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Hamel Bases, Convexity and Analytic Sets in Fréchet Spaces

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It is shown that a Hamel basis over the field of reals of an infinite dimensional linear Polish space can not be an analytic set. Furthermore, if (x_α) is an infinite linearly independent subset of a Fréchet space X and if C is the convex cone generated by (x_α), then C is not a closed set. In particular, the convex cone generated by a Hamel basis in such a space can not be closed. The notion of convex and midpoint convex functions is extended to the case when the domain of the functions is a connected open set, and analytic graph theorems are given for these functions. It is shown also that if $f : \mathbb{R}^n \to \mathbb{R}$ is an order monotone function, then f is Baire measurable, but in general, f is not universally measurable.

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1. Introduction

Hamel bases has been studied many times. Sierpinski showed in [13] that a Hamel basis of $\mathbb{R}$ over the rationals can not be an analytic set. In this work, the scalar field will be the set of reals, and in particular, Hamel bases will be considered over the scalar field of the real numbers.

It will be shown that a Hamel basis of an infinite dimensional linear Polish space can not be an analytic set. It will be shown also that if (x_α) is an infinite linearly independent subset of a Fréchet space X and if C is the (positive) convex cone generated by (x_α), then C is not closed. In particular, the (positive) convex

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cone generated by a Hamel basis in such a space can not be closed. It will be shown furthermore, that if $f : \mathbb{R}^n \to \mathbb{R}$ is an order monotone function, then f is Baire measurable, but in general, f is not universally measurable.

Some of the concepts discussed in this paper have different definitions in the literature. Here, the terminology of Rogers and Jayne [11] will be followed concerning analytic sets. Therefore the terminology will be reviewed first, together with some additional concepts and notation.

The set of all positive integers will be denoted by $\mathbb{N}$, and it will be assumed that this space is equipped with the discrete topology. The space of sequences of positive integers equipped with the product topology will be denoted by $\mathbb{N}^{\mathbb{N}}$. With the aid of a continued fraction expansion, one can show that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic with the space of irrational numbers between 0 and 1.

By exhibiting an adequate continuous map, Kuratowski showed that a Polish space, which is a complete, metrizable and separable space, can be written as a continuous image of $\mathbb{N}^{\mathbb{N}}$.

Definition 1.1. A set in a Hausdorff space is said to be an analytic set if it is empty or if it is the continuous image of $\mathbb{N}^{\mathbb{N}}$, or, equivalently, of the continuous image of a Polish space. A space is called an analytic space if it is a Hausdorff space and if it is an analytic set in itself.

If $A \subset \mathbb{R}^n$, then $\overline{A}$ denotes its closure, $\mathrm{int}(A)$ its interior, and $\mathrm{diam}(A)$ its diameter. If δ is a positive number and $x \in \mathbb{R}^n$, then $O(x,\delta) := \{y \in \mathbb{R}^n : ||y-x|| < \delta\}$ denotes the open ball of radius δ about x, where $||y-x||$ denotes the Euclidean norm of the vector $y-x$. If A is a subset of a group X, then $A-A$ denotes the set $\{x \in A : x = x_1 - x_2, \text{ for some } x_1 \in A, x_2 \in A\}$. If $A \subset X$, where X is a linear space, then $\mathrm{conv}(A)$ denotes the convex hull of A.

We will need also, some further definitions.

Definition 1.2. Let X be a linear space, a set $K \subset X$ is said to be a convex cone in X, if it has the following properties.

(i) $K + K \subset K$, and
(ii) $\lambda K \subset K$ for all $\lambda \geq 0$.

In this work the word cone will stand for convex cone, even when the word convex is skipped.

Definition 1.3. Let X be a linear space, and let $A \subset X$. Then we shall call K, the cone generated by A if

$$K = \cup_{\lambda \geq 0} \lambda\, \mathrm{conv}(A). \tag{1}$$

Definition 1.4. A linear topological space X is said to be a Fréchet space if its topology is generated by a translation invariant metric ρ such that the metric space (X,ρ) is complete.

Notice that we don't assume that a Fréchet space is locally convex.

Let X be a Fréchet space, and let W be a convex subset of X. A function $f : W \to \mathbb{R}$ is said to be midpoint convex on W if for all $x_1, x_2 \in W$, we have the inequality

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}. \tag{2}$$

In this paper, we extend the notion of convex (resp. midpoint convex) functions for a larger domain, which is not necessarily convex.

Definition 1.5. Let X be a Fréchet space, and let U be a connected open subset of X. A function $f : U \to \mathbb{R}$ is said to be convex on segments (resp. midpoint convex on segments) on U if for all $x_1, x_2 \in U$, for which the line segment $[x_1, x_2] \subset U$, $f][x_1, x_2]$ (the restriction of f to $[x_1, x_2]$) is convex (resp. midpoint convex).

