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Variational Analysis for the Bilateral Minimal Time Function

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In this paper, we derive formulas for the Fréchet (singular) subdiferentials of the bilateral minimal time function $T : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ associated with a system governed by differential inclusions. As a consequence, we give a connection between the Fréchet normals to the sub-level sets of T and to its epigraph. Finally, we show that the Fréchet normal cones to the sub-level set of T at a point (α, β) and to epi(T) at $((\alpha, \beta), T(\alpha, \beta))$ have the same dimension.

Keywords: Bilateral minimal time function, Fréchet subdifferential, singular subdifferential, normal cone

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1. Introduction

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction. We consider the system governed by the differential inclusion associated with F:

$$\begin{cases} \dot{x}(t) \in F(x(t)), & \text{a.e. } t > 0. \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

A trajectory starting at x_0 of F is a solution of the differential inclusion (1), i.e., an absolutely continuous function $x : [0, +\infty) \to \mathbb{R}^n$ satisfying $\dot{x}(t) \in F(x(t))$ for a.e. $t > 0$ and $x(0) = x_0$.

The bilateral minimal time function $T : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ associated with (1) is defined as follows: for each pair $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$, $T(\alpha, \beta)$ is the minimal time taken by the trajectories of F starting at the point α to reach the point β. If no trajectory starting at α can reach β, then $T(\alpha, \beta) = +\infty$. When we fix the final point β, we get the function $T(\cdot, \beta)$ - the well known unilateral minimal time function associated to the target $\{\beta\}$. This function is a classical and widely

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studied topic in control theory (see, e.g. [1, 2, 3, 6, 7, 8, 9, 10, 11, 16, 17, 19, 20] and references mentioned therein).

The bilateral minimal time function T was introduced in [5] by Clarke and Nour to study the Hamilton-Jacobi equation of optimal time problems in a domain containing the target. In that paper, using the function T, the authors constructed proximal solutions of the relevant equation and studied the existence of time-geodesic trajectories. After that, the bilateral minimal time function and its regularity properties were studied deeply in [5, 12, 13, 14, 15]. In these papers, the authors generalized known results for the unilateral minimal time function to the bilateral case.

Inspired by [17] where a relationship between the *proximal normal cones* to sub-level sets of the unilateral minimal time function and its epigraph is given, in the present paper, we give a similar relationship between the *Fréchet normal cones* to sub-level sets and the epigraph of the bilateral minimal time function. Our main result is presented in Theorem 3.6. By using this result, we prove a special feature of the minimal time function - evidently not true for a general, even convex, function – that is: for $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$ with $0 < T(\alpha, \beta) < +\infty$, the Fréchet normal cone to the epigraph of T at $((\alpha, \beta), T(\alpha, \beta))$ has the same (algebraic) dimension of the Fréchet normal cone to the sub-level set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : T(x, y) \leq T(\alpha, \beta)\}$ at the point (α, β).

It is worth mentioning that the proof of Theorem 3.6 relies heavily on the representations of the Fréchet (singular) subdifferentials of T (see Theorem 3.3 and Theorem 3.5) and the following interesting property of normal vectors to the sub-level sets of T: for $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$ with $0 < T(\alpha, \beta) < +\infty$, if (ζ, θ) belongs to the Fréchet normal cone of the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : T(x, y) \leq T(\alpha, \beta)\}$ at (α, β), then

$$h(\alpha, \zeta) = h(\beta, -\theta),$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the Hamintonian associated to F, is defined by

$$h(x, p) = \min_{v \in F(x)} \langle v, p \rangle, \qquad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

The paper is organized as follows. In Section 2, we give some notions, definitions and preliminaries which will be used in the sequel. Section 3 is devoted to variational analysis for the bilateral minimal time function.

2. Preliminaries

2.1. Notations and basic facts

In this section we recall some basic concepts of nonsmooth analysis. Standard references are in [4, 18].

We denote by $\| \cdot \|$ the Euclidean norm in $\mathbb{R}^n$, by $\langle \cdot, \cdot \rangle$ the inner product. We also denote by $B(x, r)$ the open ball of radius $r > 0$ centered at x, and

$\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. We will use the shortened $B = B(0,1)$. For any subset E of $\mathbb{R}^n$, we denote by $\operatorname{bdry} E$ its boundary, by $\bar{E}$ its closure and by $\mathrm{Proj_E(x)}$ the projection of $x \in \mathbb{R}^n$ on E. A subset C of $\mathbb{R}^n$ is called a cone if and only if $\lambda x \in C$ for any $x \in C$ and $\lambda \geq 0$. We say that $\kappa \in \mathbb{N}$ is the dimension of a cone C if there exist $v_1, \cdots, v_\kappa \in C$ such that they are linearly independent and for any $v \in C$ there exist nonnegative numbers $\lambda_1, \cdots, \lambda_\kappa$ such that $v = \lambda_1 v_1 + \cdots + \lambda_\kappa v_\kappa$.

Let $S \subset \mathbb{R}^n$ be a closed set and let $x \in S$. The *Fréchet normal cone* to S at x, written $\widehat{N}_S(x)$, is the set

$$\widehat{N}_S(x) := \left\{ \zeta \in \mathbb{R}^n : \limsup_{S \ni y \to x} \frac{\langle \zeta, y - x \rangle}{\|y - x\|} \leq 0 \right\}.$$

Elements in $\widehat{N}_S(x)$ are called *Fréchet normals* to S at x.

