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The Split Common Fixed Point Problem and the Shrinking Projection Method in Banach Spaces

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We consider the split common fixed point problem in Banach spaces. Using the shrinking projection method, we prove a strong convergence theorem for finding a solution of the common fixed point problem in Banach spaces. Using this result, we get well-known and new results which are connected with the split feasibility problem and the split common null point problem in Banach spaces.

Keywords: Split common fixed point problem, fixed point, metric projection, metric resolvent, shrinking projection method, duality mapping.

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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Given mappings $T : H_1 \to H_1$ and $U : H_2 \to H_2$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common fixed point problem* is to find a point $z \in H_1$ such that $z \in F(T) \cap A^{-1}F(U)$, where $F(T)$ and $F(U)$ are fixed point sets of T and U, respectively. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [6] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Defining $T = P_D$ and $U = P_Q$, where P_D and P_Q are the metric projections of H_1 onto D and H_2 onto Q, respectively, we have that $z \in D \cap A^{-1}Q$ is equivalent to $z \in F(T) \cap A^{-1}F(U)$. Furthermore, given set-valued mappings $G : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common null point problem* [5] is to find a point $z \in H_1$ such that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $G^{-1}0$ and $B^{-1}0$ are null point sets of G and B, respectively. Defining $T = J_\lambda$ and $U = Q_\mu$, where J_λ and Q_μ

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are the resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively, we have that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$ is equivalent to $z \in F(T) \cap A^{-1}F(U)$. Thus, the split common fixed point problem generalizes the split feasibility problem and the split common null point problem. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where A^* is the adjoint operator of A. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. There are many results for the problems in Hilbert spaces; see, for instance, [7, 8, 11, 22]. However, we have not known such results outside Hilbert spaces. Recently, Takahashi [19] extended the result of (1) to Banach spaces. Furthermore, by using the ideas of [12, 13, 14], Takahashi [19, 20] obtained two results for the split feasibility problem and the split common null point problem in Banach spaces; see also [15].

In this paper, motivated by the split feasibility problem and the split common null point problem in Banach spaces, we first introduce a new nonlinear operator which generalizes strict pseudo-contractions in Hilbert spaces, the metric projections and the metric resolvents in Banach spaces. Then we consider the split common fixed point problem with the operators in Banach spaces. Using the shrinking projection method, we prove a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Using this result, we get well-known and new results which are connected with the split feasibility problem and the split common null point problem in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot \ \cdot \rangle$ and norm $\| \cdot \|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [18] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle; \tag{2}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{3}$$

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{4}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_Cx\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y\rangle \tag{5}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [16].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \;:\; \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{6}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [16] and [17]. We know the following result.

Lemma 2.1 ([16]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call the mapping P_C the metric projection of E onto C.

Lemma 2.2 ([16]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;

(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of of E into 2^{E^*}. The effective domain of A is denoted by dom(A), that is, dom$(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $x, y \in$ dom(A), $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [3]; see also [17, Theorem 3.5.4].

Theorem 2.3 ([3]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. In a Hilbert space, the metric resolvents are simply called the resolvents. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [17].

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n\,, \tag{7}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [10] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [10]. The following lemma was proved by Tsukada [24].

Lemma 2.4 ([24]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

3. Main Result

In this section, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [21], we first prove a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Before proving the theorem, we need a few definitions and lemmas. Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : E \to E$ with $F(U) \neq \emptyset$ is called η-demimetric if, for any $x \in E$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U.

Examples.

(1) Let H be a Hilbert space and let k be a real number with $0 \leq k < 1$. A mapping $U : H \to H$ is called a k-strict pseudo-contraction [4] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in H$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is k-demimetric. In fact, we have that, for $x \in H$ and $q \in F(U)$,

$$\|Ux - q\|^2 \leq \|x - q\|^2 + k\|x - Ux\|^2$$

and hence

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

Therefore, we have

$$\langle x - Ux, x - q\rangle \geq \frac{1-k}{2}\|x - Ux\|^2$$

for all $x \in H$ and $q \in F(U)$. This implies that U is k-demimetric.