Analytic graph theorems for convex and midpoint convex functions on segments will be given in this work. It will be shown (Theorem 4.4) that if X is a linear Polish space, $W \subset X$ connected open, $u : W \to \mathbb{R}$ midpoint convex on segments with analytic graph, then u is continuous on W.

2. Hamel bases

In the next definition, we are following the presentation from [10].

Definition 2.1. A subset S of a Hausdorff topological space X is said to have the property of Baire, if it can be represented in the form

$$S = G \,\Delta\, E = (G \setminus E) \cup (E \setminus G),$$

where G is open and E is a set of first category in X.

The following result plays an important role in the applications of sets having the property of Baire (Lemma 2.10.3 in [11]).

Lemma 2.2. *Let A be a set of second category in a Hausdorff linear topological space X. If A has the property of Baire, then* $0 \in \operatorname{int}(A - A)$.

It is known, and it is easy to show, that the set of all sets with the property of Baire in X forms a $\sigma-$algebra. A function $f : X \to \mathbb{R}$ is said to be Baire measurable, if f is measurable with respect to the $\sigma-$algebra of all sets with the property of Baire. The following result was shown, essentially by Nikodym when $n = 1$ in [9] and by Kuratowski in the general $\mathbb{R}^n$ case in [8]. It also serves as the definition of a Baire measurable function in [15].

Theorem 2.3. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is Baire measurable, if and only if, there is a set $E \subset \mathbb{R}^n$ of first category such that the restriction of f to $\mathbb{R}^n \setminus E$ is continuous.*

In this section Hamel bases will be discussed. It is assumed, that the space is a linear Polish space. The group operation will be always commutative. The next result is an extension of a previously mentioned result of Sierpinski.

Theorem 2.4. *Let X be an infinite dimensional linear Polish space and let B be a Hamel basis of X. Then B is not an analytic set.*

Proof. Let B be a given Hamel basis of X. Let Y_n be the subset of X, whose elements can be expressed as a linear combination of at most n distinct members of B. Hence,

$$Y_n = \{x \in X : x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n, \text{ where } c_1, ..., c_n \in \mathbb{R},\ b_1, ..., b_n \in B\}.$$

Thus, $0 \in Y_n$ for all n positive integers. Furthermore, the sets $Y_n's$ have the following properties.

(i) $\bigcup_{n=1}^{\infty} Y_n = X$,
(ii) for every n positive integer $Y_n \neq X$,
(iii) $Y_n \subset Y_{n+1}$ for $n = 1, 2...$,
(iv) $Y_n - Y_n \subset Y_{2n}$ for $n = 1, 2...$,
(v) If $y \in Y_n$ and if $c \in \mathbb{R}$ then $cy \in Y_n$.

Indeed, (i) follows from the fact that B is a Hamel basis of X. Next, (ii) follows from the fact that B has infinitely many different elements. Therefore, $Y_n \neq Y_m$ if $n \neq m$. Finally, (v) follows form the fact that if $y = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n \in Y_n$, then $cy = cc_1 b_1 + cc_2 b_2 + \cdots + cc_n b_n \in Y_n$.

Assume next, to obtain a contradiction, that B is analytic. We claim that in that case every Y_n is also analytic. Indeed, since B is analytic, there is a Polish space M and a continuous function $f : M \to X$ such that $f(M) = B$. Define, for all n positive integers

$$M_n = \mathbb{R}^n \times M \times \cdots \times M, \quad \text{where M occurs n-times},$$

and where the topology of M_n is determined by the product topology. Then M_n is a Polish space. Define $F_n : M_n \to X$ as

$$F_n(c_1, c_2, ..., c_n, m_1, m_2, ..., m_n) = c_1 f(m_1) + c_2 f(m_2) + \cdots + c_n f(m_n),$$

where $c_i \in \mathbb{R}$ and $m_i \in X$ for $i = 1, ..., n$. Clearly, F_n is a continuous function with $F_n(M_n) = Y_n$. It follows that Y_n is an analytic set. By Theorem 2.2.9. of [15], an analytic set in a Hausdorff space has the property of Baire, it follows, that each Y_n has the property of Baire. Since X is a set of second category, (i) implies that there exists at least one n so that Y_n is a set of second category. Hence Lemma 2.2 implies that for this fixed n, $Y_n - Y_n$ is a neighborhood of 0. Therefore, $Y_n - Y_n$ contains an open ball $B(0, \epsilon)$, and by (iv) $B(0, \epsilon) \subset Y_{2n}$. Since $B(0, \epsilon)$ is a neighborhood of 0, it follows that $\bigcup_{c \in \mathbb{R}} cB(0, \epsilon) = X$. Thus, (v) implies that $Y_{2n} = X$, which is a contradiction. $\square$

3. Convex cones generated by Hamel bases

A cone $K \subset X$, is a generating cone if $K - K = X$. Notice that for any cone K, the set $K - K$ is a subspace of X. Next, we are stating a lemma without proof, which shows a connection between Hamel bases and generating cones. This result has been stated in [2].