In other words, $\zeta \in \widehat{N}_S(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x \rangle \leq \varepsilon \|y - x\|, \quad \forall y \in B(x, \delta).$$

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The effective domain of f is the set $\operatorname{dom}(f) := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and the epigraph of f is the set $\operatorname{epi}(f) := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{dom}(f), \alpha \geq f(x)\}$. We say that f is *lower semicontinuous* at $x_0 \in \mathbb{R}^n$ if for every $\varepsilon > 0$, there exists a neighborhood V of x_0 such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in V$ when $f(x_0) < +\infty$ and $f(x)$ tends to $+\infty$ as x tends to x_0 when $f(x_0) = +\infty$. Equivalently,

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

We say that f is lower semicontinuous if f is lower semicontinuous at every $x_0 \in \mathbb{R}^n$. Observe that if f is lower semicontinuous then its sub-level sets are closed.

Let $x \in \operatorname{dom}(f)$. The *Fréchet subdifferential* of f at x is the set

$$\widehat{\partial} f(x) := \left\{ \zeta \in \mathbb{R}^n : \liminf_{y \to x} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

In other words, $\zeta \in \widehat{\partial} f(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x \rangle \leq f(y) - f(x) + \varepsilon \|y - x\|, \quad \forall y \in B(x, \delta).$$

The Fréchet subdifferential of f at x can also be defined as follows:

$$\widehat{\partial} f(x) = \left\{ \zeta \in \mathbb{R}^n : (\zeta, -1) \in \widehat{N}_{\operatorname{epi}(f)}(x, f(x)) \right\}.$$

Elements in $\widehat{\partial} f(x)$ are called *Fréchet subgradients* of f at x.
The *Fréchet singular subdifferential* of f at x is the set

$$\widehat{\partial}^{\infty} f(x) := \left\{ \zeta \in \mathbb{R}^n : (\zeta, 0) \in \widehat{N}_{\mathrm{epi}(f)}(x, f(x)) \right\}.$$

In other words, $\zeta \in \widehat{\partial}^{\infty} f(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x \rangle \leq \varepsilon(\|y - x\| + |\beta - f(x)|), \quad \forall y \in B(x, \delta), \, (y, \beta) \in \mathrm{epi}(f).$$

Elements in $\widehat{\partial}^{\infty} f(x)$ are called *Fréchet singular subgradients* of f at x.

2.2. The bilateral minimum time function

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction. In this paper, we require the following assumptions on the multifunction F.

(F1) $F(x)$ is a nonempty compact convex set for all $x \in \mathbb{R}^n$.

(F2) F is locally Lipschitz, i.e., for any compact set K, there exists a constant $L := L(K)$ such that

$$F(x) \subset F(y) + L\|y - x\|\bar{B}, \quad \forall x, y \in K.$$

(F3) There exist some positive constants γ and c such that for all $x \in \mathbb{R}^n$,

$$v \in F(x) \Rightarrow \|v\| \leq \gamma\|x\| + c.$$

For some $\tau > 0$, we consider the differential inclusion

$$\begin{cases} \dot{x}(t) \in F(x(t)), & \text{a.e. } t \in [0, \tau]. \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2}$$

A solution of (2) is an absolutely continuous function $x(\cdot)$ defined on $[0, \tau]$ with the initial condition $x(0) = x_0$. We call $x(\cdot)$ a trajectory of F starting at x_0.

Notice that, under our assumptions on F, if $x(\cdot)$ is a trajectory of F defined on $[0, \tau]$ then by Gronwall's Lemma, there exists a constant $M > 0$ such that $\|x(t) - x_0\| \leq Mt$ for all $t \in [0, \tau]$. In this paper, for simplicity, we fix the constant M for all $\tau > 0$ and for all trajectories. The following theorem gives some information regarding C^1 trajectories of F which will be useful in the sequel.

Theorem 2.1 ([20]). *Assume* (F1)–(F3). *Let $E \subset \mathbb{R}^n$ be compact. Then there exists $\tau > 0$ such that associated to every $x \in E$ and $v \in F(x)$ is a trajectory $x(\cdot)$ defined on $[0, \tau]$ with $\dot{x}(0) = v$. Moreover, for all $t \in [0, \tau]$, we have $\|\dot{x}(t) - v\| \leq Kt$, for some constant $K > 0$ independent of x*

The bilateral minimal time function $T : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ is defined as follows: for $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$T(\alpha, \beta) = \inf\{T \geq 0 : \text{there exists some trajectory } x(\cdot) \text{ of } F \\ \text{with } x(0) = \alpha \text{ and } x(T) = \beta\}. \tag{3}$$

If there is no trajectory steering α to β, then $T(\alpha, \beta) = +\infty$. It may happen that, for $\alpha, \beta \in \mathbb{R}^n$, $T(\alpha, \beta) < +\infty$ and $T(\beta, \alpha) = +\infty$. Obviously, $T(\alpha, \alpha) = 0$, for all $\alpha \in \mathbb{R}^n$.
We have following properties of the bilateral minimal time function T (see [12]):

- T is lower semicontinuous.
- If $T(\alpha, \beta) < +\infty$, then the infimum in (3) is attained.
- For all $\alpha, \beta, \gamma \in \mathbb{R}^n$, we have the following triangle inequality

$$T(\alpha, \beta) \leq T(\alpha, \gamma) + T(\gamma, \beta).$$

For $t > 0$, the set $\mathcal{R}(t) := \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha, \beta) \leq t\}$ is called the reachable set at time t and the set

$$\mathcal{R} := \bigcup_{t \geq 0} \mathcal{R}(t) = \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha, \beta) < +\infty\},$$

is called the reachable set.

