consider the sets $J_X(\mu_1, \mu_2, ..., \mu_i)$, as given by (1), for $2 \leq i \leq n$. Using induction and very similar arguments as in (iii) of Theorem 3 from [4] (See also Theorem 2.2 in [8]), it is easily verified that

$$J_X(\mu_1, \mu_2, ..., \mu_n) \neq \phi \iff \begin{array}{l} S(\mu_i^+) \cap S(\mu_i^-) = \phi \text{ for } 1 \leq i \leq n \text{ and} \\ S(\mu_i) \backslash S(\mu_{i-1}) \text{ is closed for } 2 \leq i \leq n \end{array} \quad (2)$$

and

$$J_X(\mu_1, \mu_2, ..., \mu_n) = \bigcap_{i=1}^{n} E_i, \quad (3)$$

where $E_1 = J_X(\mu_1)$ and for $2 \leq i \leq n$, the sets E_i are given by

$$E_i = \bigcup_{\sigma \in \{-1,1\}} \left\{ g \in J_X(\mu_1) : g \equiv \sigma \text{ on } S(\mu_i^\sigma) \setminus \bigcup_{j=i}^{i-1} S(\mu_j) \right\}, \quad (4)$$

whenever the set $S(\mu_i^\sigma) \setminus \bigcup_{j=i}^{i-1} S(\mu_j)$is non-empty.

Fact 2.1. *The set $J_X(\mu_1, \mu_2, ..., \mu_n)$, if non-empty, is proximinal in X.*

Proof. Since $J_X(\mu_1, \mu_2, ..., \mu_n)$ is non-empty, (2) holds. Let $f \in X$ and set α as

$$\alpha = \max_{1\le i\le n}\left[\max_{t\in S(\mu_i^+)\setminus\bigcup_{j=i}^{i-1} S(\mu_j)} |f(t)-1|, \max_{t\in S(\mu_i^-)\setminus\bigcup_{j=i}^{i-1} S(\mu_j)} |f(t)+1|\right], \tag{5}$$

taking $S(\mu_0) = \emptyset$. By (4) and (5),

$$d(f, J_X(\mu_1, \mu_2, ..., \mu_n)) \ge \alpha. \tag{6}$$

We discuss two cases

$$\text{Case (1)}: \quad \sup_{t\in Q} d(f(t), [-1, 1]) \le \alpha. \tag{7}$$

Pick any $g \in J_X(\mu_1, \mu_2, ..., \mu_n)$ and set $h = \max\{-\alpha, \min\{f - g, \alpha\}\}$ and $\widetilde{g} = f - h$. Then $\widetilde{g}$ is continuous on Q and is in X. Also,

$$\|f - \widetilde{g}\| = \|h\| \le \alpha.$$

We will now show that $\widetilde{g} \in J_X(\mu_1, \mu_2, ..., \mu_n)$ and $\|f - \widetilde{g}\| = \alpha$. This with (6) would imply $\widetilde{g}$ is a best approximation to f from $J_X(\mu_1, \mu_2, ..., \mu_n)$.

Note that if $|(f - g)(t)| \le \alpha$, then $h(t) = (f - g)(t)$ and so

$$\widetilde{g} = (f - h)(t) = g(t).$$

Thus

$$\widetilde{g} = g \ \text{ on } \{t \in Q : |(f - g)(t)| \le \alpha\}. \tag{8}$$

Since $g \in J_X(\mu_1, \mu_2, ..., \mu_n)$, the definition of α given by (5) and the description of the sets $J_X(\mu_1, \mu_2, ..., \mu_n)$ as given by (3) and (4) imply that

$$\bigcup_{i=1}^{n} S(\mu_i) \subseteq \{t \in Q : |(f - g)(t)| \le \alpha\}.$$

Hence

$$\widetilde{g} = g \ \text{ on } \bigcup_{i=1}^{n} S(\mu_i). \tag{9}$$

Now we assume that $|(f - g)(t)| \ge \alpha$. If $(f - g)(t) \ge \alpha$, then $h(t) = \alpha$ and so