Lemma 3.1. *Let X be a linear space, and let B be a Hamel basis of X, and let K be the cone generated by B, then K is a generating cone.*

In this section, we describe some of the properties of a convex cone generated by a Hamel basis, and more generally, by an infinite linearly independent set. We start with an example of Borwein-Goebel from [2]. In this example (Example 2 in Section 1.2 from [2]) the following topics were discussed. Let X be an infinite-dimensional Banach space. Consider an ordered Hamel basis in X. Let K be the cone consisting of all the elements of X whose coordinates are nonnegative in their Hamel basis expansion with respect to the given Hamel basis. Then, it is stated that K is a closed, convex and generating cone with empty interior. Contrary to the claim that K is always closed, we will show first with the aid of an example that in an infinite dimensional Banach space, there is a cone K derived from a Hamel basis, which is not closed.

Example 3.2. Let X be an infinite dimensional Banach space, and let $\{y_n\}_0^\infty$ be a linearly independent sequence of nonzero elements in X such that

$$\lim_{n\to\infty} y_n = y_0 \neq 0.$$

Define $x_0 = -y_0$ and $x_n = y_n$ for $n \geq 1$. Next, extend $\{x_n\}_0^\infty$, by Zorn lemma, to an ordered Hamel basis in X. Let K be the convex cone, which consisting the set of elements whose all coordinates are nonnegative. Then K is not closed.

Indeed, it follows that $x_n \in K$ and $\lim x_n = y_0 = -x_0 \notin K$. Thus the cone, K, is not closed.

Next, we illustrate the previous situation, by a concrete example in $\ell_1(\mathbb{R})$, which is the Banach space of infinite absolutely convergent real sequences with norm $||x|| = \sum_{n=1}^\infty |x_n|$.

Example 3.3. Let $\{e_1, e_2, ...\}$ be the standard Schauder basis of $\ell_1(\mathbb{R})$, that is

$$e_k = (0, ..., 0, 1, 0, ...), \quad k \geq 1,$$

where e_k has zero in every position but the k-th position, where the entry is one. Let for $n \geq 1$,

$$y_n = \sum_{k=1}^{n} \frac{1}{2^k} e_k \quad \text{and} \quad y_0 = \sum_{k=1}^{\infty} \frac{1}{2^k} e_k.$$

It is clear the set $\{y_n\}_{n=1}^{\infty}$ is linearly independent. On the other hand y_0 does not belong to the linear span of $\{y_n\}_{n=1}^{\infty}$, since any vector in this span has only finitely many non-zero components. Thus $\{y_n\}_{n=0}^{\infty}$ is also linearly independent. Notice that

$$\lim_{n\to\infty} y_n = y_0 \neq 0.$$

Define, as in the previous example, $x_0 = -y_0$ and $x_n = y_n$ for $n \geq 1$. Next, extend $\{x_n\}_0^{\infty}$, by Zorn lemma, to a Hamel basis in $\ell_1(\mathbb{R})$, and let K be the cone generated by this Hamel basis. Then it is clear that $y_0 \notin K$, hence K is not closed. On the other hand, it follows from Lemma 3.1 that K is a generating cone.

Next, the statement of the Example 3.2 will be extended. We present first a lemma.

Lemma 3.4. *Let C be a closed convex cone in a Fréchet space X, and let $\phi : X \to \mathbb{R}$ be a linear form such that $\phi(x) \geq 0$ for every $x \in C$. Then $\phi|C$ (the restriction of ϕ to C) is continuous at 0, i.e., whenever $\{x_n\} \subset C$ and $x_n \to 0$, then $\phi(x_n) \to 0$.*

Proof. Let ρ be a translation invariant metric, which generates the topology of X and let 0 denote the identity element of X. Suppose that the statement is false. Thus we can assume that there exist a sequence $\{x_n\} \subset C$ and a $\delta > 0$ so that

$$\phi(x_n) \geq \delta \quad \text{and} \quad \rho(x_n, 0) \leq \frac{1}{2^n} \quad \text{for } n = 1, 2, \dots.$$

Let $S_N = x_1 + x_2 + \cdots + x_N$. Since C is a convex cone, it follows that $S_N \in C$. Because $\{S_N\}$ is a Cauchy sequence and X is complete, it follows that $\lim S_N$ exists. Let $s = \lim S_N$. Then $s \in C$, because C is closed. On the other hand, using similar arguments, we can see that $R_N = \sum_{n=N+1}^{\infty} x_n$ exists and $R_N \in C$ for all N positive integer. Thus

$$\phi(s) = \phi(s_N) + \phi(R_N) \geq \phi(s_N) \geq N\delta.$$

Hence δ must be zero, which is a contradiction. □

Now, we present a result about non closed cones.