3. Variational analysis for the bilateral minimal time function

The following theorem presents a formula for the Fréchet sudifferential of the bilateral minimal time function at a point $(\alpha, \alpha) \in \mathbb{R}^n \times \mathbb{R}^n$. The formula is similar to the one for the proximal subdifferential given in [12] (see Theorem 4.10 (1) in [12]).

Theorem 3.1. *We have*

$$\widehat{\partial} T(\alpha, \alpha) = \{(\zeta, -\zeta) \in \mathbb{R}^n \times \mathbb{R}^n : h(\alpha, \zeta) \geq -1\}, \tag{4}$$

for any $\alpha \in \mathbb{R}^n$.

Proof. Let $\alpha \in \mathbb{R}^n$ and $(\zeta, \theta) \in \widehat{\partial} T(\alpha, \alpha)$. Then for any $\varepsilon > 0$, there exists $\sigma > 0$ such that

$$\langle (\zeta, \theta), (x, y) - (\alpha, \alpha) \rangle - T(x, y) \leq \varepsilon \|(x, y) - (\alpha, \alpha)\|, \tag{5}$$

for all $(x, y) \in B((\alpha, \alpha), \sigma)$.

Taking $x = y$ in (5), we have, for all $x \in B(\alpha, \sigma/2)$, that

$$\langle (\zeta, \theta), (x - \alpha, x - \alpha) \rangle \leq \varepsilon \|(x - \alpha, x - \alpha)\|, \tag{6}$$

Let now $w \in \mathbb{R}^n$ and set $x_n := \alpha + w/n$ for $n \in \mathbb{N}^*$. Then there is some $n_0 > 0$ such that for all $n \geq n_0$ we have $x_n \in B(\alpha, \sigma/2)$. Thus, in (6), taking $x = x_n$ with $n \geq n_0$, one get

$$\langle (\zeta, \theta), (w, w) \rangle \leq \varepsilon \|(w, w)\|.$$

Letting $\varepsilon \to 0+$ in the latter inequality, we obtain $\langle (\zeta, \theta), (w, w) \rangle \leq 0$ for all $w \in \mathbb{R}^n$. This implies $\zeta = -\theta$.

Let $v \in F(\alpha)$ be such that $\langle v, \zeta \rangle = h(\alpha, \zeta)$. By Theorem 2.1, there is a C^1 trajectory $x(\cdot)$ of $-F$ such that $x(0) = \alpha$ and $\dot{x}(0) = -v$. There exists $\delta > 0$ such that $x(t) \in B(\alpha, \sigma/2)$ for all $t \in [0, \delta]$. Observe that $T(x(t), \alpha) \leq t$ for all $t \in [0, \delta]$. For $t \in [0, \delta]$, taking $x = x(t), y = \alpha$ in (5), we have

$$\langle (\zeta, \theta), (x(t), \alpha) - (\alpha, \alpha) \rangle - t \leq \varepsilon \|(x(t), \alpha) - (\alpha, \alpha)\|,$$

and then

$$\langle \zeta, z(t) - \alpha \rangle \leq t + \varepsilon M t.$$

Dividing both sides of the latter inequality by $t > 0$ then letting $t \to 0+$, we get

$$\langle \zeta, \dot{z}(0) \rangle \leq 1 + \varepsilon M.$$

Since $\varepsilon > 0$ is arbitrary, we have $\langle \zeta, v \rangle = \langle \zeta, -\dot{z}(0) \rangle \geq -1$. That is $h(\alpha, \zeta) \geq -1$.

Now let $(\alpha, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that $h(\alpha, \zeta) \geq -1$. We want to show that $(\zeta, -\zeta) \in \widehat{\partial} T(\alpha, \alpha)$. Assume to the contrary that $(\zeta, -\zeta) \notin \widehat{\partial} T(\alpha, \alpha)$. Then there exist a consant $C > 0$ and a sequence $\{(\alpha_n, \beta_n)\}$ such that $(\alpha_n, \beta_n) \to (\alpha, \alpha)$, $(\alpha_n, \beta_n) \neq (\alpha, \alpha)$ and

$$\langle (\zeta, -\zeta), (\alpha_n - \alpha, \beta_n - \alpha) \rangle - T(\alpha_n, \beta_n) > C \|(\alpha_n - \alpha, \beta_n - \alpha)\|, \quad \forall n. \tag{7}$$

It follows from (7) that for all n

$$0 < T_n := T(\alpha_n, \beta_n) \leq 2\|\zeta\|.\|(\alpha_n - \alpha, \beta_n - \alpha)\| < +\infty. \tag{8}$$

Thus, for each n, there exists a trajectory $x_n(\cdot)$ of F such that $x_n(0) = \alpha_n$ and $x_n(T_n) = \beta_n$.

We have, for all n and all $t \in [0, T_n]$, that

$$\|x_n(t) - \alpha\| \leq \|x_n(t) - \alpha_n\| + \|\alpha_n - \alpha\| \leq M T_n + \|\alpha_n - \alpha\|,$$

and then

$$\|x_n(t) - \alpha\| \leq M T_n + \|(\alpha_n - \alpha, \beta_n - \alpha)\|.$$

Let $y_n(t) := \mathrm{Proj}_{F(\alpha)}(\dot{x}_n(t))$ on $[0, T_n]$. By the Lipschitz continuity of F, we have, for all n,

$$\|y_n(t) - \dot{x}_n(t)\| \leq L \|x_n(t) - \alpha\|, \quad \forall t \in [0, T_n].$$