(2) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric. In fact, since P_C is the metric projection of E onto C, we have that, for any $x \in E$ and $q \in C$,

$$\langle P_C x - q, J(x - P_C x)\rangle \geq 0.$$

Then we get

$$\langle P_C x - x + x - q, J(x - P_C x)\rangle \geq 0$$

and hence

$$\langle x - q, J(x - P_C x)\rangle \geq \|x - P_C x\|^2 = \frac{1-(-1)}{2}\|x - P_C x\|^2.$$

This means that P_C is (-1)-demimetric.

(3) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric. In fact, since J_λ is the metric resolvent of B for $\lambda > 0$, we have that, for any $x \in E$ and $q \in B^{-1}0$,

$$\langle J_\lambda x - q, J(x - J_\lambda x)\rangle \geq 0.$$

Then we get

$$\langle J_\lambda x - x + x - q, J(x - J_\lambda x)\rangle \geq 0$$

and hence

$$\langle x - q, J(x - J_\lambda x)\rangle \geq \|x - J_\lambda x\|^2 = \frac{1-(-1)}{2}\|x - J_\lambda x\|^2.$$

This means that J_λ is (-1)-demimetric.

The following lemma is important and crucial in the proof of our main result.

Lemma 3.1. *Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of E into itself. Then $F(U)$ is closed and convex.*

Proof. Let us show that $F(U)$ is closed. For a sequence $\{q_n\}$ such that $q_n \to q$ and $q_n \in F(U)$, we have from the definition of U that

$$\langle q - q_n, J(q - Uq)\rangle \geq \frac{1-\eta}{2}\|q - Uq\|^2.$$

From $q_n \to q$, we have $0 \geq \frac{1-\eta}{2}\|q - Uq\|^2$. From $1-\eta > 0$, we have $\|q - Uq\|^2 = 0$ and hence $q = Uq$. This implies that $F(U)$ is closed. Let us prove that $F(U)$ is convex. Let $p, q \in F(U)$ and set $x = \alpha p + (1-\alpha)q$, where $\alpha \in [0, 1]$. Then we have

$$\begin{aligned}
\|x - Ux\|^2 &= \langle x - Ux, J(x - Ux)\rangle = \langle \alpha p + (1-\alpha)q - Ux, J(x - Ux)\rangle \\
&= \langle \alpha p + (1-\alpha)q - (\alpha Ux + (1-\alpha)Ux), J(x - Ux)\rangle \\
&= \alpha\langle p - Ux, J(x - Ux)\rangle + (1-\alpha)\langle q - Ux, J(x - Ux)\rangle \\
&= \alpha\langle p - x + x - Ux, J(x - Ux)\rangle + (1-\alpha)\langle q - x + x - Ux, J(x - Ux)\rangle \\
&\leq \frac{\alpha(\eta-1)}{2}\|x - Ux\|^2 + \alpha\|x - Ux\|^2 + \\
&\qquad + \frac{(1-\alpha)(\eta-1)}{2}\|x - Ux\|^2 + (1-\alpha)\|x - Ux\|^2 \\
&= \frac{(\eta-1)}{2}\|x - Ux\|^2 + \|x - Ux\|^2,
\end{aligned}$$

and hence

$$0 \leq \frac{(\eta-1)}{2}\|x - Ux\|^2.$$

We have from $0 > \eta - 1$ that $\|x - Ux\| = 0$ and hence $x = Ux$. This means that $F(U)$ is convex. $\square$

Using the idea of Maingé [8], we have the following result.