Theorem 3.5. *Let (x_α) for $\alpha \in A$ be an infinite, linearly independent subset of a Fréchet space X. Let C be the convex cone generated by (x_α) for $\alpha \in A$. Then C is not a closed subset of X.*

Proof. Notice that, it is not assumed that the set (x_α) for $\alpha \in A$ is a Hamel basis. In other words, it is not assumed that it spans the whole X. Nevertheless, (x_α), for $\alpha \in A$ can be extended to an ordered Hamel basis. There is an indexed set of vectors $(x_\beta)_{\beta\in B}$ so that $(x_\alpha)_{\alpha\in A} \cup (x_\beta)_{\beta\in B}$ is linearly independent and spans X. Denote $D = A \cup B$. It can be also assumed that $A \cap B = \emptyset$. Assume further that $A_1 := \{\alpha_1, \alpha_2, ...\} \subset A \subset D$. Let ρ be a translation invariant metric, which generates the topology of X.

Choose positive numbers t_n so that $\rho(t_n x_{\alpha(n)}) \to 0$, as $n \to \infty$. Define $\phi : X \to \mathbb{R}$ by $\phi(x_{\alpha(n)}) = 4^n/t_n$ for all $\alpha(n) \in A_1$, and $\phi(x_\gamma) = 0$ if $\gamma \in D \setminus \{\alpha(1), \alpha(2), ...\}$, and extends its definition by linearity to $X \setminus D$. It is clear that $\phi(x) \geq 0$ for $x \in C$. Since $t_n x_{\alpha(n)} \to 0$ as $n \to \infty$, Lemma 3.4 implies that $\lim_{n\to\infty} \phi(t_n x_{\alpha(n)}) = 0$, which contradicts the fact that $\phi(t_n x_{\alpha(n)}) = 4^n$ for $n \geq 1$. □

Theorem 3.6. *Let X be an infinite dimensional linear Polish space and let B be a Hamel basis of X. Let C be the convex cone generated by B. Then C is not a Borel subset of X.*

Proof. Let B be a Hamel basis, and let C be the convex cone generated by B. By a result of Christensen and Kiaer (Theorem 7.4 in [5]) at most finitely many of the coordinate linear functionals corresponding to the basic vectors can be continuous. Denote by x_0, a basic vector with the property that the corresponding coordinate linear functional, ϕ_0, is discontinuous. Let $B_0 = B \setminus \{x_0\}$. Let C_0 be the convex cone generated by B_0. Thus, we have the following properties:

$$\phi_0(x_0) = 1, \qquad \phi_0|C_0 \equiv 0, \qquad C = [0,\infty)x_0 + C_0, \qquad \ker\phi_0 = C_0 - C_0.$$

To obtain a contradiction, we assume that C is a a Borel set. Hence, its translation, $x_0 + C$ is also a Borel set. Therefore, the difference set

$$A := C \setminus (x_0 + C) = [0,1)x_0 + C_0.$$

is also a Borel set. Thus,

$$D := A - A = (-1,1)x_0 + (C_0 - C_0) = (-1,1)x_0 + \ker\phi_0.$$

Notice that D is an analytic set, because D is the image of the Borel set $A \times A$ under the continuous map $(x, y) \to x - y$. Hence,

$$D = \{x \in X : |\phi_0(x)| < 1\}.$$

Therefore,

$$nD = nA - nA = \{x \in X : |\phi_0(x)| < n\} \quad \text{for } n = 1, 2, \ldots .$$

Clearly, $\bigcup_{n=1}^{\infty}(nD) = X$, and nD is an analytic set for every positive integer n. Therefore, nD has the property of Baire for every positive n. Since X is a set of second category, there is a positive integer n_0 such that n_0D is a set of second category. Hence Lemma 2.2 implies that $n_0D - n_0D = 2n_0D$ is a neighborhood of 0. It follows that $|\phi_0(x)|$ is bounded by $2n_0$ on the open set $2n_0D$, and thus ϕ_0 is continuous, which is a contradiction. □

4. Analytic graphs

The following definition will be needed in the sequel.

Definition 4.1. Let X be a linear topological space. Let A be a subset of X. Then A is called balanced, if for every $x \in A$ and for every $-1 \leq t \leq 1$ the point $tx \in A$. Let $x_0 \in X$ and let $B \subset X$. Then B is balanced with respect to x_0, if the set $B - x_0$ is balanced.

In what follows X is a linear metric space. The next well-known lemma is needed in the sequel, and it will be stated without proof.

Lemma 4.2. *Let X be a linear metric space. Then every $x_0 \in X$ has a basis of open neighborhoods consisting of balanced sets with respect to x_0.*

Next, a result about midpoint convex functions on segments will be stated. Its proof will be given in the Appendix.

Theorem 4.3. *Let E be a connected open subset of a linear metric space X and let $f : E \to \mathbb{R}$ be a midpoint convex function on segments. If f is locally upper bounded on E, then f is continuous on E.*

Now, we prove an analytic graph theorem for midpoint convex functions on segments.