$$\begin{aligned}\widetilde{g}(t) = f(t) - h(t) &= f(t) - \alpha \\ &\ge g(t) + \alpha - \alpha \\ &= g(t) \\ &\ge\ -1,\end{aligned}$$

since $\|g\| = 1$. Further $\widetilde{g}(t) = f(t) - h(t) = f(t) - \alpha \leq 1$ by (7). On the other hand, if $(f - g)(t) \leq -\alpha$ then $h(t) = -\alpha$ and

$$\begin{aligned} \widetilde{g}(t) = f(t) - h(t) &= f(t) + \alpha \\ &\leq g(t) - \alpha + \alpha \\ &= g(t) \\ &\leq 1, \end{aligned}$$

as $\|g\| = 1$. Further $\widetilde{g}(t) = f(t) - h(t) = f(t) + \alpha \geq -1$ by (7). Thus $|\widetilde{g}| \leq 1$ on $\{t \in Q : |(f - g)(t)| \geq \alpha\}$. This with (8) implies $\|\widetilde{g}\| = 1$. Since g is in $J_X(\mu_1, \mu_2, ..., \mu_n)$, this together with (9), (3) and (4) further implies that $\widetilde{g} \in J_X(\mu_1, \mu_2, ..., \mu_n)$.

We recall that $\|f - \widetilde{g}\| = \|h\| \leq \alpha$ and therefore,

$$\|f - \widetilde{g}\| \leq \alpha \leq d(f, J_X(\mu_1, \mu_2, ..., \mu_n)).$$

Thus $\widetilde{g}$ is a best approximation to f from $J_X(\mu_1, \mu_2, ..., \mu_n)$. We now discuss the second case.

$$\text{Case (2):} \quad \beta = \sup\left\{ d(f(t), [-1, 1]) : t \in Q \setminus \bigcup_{i=1}^{n} S(\mu_i) \right\} > \alpha.$$

In this case, note that since $J_X(\mu_1, \mu_2, ..., \mu_n)$, is a subset of the unit sphere S_X,

$$d(f, J_X(\mu_1, \mu_2, ..., \mu_n)) \geq \sup\left\{ d(f(t), [-1, 1]) : t \in Q \setminus \bigcup_{i=1}^{n} S(\mu_i) \right\} = \beta. \quad (10)$$

Define $\widetilde{f}$ by

$$\widetilde{f} = \max\{-1 - \alpha, \min\{f, 1 + \alpha\}\}. \quad (11)$$

We have $\widetilde{f} \in C(Q)$ and by (7)

$$\widetilde{f} = f \quad \text{on} \quad \bigcup_{i=1}^{n} S(\mu_i) \subseteq \{t \in Q : |f(t)| \leq 1 + \alpha\}. \quad (12)$$

Clearly $\sup_{t \in Q} d(\widetilde{f}(t), [-1, 1]) \leq \alpha$ and hence $d(\widetilde{f}, J_X(\mu_1, \mu_2, ..., \mu_n)) = \alpha$. By Case (1), there exists $\widetilde{g} \in P_{J_X(\mu_1, \mu_2, ..., \mu_n)}(\widetilde{f})$. Thus

$$\|\widetilde{f} - \widetilde{g}\| = d(\widetilde{f}, J_X(\mu_1, \mu_2, ..., \mu_n)) = \alpha. \quad (13)$$

We now claim that $\widetilde{g}$ is a best approximation to f from $J_X(\mu_1, \mu_2, ..., \mu_n)$. For this, we would show that $\|f - \widetilde{g}\| \leq \beta$. This with (10) would prove the claim. Now using (12) and (13) we note that

$$\sup\{|(f - \widetilde{g})(t)| : t \in Q : |f(t)| \leq 1 + \alpha\} \leq \|\widetilde{f} - \widetilde{g}\| \leq \alpha < \beta. \quad (14)$$