Lemma 3.2. *Let H be a Hilbert space and let E be a smooth, strictly convex and reflexive Banach space. Let J_E be the duality mapping on E. Let η be a real number with $\eta \in (-\infty, 1)$ and let $U : E \to E$ be an η-demimetric mapping with $F(U) \neq \emptyset$. Let $A : H \to E$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Then, for any $x \in H$ and $q \in A^{-1}F(U)$,*

$$2\langle x - q, A^* J_E(Ax - UAx)\rangle \geq (1 - \eta)\|Ax - UAx\|^2.$$

Furthermore, define a mapping $L : H \to [0, \infty)$ as follows:

$$L(x) = \begin{cases} \frac{\|Ax - UAx\|^2}{\|A^* J_E(Ax - UAx)\|^2}, & \text{if } (I - U)Ax \neq 0, \\ \|AA^*\|^{-1}, & \text{otherwise}. \end{cases}$$

Then, for any $x \in H$ and $q \in A^{-1}F(U)$, and for any nonnegative number γ,

$$\|q - y\|^2 \leq \|q - x\|^2 - \gamma((1 - \eta)L(x) - \gamma)\|A^* J_E(Ax - UAx)\|^2,$$

where $y = x - \gamma A^* J_E(Ax - UAx)$. Moreover,

$$L(x) \geq \|AA^*\|^{-1}, \quad \forall x \in H.$$

Proof. Let $x \in H$ and $q \in A^{-1}F(U)$ be arbitrarily chosen. It follows from the definition of U that

$$\begin{aligned} 2\langle x - q, A^* J_E(Ax - UAx)\rangle &= 2\langle Ax - Aq, J_E(Ax - UAx)\rangle \\ &\geq (1 - \eta)\|Ax - UAx\|^2. \end{aligned} \tag{8}$$

In view of (8) we conclude that $(I - U)Ax \neq 0$ yields $A^* J_E(Ax - UAx) \neq 0$. Hence the mapping L given in Lemma 3.2 is well-defined. On the other hand, using (8) and the definition of L, we have the following two possible cases:

If $(I - U)Ax \neq 0$, then we get

$$L(x) = \frac{\|Ax - UAx\|^2}{\|A^* J_E(Ax - UAx)\|^2},$$

which, together with (8), implies that

$$\begin{aligned} (1 - \eta)L(x)\|A^* J_E(Ax - UAx)\|^2 &= (1 - \eta)\|Ax - UAx\|^2 \\ &\leq 2\langle x - q, A^* J_E(Ax - UAx)\rangle. \end{aligned}$$

If $(I - U)Ax = 0$, then we get

$$L(x) = \|AA^*\|^{-1} \text{ and } \|A^* J_E(Ax - UAx)\|^2 = 0,$$

which trivially implies that

$$(1 - \eta)L(x)\|A^* J_E(Ax - UAx)\|^2 = 2\langle x - q, A^* J_E(Ax - UAx)\rangle.$$

Therefore, we have

$$(1-\eta)L(x)\|A^*J_E(Ax-UAx)\|^2 \le 2\langle x-q, A^*J_E(Ax-UAx)\rangle, \quad \forall x \in H.$$

Furthermore, setting $y = x - \gamma A^*J_E(Ax-UAx)$, we obtain that

$$\begin{aligned}\|q-y\|^2 &= \|q-x\|^2 + \|x-y\|^2 + 2\langle q-x, x-y\rangle \\ &= \|q-x\|^2 + \gamma^2\|A^*J_E(Ax-UAx)\|^2 + \\ &\qquad + 2\langle q-x, \gamma A^*J_E(Ax-UAx)\rangle \\ &\le \|q-x\|^2 + \gamma^2\|A^*J_E(Ax-UAx)\|^2 - \\ &\qquad - \gamma(1-\eta)L(x)\|A^*J_E(Ax-UAx)\|^2 \\ &= \|q-x\|^2 - \gamma((1-\eta)L(x)-\gamma)\|A^*J_E(Ax-UAx)\|^2.\end{aligned}$$