Theorem 4.4. *Let X be a linear Polish space, and let W be a connected open subset of X. Let $f : W \to \mathbb{R}$ be a midpoint convex function on segments with analytic graph. Then f is continuous on W.*

Proof. It is enough to show that the midpoint convex function on segments $f : W \to \mathbb{R}$ is locally bounded from above at each point of W. Indeed, in that case Theorem 4.3 indicates that f is continuous on W. By Lemma 4.2, every $x \in W$ has a balanced open neighborhood $U \subset W$, which is symmetric with respect to x, and so that $x_0 + U + U = x_0 + U - U \subset W$. It can be assumed, without loss of generality, that the point is $x = 0$, and U is a balanced neighborhood of 0. Thus,

$$U + U = U - U \subset W. \tag{3}$$

By assumption, $\Gamma_f := \{(x,t) : f(x) = t\}$, the graph of f, is an analytic set in $X \times \mathbb{R}$. Let k be a positive integer. The set $A_k := \{(x,t) : t \leq k\}$ is an analytic set in $X \times \mathbb{R}$. It follows that $B_k := \Gamma_f \cap A_k$ is analytic in $X \times \mathbb{R}$. Hence, the projection of B_k into X is also analytic. Denote this projection by V_k. Clearly, $V_k := \{x \in W : f(x) \leq k\}$. Let $W_k := U \cap (-V_k) \cap V_k$. Then W_k is an analytic set, with $W_k = -W_k$. As it was discussed in the proof of Theorem 2.4, an analytic set has the property of Baire. Given that X is a complete metric space, the Baire category theorem implies that the open set U is a set of second category. It is clear that the union of the $W_k's$ is U, and so one of them, say W_k, is a set of second category and has the property of Baire. Thus, by Lemma 2.2, $W_k - W_k = W_k + W_k$ is a a neighborhood of 0. Let $H := (1/2)(W_k - W_k)$. Then H is a neighborhood of 0. The proof will be completed by showing the statement $f|H \leq k$.

Let z be a point in H. Then, in view of the definition of H, there are $x, y \in W_k$ so that $z = (1/2)x + (1/2)y$. We claim that the whole segment $[x, y]$ is contained in W.

Consider any $\xi \in [x, y]$, i.e.

$$\xi = tx + (1-t)y, \quad \text{for some } 0 \leq t \leq 1.$$

Given that $x, y \in U$ and U is balanced, it follows that $tx, (1-t)y \in U$. Hence $\xi \in U + U \subset W$. Given that f is a midpoint convex function on segments, and f is a midpoint convex function on $[x, y]$, it follows that

$$f(z) = f\left(\frac{x+y}{2}\right) \leq \frac{f(x)}{2} + \frac{f(y)}{2} \leq \frac{k}{2} + \frac{k}{2} = k. \qquad \square$$

Remark 4.5. The statement about the continuity of f in Theorem 4.4 is new, even in the case when f is convex on segments, or f is a convex function (in the ordinary sense) on a separable Banach space.

5. Order monotone functions

The following partial order on $\mathbb{R}^n$ will be used in this paper.

Definition 5.1. Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$. We will say that $x \leq y$ if $x_i \leq y_i$ for $i = 1, ..., n$.

Corresponding to this partial order, a notion of (order) monotonicity can be defined in $\mathbb{R}^n$.

Definition 5.2. Let $B \subset \mathbb{R}^n$ and let $f : B \to \mathbb{R}$. The function f is said to be (order) nondecreasing on B, if for any x, $y \in B$, such that $x \leq y$, the inequality $f(x) \leq f(y)$ holds. We shall also call such an f an order monotone function.

The following result was obtained by Chabrillac and Crouzeix in [4] about order monotone functions.

Theorem 5.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an order monotone function, let $x_0 \in \mathbb{R}^n$, and let $a = (a_1, ..., a_n) \in \mathbb{R}^n$, where $a_i > 0$ for $i = 1, ..., n$. Then*

(*a*) *f is Lebesgue measurable,*

(*b*) *f is continuous at x_0, if and only if, the real variable function $t \to f(x_0 + ta)$ is continuous at $t = 0$,*

(*c*) *f is almost everywhere Fréchet differentiable.*

The next result will be also used in the sequel. It can be found in [10], in the case when f is a real-valued function on $\mathbb{R}$. The proof given in [10] can be extended, without any difficulties, to the case when $f : \mathbb{R}^n \to \mathbb{R}$.

Lemma 5.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$, and let E be the set of points of discontinuity of f. Then E is an F_σ set, that is a set which is the countable union of closed sets.*

We will need also in the sequel the notion of universally measurable sets.

Definition 5.5. Let X be a Hausdorff space. A set $A \subset X$ is said to be universally measurable, if every finite (or equivalently every σ- finite) measure m on the Borel sets of X, the set A belongs to the m-completion of the σ algebra of the Borel sets of X.