Moreover, since $h(\alpha,\zeta) \geq -1$, we have $\langle \zeta, y_n(t)\rangle \geq -1$ for all n and for all $t \in [0,T_n]$. Then using (8),

$$\begin{aligned}
&\langle (\zeta,-\zeta),(\alpha_n-\alpha,\beta_n-\alpha)\rangle - T(\alpha_n,\beta_n)\\
&= \langle \zeta, \alpha_n-\beta_n\rangle - T_n\\
&\leq \left\langle \zeta, \int_0^{T_n} \dot{x}_n(t)dt \right\rangle - \int_0^{T_n} \langle \zeta, y_n(t)\rangle dt\\
&\leq L\|\zeta\| \int_0^{T_n} \|y_n(t)-\dot{x}_n(t)\|dt\\
&\leq L\|\zeta\|(MT_n^2 + \|(\alpha_n-\alpha,\beta_n-\alpha)\|T_n)\\
&\leq 2L\|\zeta\|^2(M\|\zeta\|+1)\|(\alpha_n-\alpha,\beta_n-\alpha)\|^2.
\end{aligned}$$

Combining with (7) we have, for all n, that

$$C\|(\alpha_n-\alpha,\beta_n-\alpha)\| < 2L\|\zeta\|^2(M\|\zeta\|+1)\|(\alpha_n-\alpha,\beta_n-\alpha)\|^2.$$

Dividing both sides of the latter inequality by $\|(\alpha_n-\alpha,\beta_n-\alpha)\| > 0$ then letting $n\to\infty$, we obtain $C \leq 0$. This is a contradiction. Therefore $(\zeta,-\zeta) \in \widehat{\partial} T(\alpha,\alpha)$. □

We can also derive a formula for the Fréchet subdifferential of the bilateral minimal time function at a point $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\alpha \neq \beta$. Again, the formula is similar to the one for the proximal subdifferential given in [12] (see Theorem 4.10 (2) in [12]). Before stating the result, in the next proposition, we present a characterisation of a Fréchet normals to sub-level sets of the bilateral minimal time function which will be useful in the sequel

Proposition 3.2. *Let $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that $0 < r := T(\alpha,\beta) < \infty$. If $(\zeta,\theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ then $h(\alpha,\zeta) = h(\beta,-\theta) \leq 0$.*

Proof. Since $(\zeta,\theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$, for any $\varepsilon > 0$, there exists $\sigma > 0$ such that, for all $(x,y) \in \mathcal{R}(r) \cap B((\alpha,\beta),\sigma)$, one has

$$\langle (\zeta,\theta),(x,y)-(\alpha,\beta)\rangle \leq \varepsilon\|(x,y)-(\alpha,\beta)\|. \tag{9}$$

Let $x(\cdot)$ be a trajectory of F such that $x(0)=\alpha$ and $x(r)=\beta$. Then for all $t \in [0,r]$ sufficiently small, we have $(x(t),\beta) \in \mathcal{R}(r) \cap B((\alpha,\beta),\sigma)$. Hence, by (9), we have, for $t > 0$ sufficiently small, that

$$\langle (\zeta,\theta),(x(t),\beta)-(\alpha,\beta)\rangle \leq \varepsilon\|(x(t),\beta)-(\alpha,\beta)\|.$$

Thus,

$$\int_0^t \langle \zeta, \dot{x}(s)\rangle ds \leq \varepsilon\|x(t)-x(0)\| \leq \varepsilon M t. \tag{10}$$

Let $g(\cdot)$ be the projection of $\dot{x}(\cdot)$ on $F(\alpha)$ restricted to $[0, r]$. Then by the Lipschitz continuity of F,

$$\|\dot{x}(s) - g(s)\| \leq L\|x(s) - \alpha\| \leq LMs, \quad \text{for all } s \in [0, r]. \tag{11}$$

Using (10)–(11), we have, for $t \in [0, r]$ sufficiently small, that

$$\begin{aligned} h(\alpha, \zeta)t &\leq \int_0^t \langle \zeta, g(s) \rangle ds = \int_0^t \langle \zeta, \dot{x}(s) \rangle ds + \int_0^t \langle \zeta, g(s) - \dot{x}(s) \rangle ds \\ &\leq \varepsilon Mt + \|\zeta\| \int_0^t \|g(s) - \dot{x}(s)\| ds \leq \varepsilon Mt + LM\|\zeta\|t^2. \end{aligned} \tag{12}$$

Dividing (12) by $t > 0$ then letting $t \to 0+$, we get $h(\alpha, \zeta) \leq \varepsilon M$. Since $\varepsilon > 0$ is arbitrary, we conclude that $h(\alpha, \zeta) \leq 0$.

Let $p(\cdot)$ be the projection of $\dot{x}(\cdot)$ on $F(\beta)$ restricted to $[0, r]$. We have

$$\|\dot{x}(s) - p(s)\| \leq L\|x(s) - \beta\| \leq ML(r - s), \quad \forall s \in [0, r]. \tag{13}$$

Now let $w \in F(\alpha)$ be such that

$$\langle w, \zeta \rangle = h(\alpha, \zeta) = \min_{v \in F(\alpha)} \langle v, \zeta \rangle.$$