We observe that, by (13), $(f-\widetilde{g})(t) \leq \alpha$. Hence by (11) we have

$$\widetilde{g}(t)=1 \quad \text{if } \widetilde{f}(t)=1+\alpha \quad \text{or equivalently } f(t) \geq 1+\alpha$$

and similarly

$$\widetilde{g}(t)=-1 \quad \text{if } \widetilde{f}(t)=-1-\alpha \quad \text{or equivalently } f(t) \leq -1-\alpha.$$

Now using (10) and (12), we conclude that

$$\begin{aligned} &\sup\{|(f-\widetilde{g})(t)| : t \in Q : |f(t)| > 1+\alpha\} \\ \leq\ &\sup\left\{d(f(t),[-1,1]) : t \in Q \setminus \bigcup_{i=1}^{n} S(\mu_i)\right\} = \beta. \end{aligned}$$

Clearly, this with (10) and (14) imply that

$$\|f-\widetilde{g}\| = \beta \leq d(\widetilde{f}, J_X(\mu_1,\mu_2,...,\mu_n))$$

and $\widetilde{g}$ is a best approximation to f from $J_X(\mu_1,\mu_2,...,\mu_n)$. □

Fact 2.2. *The set $C = J_X(\mu_1,\mu_2,...,\mu_n)$ is strongly proximinal and P_C is Hausdorff metric continuous. In particular P_C has a continuous selection.*

Proof. Let $f \in X$ and α as given by (5). If $d = d(f,C)$, then $d \geq \alpha$. Recall that

$$C = J_X(\mu_1,\mu_2,...,\mu_n) = \bigcap_{i=1}^{n} E_i$$

where $E_1 = J_X(\mu_1)$ and for $2 \leq i \leq n$, the sets E_i are given by (4).

By Fact 2.2 above, C is proximinal. Pick any $f_1 \in P_C(f)$. Since C is a subset of the unit sphere, we have $\|f_1\| = 1$ and $\|f - f_1\| = d$. Hence

$$\|f\| \leq \|f_1\| + d = 1 + d. \tag{15}$$

Pick $h \in C$ such that $\|f-h\| < d+\delta$ for some $\delta > 0$. We will construct $\widetilde{h} \in P_C(f)$ such that $\|h-\widetilde{h}\| \leq \delta$. This would prove C is strongly proximnal.

Define

$$g = \max\{-d, \min\{f-h, d\}\}. \tag{16}$$

Then $g \in C(Q)$ and for any $t \in Q$ we have

$$(f-h)(t) \geq d \Rightarrow g(t) = d, \qquad (f-h)(t) \leq -d \Rightarrow g(t) = -d$$

and

$$|(f-h)(t)| \leq d \Rightarrow g(t) = (f-h)(t).$$

Define $\widetilde{h} = f - g$. Then

$$\widetilde{h} = h \ \text{ on } \{t \in Q : |(f-h)(t)| \leq d\}. \tag{17}$$

If $(f-h)(t) > d$ then $g(t) = d$ and $\widetilde{h}(t) = f(t) - g(t) \leq 1 + d - d \leq 1$ by (15). Also

$$\begin{aligned} \widetilde{h}(t) &= f(t) - g(t) \\ &= h(t) + (f(t) - h(t)) - g(t) \\ &\geq h(t) + d - d \\ &= h(t) \\ &\geq -1. \end{aligned}$$

If $(f-h)(t) < -d$, then $g(t) = -d$ and $\widetilde{h}(t) = f(t) - g(t) \geq -1 - d + d = -1$ by (15). Further in this case,

$$\begin{aligned} \widetilde{h}(t) &= f(t) - g(t) \\ &= h(t) + (f(t) - h(t)) - g(t) \\ &\leq h(t) - d + d \\ &= h(t) \\ &\leq 1. \end{aligned}$$

Thus

$$|\widetilde{h}(t)| \leq 1 \ \text{ if } |(f-h)(t)| > d$$

and

$$|\widetilde{h}(t)| = |h(t)| \leq 1 \ \text{ if } |(f-h)(t)| \leq d.$$

Hence $\|\widetilde{h}\| \leq 1$.