Next, the measurability of order monotone functions will be discussed in $\mathbb{R}^n$. Part (a) of Theorem 5.3 shows that an order monotone function is Lebesgue measurable. It was shown in [7] that an order monotone function in $\mathbb{R}^n$ does not need to be Borel measurable. Improving that result, it will be shown, that an order monotone function is always Baire measurable, but it does not need to be universally measurable. To simplify the notation, I_0 will denote the open interval in $\mathbb{R}$ with left endpoint $1/2$ and right endpoint $5/4$.

Example 5.6. Let $X = [0,1] \times [0,1]$. Let $E \subset [0,1]$ be a non universally measurable set. Let

$$f(x,y) = \begin{cases} 0 & x+y<1; \\ 1 & x+y=1,\, x \in E; \\ 3/2 & x+y=1,\, x \notin E; \\ 2 & x+y>1. \end{cases} \tag{4}$$

Then f is Lebesgue measurable, but f is not universally measurable.

Proof. If the function f were universally measurable, then the level set $f^{-1}(I_0) = \{(x,y) : x+y = 1,\, x \in E\}$ were universally measurable. Consider $g : x \to (x,y)$ defined as $g(x) = (x, -x+1)$. Then $g([0,1]) = M := \{(x,y) : x+y = 1,\, 0 \le x \le 1\}$, and g is a homeomorphism which maps the metric space $[0,1]$ onto the metric space M. Furthermore, $f^{-1}(I_0) = g(E)$, and $E = g^{-1}(g(E))$. Since g is a homeomorphism, in particular, g^{-1} is continuous, it follows directly from a result of Thibault (Remark 1.1 in [14]) that $f^{-1}(I_0) = g(E)$ is not universally measurable, because $g(E)$ is universally measurable, if $E = g^{-1}(g(E))$ is universally measurable. The Lebesgue-measurability of f follows from Theorem 5.3. $\square$

The next result shows that an order monotone function is Baire measurable.

Theorem 5.7. *Let $f : X \to \mathbb{R}$ be an order monotone function, where $X = [a_1, b_1] \times ... \times [a_n, b_n]$, with $a_i < b_i$ for $i = 1, ..., n$. Then f is Baire measurable.*

Proof. Let E be the set of points of discontinuity of f. It follows from Lemma 5.4 that E is an F_σ set. By the previously mentioned results of Nikodym and Kuratowski, it needs to be shown that E is a set of first category. To do that, two cases will be separated. In the first case, assume that E is a closed set in X. Let $x \in X$ and let $a = (a_1, ..., a_n) \in \mathbb{R}^n$, where $a_i > 0$ for $i = 1, ..., n$. Consider the line $\ell_x(t) := x + ta$, where t is a real variable. Since, f is a single variable and monotone function of t when it is restricted to $\ell_x(t)$, it follows from Theorem 5.3(b) that $f : X \to \mathbb{R}$ has at most countable many discontinuities on this line. It follows also, the set of discontinuities of f is closed on this line in the Euclidean topology of $\ell_x(t)$. Therefore, the points of discontinuities of f on any subline in X of the form $X \cap (x + \mathbb{R}a)$ is a nowhere dense set. It has to be shown yet, that E is a nowhere dense set in $\mathbb{R}^n$. Given that E is closed, it is enough to show that E has no interior point. Assume, to obtain a contradiction, there is an $x_0 \in \text{int}(E)$. Consider the subline $X \cap (x_0 + \mathbb{R}a)$, where $a = (a_1, ..., a_n) \in \mathbb{R}^n$, with $a_i > 0$ for $i = 1, ..., n$. Consider an arbitrary neighborhood $O(x_0, \epsilon)$ of x_0, where $\epsilon > 0$. Since the intersection of this line with $O(x_0, \epsilon)$ is nowhere dense, there is an interval on that line which doesn't belong to E. Hence E must be a nowhere dense set in $\mathbb{R}^n$.

In the second case, assume, that E is an F_σ set. Then E can be written in the form $E = \cup F_i$, where F_i is closed for $i = 1, ..., m, ...$. It follows, like in the first case that F_i is a nowhere dense set for $i = 1, ...$. Therefore, E is a set of first category. □

It follows from Theorem 5.7 and from Example 5.6 that the set $f^{-1}(I_0)$ is not universally measurable, but it has the property of Baire.

Order increasing functions on $\mathbb{R}^n$ were extended to cone monotone functions by Borwein, Burke and Lewis in [1]. Here, we define this concept only in a special case.

Definition 5.8. Let X be a Banach space, let A be an open subset of X, and let K be a cone in X. Let $f : A \to \mathbb{R} \cup \{\infty\}$. Then f is said to be K monotone on A if $f(x + k) \geq f(x)$, whenever $x, x + k \in A$ and $k \in K$.