By Theorem 2.1, there exist $\tau > 0$ and a C^1 trajectory $z(\cdot)$ of $-F$ on $[0, \tau]$ with $z(0) = \alpha$ and $\dot{z}(0) = -w$. For $t \in [0, \tau]$, set $q(t) = z(\tau - t)$. Then $q(\cdot)$ is a trajectory of F with $q(0) = z(\tau)$ and $q(\tau) = \alpha$. By the principle of optimality, we have, for all $t \in [0, \tau]$, that

$$T(z(t), \alpha) = T(q(\tau - t), \alpha) = T(q(\tau - t), q(\tau)) \leq t.$$

Fixed $0 < t < \min\{r, \tau\}$. By the triangle inequality, we have $(z(t), x(r - t)) \in \mathcal{R}(r)$. We may choose $\tau > 0$ sufficiently small such that $(z(t), x(r - t)) \in \mathcal{R}(r) \cap B((\alpha, \beta), \sigma)$. It follows from (9) that

$$\langle (\zeta, \theta), (z(t), x(r - t)) - (\alpha, \beta) \rangle \leq \varepsilon \|(z(t), x(r - t)) - (\alpha, \beta)\|.$$

and then

$$\langle \zeta, z(t) - z(0) \rangle + \int_{r-t}^r \langle -\theta, \dot{x}(s) \rangle ds \leq 2\varepsilon Mt. \tag{14}$$

From (13) and (14), for $0 < t < \min\{r, \tau\}$, one has

$$\begin{aligned} &\langle \zeta, z(t) - z(0) \rangle + h(\beta, -\theta)t \\ &\leq \langle \zeta, z(t) - z(0) \rangle + \int_{r-t}^r \langle -\theta, \dot{x}(s) \rangle ds + \int_{r-t}^r \langle -\theta, p(s) - \dot{x}(s) \rangle ds \\ &\leq 2\varepsilon Mt + \|\theta\| \int_{r-t}^r \|p(s) - \dot{x}(s)\| ds \leq 2\varepsilon Mt + ML\|\theta\|t^2. \end{aligned} \tag{15}$$

Dividing (15) by $t > 0$ then letting $t \to 0+$, we obtain

$$-h(\alpha,\zeta) + h(\beta,-\theta) = \langle \zeta, -w\rangle + h(\beta,-\theta) \le 2\varepsilon M.$$

Letting $\varepsilon \to 0+$ in the latter inequality, we get $-h(\alpha,\zeta) + h(\beta,-\theta) \le 0$, i.e., $h(\beta,-\theta) \le h(\alpha,\zeta)$.

Similarly, one can show that $h(\alpha,\zeta) \le h(\beta,-\theta)$. Thus

$$h(\alpha,\zeta) = h(\beta,-\theta) \le 0.$$

The proof is complete. □

Theorem 3.3. *Let $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that $0 < r := T(\alpha,\beta) < \infty$. One has*

$$\widehat{\partial} T(\alpha,\beta) = \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta) \cap \{(\zeta,\theta) \in \mathbb{R}^n \times \mathbb{R}^n : h(\alpha,\zeta) = h(\beta,-\theta) = -1\}.$$

Proof. Let $(\zeta,\theta) \in \widehat{\partial} T(\alpha,\beta)$. Then for any $\varepsilon > 0$ there exists $\sigma > 0$ such that

$$\langle (\zeta,\theta),(x,y)-(\alpha,\beta)\rangle \le T(x,y) - r + \varepsilon\|(x,y)-(\alpha,\beta)\|, \tag{16}$$

for all $(x,y) \in B((\alpha,\beta),\sigma)$.

It follows from (16) that

$$\langle (\zeta,\theta),(x,y)-(\alpha,\beta)\rangle \le \varepsilon\|(x,y)-(\alpha,\beta)\|, \quad \forall (x,y) \in \mathcal{R}(r) \cap B((\alpha,\beta),\sigma).$$

This means that $(\zeta,\theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$.

Let $z(\cdot)$ be as in the proof of Proposition 3.2. By the triangle inequality, we have

$$T(z(t),\beta) \le T(z(t),\alpha) + T(\alpha,\beta) \le t + r, \quad \forall t \in [0,\tau].$$

In (16), taking $x = z(t), y = \beta$, we obtain

$$\langle \zeta, z(t)-\alpha\rangle \le t + \varepsilon\|z(t)-\alpha\| \le t + \varepsilon M t.$$

Dividing the latter inequality by $t > 0$ then letting $t \to 0$, we have

$$-h(\alpha,\zeta) = \langle \zeta,-w\rangle = \langle \zeta, \dot{z}(0)\rangle \le 1 + M\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$h(\alpha,\zeta) \ge -1. \tag{17}$$

Let $x(\cdot)$ and $g(\cdot)$ be also as in the proof of Proposition 3.2. We have that $T(x(t),\beta) = r - t$ for all $t \in [0,r]$. In (16), taking $x = x(t), y = \beta$, one has

$$\begin{aligned}\int_0^t \langle \zeta, \dot{x}(t)\rangle &= \langle (\zeta,\beta),(x(t),\beta)-(\alpha,\beta)\rangle \\ &\le\ -t + \varepsilon\|(x(t),\beta)-(\alpha,\beta)\| \le -t + \varepsilon M t.\end{aligned}$$

Then

$$
\begin{aligned}
h(\alpha,\zeta)t \leq & \int_0^t \langle \zeta, g(s)\rangle ds = \int_0^t \langle \zeta, \dot{x}(s)\rangle ds + \int_0^t \langle \zeta, g(s)-\dot{x}(s)\rangle ds \\
\leq & -t + \varepsilon M t + \|\zeta\| \int_0^t \|g(s)-\dot{x}(s)\| ds \leq -t + \varepsilon M t + LM\|\zeta\| t^2.
\end{aligned}
$$

This implies that $h(\alpha,\zeta) \leq -1$. Together with (17) and Proposition 3.2 we get $h(\alpha,\zeta) = h(\beta,-\theta) = -1$.