We have $d = d(f, C) \geq \alpha$ by (6). Since $h \in C$, using (3), (4) and (5) we have

$$\bigcup_{i=1}^{n} S(\mu_i) \subseteq \{t \in Q : |(f-h)(t)| \leq \alpha \leq d\}.$$

Hence, by (17), $\widetilde{h} = h$ on $\bigcup_{i=1}^{n} S(\mu_i)$ and therefore, (3) and (4) imply that $\widetilde{h} \in C$ since $\|h\| \leq 1$. Now $f - \widetilde{h} = f - (f - g) = g$ and by (16), $\|g\| \leq d$. So $\widetilde{h} \in P_C(f)$.

We now show that $\|h - \widetilde{h}\| \leq \delta$. If $|(f-h)(t)| \leq d$ then $\widetilde{h}(t) = h(t)$. Hence

$$|(\widetilde{h} - h)(t)| \leq \delta \ \text{ if } |(f-h)(t)| \leq d. \tag{18}$$

So we assume that $|(f-h)(t)| > d$. If $(f-h)(t) > d$ then by (16), $g(t) = d < (f-h)(t)$. Now $\widetilde{h}(t) = (f-g)(t)$ and therefore

$$(\widetilde{h} - h)(t) = (f-g)(t) - h(t) > h(t) - h(t) = 0.$$

Also since $\|f-h\| < d+\delta$, we have

$$0 < (\widetilde{h}-h)(t) = (f-h)(t) - g(t) \leq d+\delta-d = \delta.$$

If $(f-h)(t) < -d$, then

$$g(t) = -d > (f-h)(t).$$

This implies $(f-g)(t) < \ h(t)$ and therefore

$$(\widetilde{h}-h)(t) = (f-g)(t) - h(t) < h(t) - h(t) = 0.$$

Further

$$\begin{aligned} 0 > (\widetilde{h}-h)(t) &= (f-h)(t) - g(t) \\ &\geq \ -d-\delta+d = -\delta. \end{aligned}$$

Thus if $|(f-h)(t)| > d$ then either $0 \leq (\widetilde{h}-h)(t) \leq \delta$ or $0 \geq (\widetilde{h}-h)(t) \geq -\delta$. So $|(\widetilde{h}-h)(t)| \leq \delta$ if $|(f-h)(t)| > d$. This with (18) implies that

$$\|h-\widetilde{h}\| \leq \delta$$

and completes the proof for strong proximinality of C.

We now show that the metric projection P_C is lHsc. Since C is strongly proximinal, P_C is upper Hausdorff semi-continuous. We only need to show that P_C is lower Hausdorff semi-continuous. From the above for any $f \in C(Q)$ and $\delta > 0$, we have

$$h \in C, \|f-h\| < d(f,C)+\delta \Rightarrow \exists\, \widetilde{h} \in P_C(f) \ \text{ such that } \|\widetilde{h}-h\| \leq \delta.$$

Let $f \in C(Q)$ and $h \in P_C(f)$. Let $\epsilon > 0$ be given, we need to show that there exists $\delta > 0$ such that

$$\|f-g\| < \delta \Rightarrow B(h,\epsilon) \cap P_C(g) \neq \phi. \tag{19}$$

Choose $0 < \delta < \frac{\epsilon}{4}$. Then $\|f-g\| < \delta$ implies

$$|d(f,C) - d(g,C)| \leq \|f-g\| < \delta$$

and we have

$$\begin{aligned} \|g-h\| &\leq \|g-f\| + \|f-h\| \\ &< \delta + d(f,C) \\ &\leq d(g,C) + 2\delta. \end{aligned}$$

Hence there exists $\widetilde{h} \in P_C(g)$ such that $\|h-\widetilde{h}\| \leq 2\delta < \epsilon$ and (19) is satisfied. Thus P_C is Hausdorff metric continuous and by the well known Michael selection theorem [5, 6], P_C has a continuous selection. $\square$

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