We close this section with an interesting example of a Borel measurable cone monotone function.

The following example (Example 13) was considered in [3]. For the Hilbert space $\ell_2(\mathbb{R})$ with norm $||x|| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$, the closed convex cone ℓ_2^+, i.e., the set of all nonnegative sequences in that space, has an empty interior. Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \ell_2 \text{ has infinitely many positive terms,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

It was shown in [3] that f is ℓ_2^+-increasing, and it is nowhere continuous on ℓ_2.

Example 5.9. The function $f : \ell_2(\mathbb{R}) \to \mathbb{R}$ is a Borel measurable function, where f is defined by (5).

Proof. Indeed, let $x = (x_1, x_2, ...) \in \ell_2(\mathbb{R})$, and define $g_N(x) : \ell_2(\mathbb{R}) \to \mathbb{R}$ by

$$g_N(x) = \sum_{k=N}^{\infty} (\max(x_k, 0))^2 \quad \text{for } N = 1, 2, , , , .$$

Clearly, g_N is a continuous function for all positive integer N. Denote the characteristic function of the set $(0, \infty)$ by χ, i.e.,

$$\chi(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

It will be shown next, that $\chi \circ g_N$ is a Baire class 1 function. Indeed, let $A_0 := \{x \in \ell_2(\mathbb{R}) : \chi \circ g_N(x) = 0\}$ and let $A_1 := \{x \in \ell_2(\mathbb{R}) : \chi \circ g_N(x) = 1\}$. It follows that A_0 is a closed set in $\ell_2(\mathbb{R})$ and A_1 is an open set in $\ell_2(\mathbb{R})$, which is also an F_σ set. Hence, $A_1 = \cup_{n=1}^{\infty} B_n$, where B_n is a closed set for $n = 1, \dots$. An application of the Tietze extension theorem to the function $\chi \circ g_N(x)|C_k$ extended from the closed set $C_k := A_0 \cup \cup_{n=1}^{k} B_n$ to $\ell_2(\mathbb{R})$ for $k = 1, 2, \dots$, yields a sequence of continuous functions, which is converging pointwise to $\chi \circ g_N(x)$ on $\ell_2(\mathbb{R})$.

It is easy to see that $f(x) = \lim_{N\to\infty} \chi \circ g_N(x), \quad x \in \ell_2$. Therefore, f is a Baire class 2 function, hence it is a Borel function. Combining that with the result of Borwein and Wang, it follows that (5) is a nowhere continuous Borel measurable function. □

6. Appendix

The following result can be found in Donoghue [6], in the one dimensional case and in a blog of Matt Rosenzweig [12], in the form stated in the next theorem.

Theorem 6.1. *Let X be a linear metric space. Let E be an open and convex subset of X. Let $f : E \to \mathbb{R}$ be a midpoint convex function. If f is discontinuous at a point $x_0 \in E$, then f is unbounded on every non empty convex open set contained in E.*

The proof of the previous theorem yields also the following stronger result.

Theorem 6.2. *Let X be a linear metric space. Let E be an open and convex subset of X. Let $f : E \to \mathbb{R}$ be a midpoint convex function. If f is bounded above in a neighborhood of a point $x_0 \in E$, then f is continuous on E.*

Next, we state a generalization of Theorem 6.1 for midpoint convex functions on segments.

Lemma 6.3. *Let E be a connected open subset of a linear metric space X, and let $f : E \to \mathbb{R}$ be a midpoint convex function on segments, and let $x_0 \in E$. If* $\limsup_{x\to x_0} f(x) < \infty$, *then f is continuous at x_0.*

Proof. By shifting the set E by x_0 and replacing f by f plus a constant, it can be assumed that $x_0 = 0$ and $f(0) = 0$. Let $m =: \limsup_{x\to 0} f(x)$. Hence $-\infty \le m < \infty$. Let U be a balanced open set with $0 \in U \subset E$. The existence of such a neighborhood of 0 follows from Lemma 4.2. For $x \in U$, the closed line segment connecting $-x$ and x is contained in U. Therefore, the definition of midpoint convexity implies that

$$2f(0) = 0 \le f(x) + f(-x), \quad x \in U. \tag{6}$$

Now, if $2x \in U$, then the interval $[0, 2x] \subset U$. Thus by midpoint convexity

$$f(x) = f\left(\frac{0+2x}{2}\right) \le \frac{f(0)}{2} + \frac{f(2x)}{2} = \frac{f(2x)}{2}.$$

Hence

$$m =: \limsup_{x\to 0} f(x) \le \limsup_{x\to 0} \frac{f(2x)}{2} \le \frac{m}{2},$$

and thus $m \le 0$. Let $\ell := \liminf_{x\to 0} f(x)$. It follows that $-\infty \le \ell \le m \le 0$. Let $\{x_n\} \subset U$ be a sequence such that, $x_n \to 0$ with $\ell = \lim_{n\to\infty} f(x_n)$. Then

$$\begin{aligned} 0 \le\ & \limsup_n [f(-x_n) + f(x_n)] \le \limsup_n f(-x_n) + \limsup_n f(x_n) \\ & \le m + \lim_n f(x_n) = m + l. \end{aligned}$$

Thus, $0 \ge m \ge \ell \ge -m \ge 0$. Hence $m = \ell = 0$. Therefore, f is continuous at $x = 0$. □

The proof of Theorem 4.3 is now obvious.