Conversely, let $(\zeta,\theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ with $h(\alpha,\zeta) = h(\beta,-\theta) = -1$. We attempt to show that $(\zeta,\theta) \in \widehat{\partial} T(\alpha,\beta)$. Assume to the contrary that there exist a constant $C > 0$ and a sequence $\{(\alpha_n,\beta_n)\}$ such that $(\alpha_n,\beta_n) \to (\alpha,\beta)$ as $n \to \infty$ and $(\alpha_n,\beta_n) \neq (\alpha,\beta)$ and

$$
\langle (\zeta,\theta), (\alpha_n-\alpha, \beta_n-\beta)\rangle + r - T(\alpha_n,\beta_n) > C\|(\alpha_n-\alpha,\beta_n-\beta)\|, \quad \forall n. \quad (18)
$$

For each n, set $T_n := T(\alpha_n,\beta_n)$ and $\Lambda_n := \|(\alpha_n-\alpha,\beta_n-\beta)\|$. There are 3 possible cases.

Case 1. $T_n = r$ for infinitely many n. It follows from (18) that

$$
\langle (\zeta,\theta), (\alpha_n-\alpha,\beta_n-\beta)\rangle > C\|(\alpha_n-\alpha,\beta_n-\beta)\|.
$$

It is evident that this contradicts to $(\zeta,\theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$.

Case 2. $T_n > r$ for infinitely many n. It follows from (18) that

$$
0 < T_n - r < \|(\zeta,\theta)\|.\|(\alpha_n-\alpha,\beta_n-\beta)\| < +\infty. \quad (19)
$$

Set $d_n := (T_n - r)/2$. Let $x_n(\cdot)$ be a trajectory of F such that $x_n(0) = \alpha_n$ and $x_n(T_n) = \beta_n$. Set $u_n := x_n(d_n), w_n := x_n(T_n - d_n)$. Then $T(u_n,w_n) = r$, i.e., $(u_n,w_n) \in \mathcal{R}(r)$.

For $t \in [0,T_n]$, we have that

$$
\|x_n(t)-\alpha\| \leq \|x_n(t)-\alpha_n\| + \|\alpha_n-\alpha\| \leq Mt + \|\alpha_n-\alpha\|, \quad (20)
$$

and

$$
\|x_n(T_n-t)-\beta\| \leq \|x_n(T_n-t)-\beta_n\| + \|\beta_n-\beta\| \leq Mt + \|\beta_n-\beta\|. \quad (21)
$$

Let $p_n(\cdot)$ and $q_n(\cdot)$ be the projections of $\dot{x}_n(\cdot)$ on $F(\alpha)$ and $F(\beta)$ restricted on $[0,T_n]$, respectively. By Lipschitz continuity of F and (20), (21), one has, for all $t \in [0,T_n]$, that

$$
\|\dot{x}_n(t)-p_n(t)\| \leq L\|x_n(t)-\alpha\| \leq LMt + L\|\alpha_n-\alpha\|, \quad (22)
$$

and

$$
\|\dot{x}_n(T_n-t)-q_n(T_n-t)\| \leq L\|x_n(T_n-t)-\beta\| \leq LMt + L\|\beta_n-\beta\|. \quad (23)
$$

Using (19)–(21) and the facts $(u_n, w_n) \in \mathcal{R}(r), (\zeta, \theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha, \beta)$, for any $\varepsilon > 0$, for n sufficiently large, we have

$$\begin{aligned}
&\langle (\zeta, \theta), (u_n, w_n) - (\alpha, \beta) \rangle \leq \varepsilon \|(u_n - \alpha, w_n - \beta)\| \\
&\leq \varepsilon (M d_n + \|\alpha_n - \alpha\| + M d_n + \|\beta_n - \beta\|) \\
&= \varepsilon (2 M d_n + 2\|(\alpha_n - \alpha, \beta_n - \beta)\| \leq \varepsilon (M \|(\zeta, \theta)\| + 2) \Lambda_n.
\end{aligned} \tag{24}$$

Moreover, using (19), (22), (23) and the fact $h(\alpha, \zeta) = h(\beta, -\theta) = -1$, one has

$$\begin{aligned}
&\langle (\zeta, \theta), (\alpha_n, \beta_n) - (u_n, w_n) \rangle = \langle \zeta, \alpha_n - u_n \rangle + \langle \theta, \beta_n - w_n \rangle \\
&= \int_0^{d_n} \langle \zeta, -\dot{x}_n(t) \rangle dt + \int_{T_n - d_n}^{T_n} \langle \theta, \dot{x}_n(t) \rangle dt \\
&= \int_0^{d_n} \langle \zeta, -p_n(t) \rangle dt + \int_0^{d_n} \langle \zeta, p_n(t) - \dot{x}_n(t) \rangle dt \\
&\qquad + \int_{T_n - d_n}^{T_n} \langle \theta, q_n(t) \rangle dt + \int_{T_n - d_n}^{T_n} \langle \theta, \dot{x}_n(t) - q_n(t) \rangle dt \\
&\leq -h(\alpha, \zeta) d_n + \|\zeta\| \int_0^{d_n} \|p_n(t) - \dot{x}_n(t)\| dt \\
&\qquad - h(\beta, -\theta) d_n + \|\theta\| \int_{T_n - d_n}^{T_n} \|\dot{x}_n(t) - q_n(t)\| dt \\
&\leq 2 d_n + L \|\zeta\| (M d_n^2 + \|\alpha_n - \alpha\| d_n) + L \|\theta\| (M d_n^2 + \|\beta_n - \beta\| d_n) \\
&\leq 2 d_n + \kappa \Lambda_n^2,
\end{aligned} \tag{25}$$

for some constant $\kappa > 0$ independent of n.