From

$$\|A^*J_E(Ax-UAx)\|^2 = \langle AA^*J_E(Ax-UAx), J_E(Ax-UAx)\rangle,$$

we have that

$$\|A^*J_E(Ax-UAx)\|^2 \le \|AA^*\|\|J_E(Ax-UAx)\|^2.$$

If $(I-U)Ax \ne 0$, then

$$\|AA^*\|^{-1} \le \frac{\|J_E(Ax-UAx)\|^2}{\|A^*J_E(Ax-UAx)\|^2} = \frac{\|Ax-UAx\|^2}{\|A^*J_E(Ax-UAx)\|^2} = L(x).$$

If $(I-U)Ax = 0$, then we get $\|AA^*\|^{-1} = L(x)$ by the definition of $L(x)$. Therefore, we have that

$$L(x) \ge \|AA^*\|^{-1}, \quad \forall x \in H.$$

This completes the proof. □

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. A mapping $U : C \to E$ is called demiclosed if for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds.

Theorem 3.3. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : H \to H$ be a nonexpansive mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \ne \emptyset$. Let $A : H \to F$ be a bounded linear operator such that $A \ne 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \ne \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in H$ and $C_1 = H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T\Big(x_n - rA^*J_F(Ax_n - UAx_n)\Big), \\ C_{n+1} = \{z \in H : \|z_n - z\| \le \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}}u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|AA^*\| < (1-\eta)$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $z_0 \in F(T) \cap A^{-1}F(U)$, *where* $z_0 = P_{F(T)\cap A^{-1}F(U)}u$.

Proof. We first show that the sequence $\{x_n\}$ is well defined. Let $x_1 \in H$ and $C_1 = H$. Let $z_n = T\Big(x_n - rA^*J_F(Ax_n - UAx_n)\Big)$ with $0 < r < \frac{(1-\eta)}{\|AA^*\|}$. Define a mapping $L : H \to [0, \infty)$ as follows:

$$L(x) = \begin{cases} \frac{\|Ax-UAx\|^2}{\|A^*J_E(Ax-UAx)\|^2}, & \text{if } (I-U)Ax \neq 0, \\ \|AA^*\|^{-1}, & \text{otherwise.} \end{cases}$$

From Lemma 3.2, the definition of $L(x)$ is well defined. We have from Lemma 3.2 that for $z \in F(T) \cap A^{-1}F(U)$,

$$\begin{aligned} \|z_n - z\|^2 &= \|T\Big(x_n - rA^*J_F(Ax_n - UAx_n)\Big) - Tz\|^2 \\ &\leq \|x_n - rA^*J_F(Ax_n - UAx_n) - z\|^2 \\ &= \|x_n - z\|^2 - 2\langle x_n - z, rA^*J_F(Ax_n - UAx_n)\rangle + \\ &\qquad + \|rA^*J_F(Ax_n - UAx_n)\|^2 \\ &\leq \|x_n - z\|^2 - r(1-\eta)L(x_n)\|A^*J_F(Ax_n - UAx_n)\|^2 + \\ &\qquad + r^2\|A^*J_F(Ax_n - UAx_n)\|^2 \\ &= \|x_n - z\|^2 - r((1-\eta)L(x_n) - r)\|A^*J_F(Ax_n - UAx_n)\|^2 \\ &\leq \|x_n - z\|^2 - r((1-\eta)\|AA^*\|^{-1} - r)\|A^*J_F(Ax_n - UAx_n)\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned} \tag{9}$$