Theorem 6.4. *Let X be a linear metric space, and let E be a connected open subset of X. Let $f : E \to \mathbb{R}$ be a midpoint convex function on segments. If there is a point $x_0 \in E$ such that* $\limsup_{x\to x_0} f(x) < \infty$, *then f is a continuous and convex function on segments on E.*

Proof. Whenever $\limsup_{x\to x_0} f(x) < \infty$, then f is bounded from above on a neighborhood of the point x_0. Thus the set

$$W := \{x \in E : \limsup_{y\to x} f(y) < \infty\},$$

is a non-empty open subset of E on which, by Lemma 6.3, f is continuous. It is enough to show that ∂W, the boundary of W in E, is empty. Suppose the contrary. That is, assume that there is an $x_0 \in \partial W \cap E$. Let $d := \operatorname{dist}(x_0, X \setminus E)$ if $X \neq E$, and let $d := \min(1, (1/2)\operatorname{diam}(X))$ if $X = E$. Clearly, $d > 0$ because x_0 is an interior point of E. By Lemma 4.2, there is a balanced open set U (with respect to 0) such that

$$U \subset \left\{ x \in X : \rho(x, 0) < \frac{d}{2} \right\}.$$

It follows that

$$U + U = U - U \subset \{x \in X : \rho(x, 0) < d\},$$

and therefore

$$x_0 + U + U = x_0 + U - U \subset E. \tag{7}$$

Now $(x_0 + U)$ intersects W, because $x_0 \in \partial W$. Fix a point $a \in (x_0 + U) \cap W$ such that $a \neq x_0$. The intersection of the interval (a, x_0) (i.e. $\{\alpha a + \beta x_0 : \alpha + \beta = 1,\ \alpha, \beta > 0\}$) and $(x_0 + U) \cap W$ is relatively open in (a, x_0), and it contains a point $p \in (a, x_0) \cap (x_0 + U) \cap W$ such that $p - x_0 = t(a - x_0)$, where t is a binary fraction in $(0, 1)$, i.e. $t = m/2^n, m, n \in \mathbb{N}$. We fix such a t for the remainder of the proof. Consider the vector

$$v := a - p = (1 - t)(a - x_0),$$

and shift the neighborhood $x_0 + U$ by v. That is, consider $U + x_0 - v$, which is a balanced neighborhood of $x_0 - v$. Since U is balanced, $v \in U$, we obtain from (7)

$$U + x_0 - v \subset E. \tag{8}$$

Observe also that

$$a - x_0 \in U, \qquad p - x_0 = t(a - x_0) \in U,$$

because U is balanced. Hence,

$$p - x_0 = (a - x_0) - (1 - t)(a - x_0) = a - x_0 - v \in U - v.$$

Thus $p \in (U + x_0 - v) \cap W$. By Lemma 4.2, there is a small balanced neighborhood B of 0 with

$$p + B \subset (U + x_0 - v) \cap W$$

and such that f is bounded from above on $p + B$, say $f(p + x) \leq M < \infty$ for all $x \in B$. Consider now any $b \in B$. Since $p + b \in U + x_0 - v$, which is balanced with respect to $x_0 - v$, the whole closed line segment $[p + b, x_0 - v]$ is contained in $U + x_0 - v$, and so in E by (8). With the aid of a computation one can derive that

$$x_0 = (1 - t)p + t(x_0 - v).$$

Therefore

$$x_0 + (1-t)b = (1-t)(p+b) + t(x_0 - v).$$

Since t is a binary fraction, and f is midpoint convex on segments on E one can deduce that

$$\begin{aligned} f(x_0 + (1-t)b) &\leq (1-t)f(p+b) + tf(x_0 - v) \\ &\leq (1-t)M + tf(x_0 - v) =: M_1 \end{aligned}$$

Thus f is bounded above by M_1 on the set $x_0+(1-t)B = \{x_0+(1-t)b : b \in B\}$, which is a neighborhood of x_0 as $1-t \neq 0$. Thus $x_0 \in W$. Hence $x_0 \notin \partial W$, which is a contradiction. □

The following corollary is now obvious.

Corollary 6.5. *Let f be a midpoint convex function on segments on a connected open subset E of a linear metric space X. Then f is either everywhere convex by segments and continuous or* $\limsup_{x \to x_0} f(x) = \infty$ *for every point* $x_0 \in E$.

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