From (18), (24) and (25), we have

$$\begin{aligned}
C \Lambda_n &< -T_n + r + \langle (\zeta, \theta), (\alpha_n - \alpha, \beta_n - \beta) \rangle \\
&= -2 d_n + \langle (\zeta, \theta), (u_n, w_n) - (\alpha, \beta) \rangle + \langle (\zeta, \theta), (\alpha_n, \beta_n) - (u_n, w_n) \rangle \\
&\leq \varepsilon (M \|(\zeta, \theta)\| + 2) \Lambda_n + \kappa \Lambda_n^2
\end{aligned} \tag{26}$$

Since $\Lambda_n > 0$ for all n, it follows from (26) that $C < \varepsilon (M \|(\zeta, \theta)\| + 1) + \kappa \Lambda_n$. Letting $n \to \infty$ and then letting $\varepsilon \to 0+$ in the latter inequality, we get $C \leq 0$. This is a contradiction.

Case 3. $T_n < r$ for infinitely many n. Set $h_n := (r - T_n)/2$. Let $v \in F(\alpha)$ and $w \in F(\beta)$ be such that $\langle v, \zeta \rangle = \langle w, -\theta \rangle = -1$. Let

$$v_n := \mathrm{Proj}_{F(\alpha_n)}(v) \quad \text{and} \quad w_n := \mathrm{Proj}_{F(\beta_n)}(w).$$

Then by the Lipschitz continuity of F, we have

$$\|v_n - v\| \leq L \|\alpha_n - \alpha\| \quad \text{and} \quad \|w_n - w\| \leq L \|\beta_n - \beta\|. \tag{27}$$

For each n, by Theorem 2.1, there exist a C^1 trajectory $\alpha_n(\cdot)$ of $-F$ and a C^1 trajectory $\beta_n(\cdot)$ of F such that

$$\alpha_n(0) = \alpha_n, \qquad \dot{\alpha_n}(0) = -v_n, \qquad \beta_n(0) = \beta_n \quad \text{and} \quad \dot{\beta}_n(0) = w_n,$$

and for some $K > 0$,

$$\|\dot{\alpha}_n(t) + v_n\| \le Kt, \qquad \|\dot{\beta}_n(t) - w_n\| \le Kt, \quad \forall t \in [0, h_n]. \tag{28}$$

Set $\gamma_n := \alpha_n(h_n)$ and $\lambda_n := \beta_n(h_n)$. Then by the triangle inequality,

$$T(\gamma_n, \lambda_n) \le T(\gamma_n, \alpha_n) + T(\alpha_n, \beta_n) + T(\beta_n, \lambda_n) \le h_n + T_n + h_n = r.$$

This means that $(\gamma_n, \lambda_n) \in \mathcal{R}(r)$. Then for any $\varepsilon > 0$, for n sufficiently large, we have that

$$\begin{aligned}
\langle (\zeta, \theta), (\gamma_n, \lambda_n) - (\alpha, \beta) \rangle &\le \varepsilon \|(\gamma_n, \lambda_n) - (\alpha, \beta)\| \\
&\le \varepsilon \left(\|(\gamma_n, \lambda_n) - (\alpha_n, \beta_n)\| + \|(\alpha_n, \beta_n) - (\alpha, \beta)\| \right) \\
&\le \varepsilon \left(\left\| \int_0^{h_n} \dot{\alpha}_n(t) dt \right\| + \left\| \int_0^{h_n} \dot{\beta}_n(t) dt \right\| + \Lambda_n \right) \\
&\le \varepsilon \left(2M h_n + \Lambda_n \right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\langle (\zeta, \theta), (\gamma_n, \lambda_n) - (\alpha_n, \beta_n) \rangle \\
&= \langle (\zeta, \theta), (\gamma_n, \lambda_n) - (\alpha, \beta) \rangle + \langle (\zeta, \theta), (\alpha, \beta) - (\alpha_n, \beta_n) \rangle \\
&\le \varepsilon \left(2M h_n + \Lambda_n \right) + \|(\zeta, \theta)\| . \|(\alpha_n, \beta_n) - (\alpha, \beta)\| \\
&= 2M\varepsilon h_n + \left(\varepsilon + \|(\zeta, \theta)\| \right) \Lambda_n.
\end{aligned} \tag{29}$$

We now have that

$$\begin{aligned}
& r - T_n \\
&= 2h_n = -\int_0^{h_n} \langle \zeta, v \rangle dt + \int_0^{h_n} \langle \theta, w \rangle dt \quad (\text{since } \langle \zeta, v \rangle = \langle -\theta, w \rangle = -1) \\
&= -\int_0^{h_n} \langle \zeta, v_n \rangle dt + \int_0^{h_n} \langle \zeta, v_n - v \rangle dt + \int_0^{h_n} \langle \theta, w_n \rangle dt + \int_0^{h_n} \langle \theta, w - w_n \rangle dt \\
&\le \int_0^{h_n} \langle \zeta, \dot{\alpha}_n(t) \rangle dt + \int_0^{h_n} \langle \zeta, -\dot{\alpha}_n(t) - v_n \rangle dt + L\|\zeta\| h_n \|\alpha_n - \alpha\| \\
&\qquad + \int_0^{h_n} \langle \theta, \dot{\beta}_n(t) \rangle dt + \int_0^{h_n} \langle \theta, w_n - \dot{\beta}_n(t) \rangle dt + L\|\theta\| h_n \|\beta_n - \beta\| \\
&\le \langle \zeta, \gamma_n - \alpha_n \rangle + K\|\zeta\| h_n^2 + \langle \theta, \lambda_n - \beta_n \rangle \\
&\qquad + K\|\theta\| h_n^2 + 2L\|(\zeta, \theta)\| . \|(\alpha_n - \alpha, \beta_n - \beta)\| h_n \\
&\le \langle (\zeta, \theta), (\gamma_n, \lambda_n) - (\alpha_n, \beta_n) \rangle + K\|(\zeta, \theta)\| h_n^2 + 2L\|(\zeta, \theta)\| \Lambda_n h_n \\
&\le 2M\varepsilon h_n + \left(\varepsilon + \|(\zeta, \theta)\| \right) \Lambda_n + K\|(\zeta, \theta)\| h_n^2 + 2L\|(\zeta, \theta)\| \Lambda_n h_n \\
&= \left[2M\varepsilon + K\|(\zeta, \theta)\| h_n + 2L\|(\zeta, \theta)\| \Lambda_n \right] h_n + (\varepsilon + \|(\zeta, \theta)\|) \Lambda_n.
\end{aligned} \tag{30}$$