Therefore, $F(T) \cap A^{-1}F(U) \subset C_n$ for all $n \in \mathbb{N}$. Moreover, since

$$\begin{aligned} \{z \in H : \|z_n - z\| \leq \|x_n - z\|\} &= \{z \in H : \|z_n - z\|^2 \leq \|x_n - z\|^2\} \\ &= \{z \in H : \|z_n\|^2 - \|x_n\|^2 \leq 2\langle z_n - x_n, z\rangle\}, \end{aligned}$$

it is closed and convex. Applying these facts inductively, we obtain that C_n are nonempty, closed, and convex for all $n \in \mathbb{N}$, and hence $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^\infty C_n$. Since $C_0 \supset F(T) \cap A^{-1}F(U) \neq \emptyset$, C_0 is also nonempty. Let $w_n = P_{C_n}u$ for every $n \in \mathbb{N}$. Then, by Lemma 2.4, we have $w_n \to w = P_{C_0}u$. Since a metric projection on H is nonexpansive, it follows that

$$\begin{aligned} \|x_n - w\| &\leq \|x_n - w_n\| + \|w_n - w\| \\ &= \|P_{C_n}u_n - P_{C_n}u\| + \|w_n - w\| \\ &\leq \|u_n - u\| + \|w_n - w\| \end{aligned}$$

and hence $x_n \to w$. Since $w \in C_0 \subset C_{n+1}$, we have $\|z_n - w\| \leq \|x_n - w\|$ for all $n \in \mathbb{N}$. Tending $n \to \infty$, we get that $z_n \to w$. Then we have

$$\|x_n - z_n\| \leq \|x_n - w\| + \|w - z_n\| \to 0.$$

We also have from (9) that

$$\begin{aligned} r((1-\eta)\|AA^*\|^{-1}-r)\|A^*J_F(Ax_n-UAx_n)\|^2 &\le \|x_n-z\|^2-\|z_n-z\|^2 \\ &= (\|x_n-z\|-\|z_n-z\|)(\|x_n-z\|+\|z_n-z\|) \\ &\le \|x_n-z_n\|(\|x_n-z\|+\|z_n-z\|). \end{aligned}$$

Since $\|x_n-z_n\|\to 0$ and $0<r\|AA^*\|<(1-\eta)$, we have that

$$\|A^*J_F(Ax_n-UAx_n)\|\to 0.$$

Then we get from Lemma 3.2 that

$$\lim_{n\to\infty}\|Ax_n-UAx_n\|=0.$$

Since $x_n\to w$ and A is bounded and linear, we have that $\{Ax_n\}$ converges strongly to Aw and hence $\{Ax_n\}$ converges weakly to Aw. Since U is demi-closed, we have from $\lim_{n\to\infty}\|Ax_n-UAx_n\|=0$ that $Aw=UAw$. We also have that

$$\begin{aligned} \|x_n-Tx_n\| &= \|x_n-z_n+z_n-Tx_n\| \\ &= \|x_n-z_n+T\Big(x_n-rA^*J_F(Ax_n-UAx_n)\Big)-Tx_n\| \\ &\le \|x_n-z_n\|+\|x_n-rA^*J_F(Ax_n-UAx_n)-x_n\| \\ &= \|x_n-z_n\|+\|rA^*J_F(Ax_n-UAx_n)\|\to 0. \end{aligned}$$

Since $x_n\to w$ and T is nonexpansive, we have $w=Tw$. This implies that $w\in F(T)\cap A^{-1}F(U)$.

Since $F(U)$ is nonempty, closed and convex from Lemma 3.1, $F(T)\cap A^{-1}F(U)$ is also nonempty, closed and convex. Then there exists $z_0\in F(T)\cap A^{-1}F(U)$ such that $z_0=P_{F(T)\cap A^{-1}F(U)}u$. From $x_{n+1}=P_{C_{n+1}}u_{n+1}$, we have that

$$\|u_{n+1}-x_{n+1}\|\le\|u_{n+1}-y\|$$

for all $y\in C_{n+1}$. Since $z_0\in F(T)\cap A^{-1}F(U)\subset C_{n+1}$, we have that

$$\|u_{n+1}-x_{n+1}\|\le\|u_{n+1}-z_0\|. \tag{10}$$

From $z_0=P_{F(T)\cap A^{-1}F(U)}u$, $w\in F(T)\cap A^{-1}F(U)$ and (10), we have that

$$\begin{aligned} \|u-z_0\|\le\|u-w\| &= \lim_{n\to\infty}\|u_{n+1}-x_{n+1}\| \\ &\le \lim_{n\to\infty}\|u_{n+1}-z_0\|=\|u-z_0\|. \end{aligned}$$

Then we get that $\|u-w\|=\|u-z_0\|$ and hence $z_0=w$. Therefore, we have $x_n\to w=z_0$. This completes the proof. □

4. Applications

In this section, using Theorem 3.3, we get well-known and new strong convergence theorems which are connected with the split common fixed point problems in Banach spaces. We know the following result obtained by Marino and Xu [9]; see also [23].

Lemma 4.1 ([9]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 4.2. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0, 1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T\Big(x_n - rA^*(Ax_n - UAx_n)\Big), \\ C_{n+1} = \{z \in H_1 : \|z_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|AA^*\| < (1 - k)$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)} u$.

Proof. Since U be a k-strict pseudo-contraction of H_2 into itself such that $F(U) \neq \emptyset$, from (1) in Examples, U is k-demimetric. Furthermore, from Lemma 4.1, U is demiclosed. Therefore, we have the desired result from Theorem 3.3. □

Theorem 4.3. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in H$ and $C_1 = H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C\Big(x_n - rA^*J_F(Ax_n - P_DAx_n)\Big), \\ C_{n+1} = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|AA^*\| < 2$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D} u$.

Proof. Since P_C is the metric projection of H onto C, P_C is nonexpansive. Furthermore, since P_D is the metric projection of F onto D, from (2) in Examples, P_D is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in F such that $x_n \rightharpoonup p$ and $x_n - P_D x_n \to 0$, then $p = P_D p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - P_D x_n \to 0$. It is clear that $P_D x_n \rightharpoonup p$ and $\|J_F(x_n - P_D x_n)\| = \|x_n - P_D x_n\| \to 0$. Since P_D is the metric projection of F onto D, we have that

$$\langle P_D x_n - P_D p, J_F(x_n - P_D x_n) - J_F(p - P_D p)\rangle \geq 0.$$

Therefore, $-\|p - P_D p\|^2 = \langle p - P_D p, -J_F(p - P_D p)\rangle \geq 0$ and hence $p = P_D p$. Therefore, we have the desired result from Theorem 3.3. □

Theorem 4.4. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into H and F into F^*, respectively. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$, respectively. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in H$ and $C_1 = H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_\lambda\Big(x_n - rT^* J_F(Tx_n - Q_\mu Tx_n)\Big), \\ C_{n+1} = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\lambda > 0$, $\mu > 0$ and $0 < r\|AA^*\| < 2$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} u$.

Proof. Since J_λ is the resolvent of A on H, J_λ is nonexpansive. Furthemore, since Q_μ is the metric resolvent of B on F, from (3) in Examples, Q_μ is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in F such that $x_n \rightharpoonup p$ and $x_n - Q_\mu x_n \to 0$, then $p = Q_\mu p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - Q_\mu x_n \to 0$. It is clear that $Q_\mu x_n \rightharpoonup p$ and $\|J_F(x_n - Q_\mu x_n)\| = \|x_n - Q_\mu x_n\| \to 0$. Since Q_μ is the metric resolvent of B on D, we have from [2] that

$$\langle Q_\mu x_n - Q_\mu p, J_F(x_n - Q_\mu x_n) - J_F(p - Q_\mu p)\rangle \geq 0.$$

Therefore, $-\|p - Q_\mu p\|^2 = \langle p - Q_\mu p, -J_F(p - Q_\mu p)\rangle \geq 0$ and hence $p = Q_\mu p$. Therefore, we have the desired result from Theorem 3.3. □

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