Since $h_n \to 0$, $\Lambda_n \to 0$ as $n \to \infty$ and $\varepsilon > 0$ is arbitrary, we can choose $\varepsilon > 0$ small enough such that for n sufficiently large, $[2M\varepsilon + K\|(\zeta,\theta)\|h_n + 2L\|(\zeta,\theta)\|\Lambda_n] < 1$. Then there is some constant $Q > 0$ depending only on ζ, θ such that for n sufficiently large,

$$h_n \leq Q\Lambda_n. \tag{31}$$

Now

$$\begin{aligned}
&T_n - r - \langle(\zeta,\theta),(\alpha_n,\beta_n)-(\alpha,\beta)\rangle \\
= &-2h_n + \langle(\zeta,\theta),(\gamma_n,\lambda_n)-(\alpha_n,\beta_n)\rangle - \varepsilon\|(\gamma_n,\lambda_n)-(\alpha,\beta)\| \\
\geq &-2h_n + \langle\zeta,\gamma_n-\alpha_n\rangle + \langle\theta,\lambda_n-\beta_n\rangle - \varepsilon\|(\gamma_n,\lambda_n)-(\alpha,\beta)\| \\
= &-2h_n + \int_0^{h_n}\langle\zeta,\dot{\alpha_n}(t)\rangle dt + \int_0^{h_n}\langle\theta,\dot{\beta}_n(t)\rangle dt - \varepsilon(2Mh_n+\Lambda_n) \\
= &\int_0^{h_n}\langle\zeta,v-v_n\rangle dt + \int_0^{h_n}\langle\zeta,\dot{\alpha}_n(t)+v_n\rangle + \int_0^{h_n}\langle\theta,w_n-w\rangle dt \\
&\quad + \int_0^{h_n}\langle\theta,\dot{\beta}_n(t)-w_n\rangle dt - \varepsilon(2Mh_n+\Lambda_n) \\
\geq &-L\|\zeta\|.\|\alpha_n-\alpha\|h_n - K\|\zeta\|.\|\beta_n-\beta\|h_n^2 \\
&\quad - L\|\theta\|h_n - K\|\theta\|h_n^2 - \varepsilon(2Mh_n+\Lambda_n) \\
\geq &-2Q\|(\zeta,\theta)\|[L+KQ]\Lambda_n^2 - \varepsilon(2MQ+1)\Lambda_n.
\end{aligned} \tag{32}$$

Then, by (18) and (32),

$$C\Lambda_n < 2Q\|(\zeta,\theta)\|[L+KQ]\Lambda_n^2 + \varepsilon(2MQ+1)\Lambda_n.$$

Dividing both sides of the latter inequality by $\Lambda_n > 0$ then letting $n \to \infty$, we get

$$C \leq \varepsilon(2MQ+1).$$

Letting $\varepsilon \to 0+$, we obtain $C \leq 0$. This leads to a contradiction. The proof is complete. □

Singular subdifferentials are connected to the non-Lipschitzianity of a function. It is evident that if the Fréchet singular subdifferential of a function f at a point is nonempty, then f is not Lipschitz around that point. In the next two theorems, we derive formulas for the Fréchet singular subdifferentials of the bilateral minimal time function T. These formulas may be useful when we study the non-Lipschitz set of T. In this paper, the representations of the Fréchet singular subdifferentails, together with the representations of the Fréchet subdifferentials, are used to study the connection between the Fréchet normals to sub-level sets of T and to its epigraph.

Theorem 3.4. *Let $\alpha \in \mathbb{R}^n$. We have*

$$\widehat{\partial}^\infty T(\alpha,\alpha) = \{(\zeta,-\zeta) \in \mathbb{R}^n \times \mathbb{R}^n : h(\alpha,\zeta) \geq 0\}.$$

Proof. Let $(\zeta,\theta) \in \widehat{\partial}^\infty T(\alpha,\alpha)$. Then for any $\varepsilon > 0$, there exists $\sigma > 0$ such that

$$\langle (\zeta,\theta),(x,y)-(\alpha,\alpha)\rangle \le \varepsilon(\|(x,y)-(\alpha,\alpha)\|+\lambda), \tag{33}$$

for all $(x,y) \in B((\alpha,\alpha),\sigma)$ and $\lambda \ge T(x,y)$.

Let $v \in \mathbb{R}^n$. For each $n \in \mathbb{N}^*$, taking $x = y = \alpha + v/n$ and $\lambda = 0$ in (33), we have

$$\langle (\zeta,\theta),(v/n,v/n)\rangle \le \varepsilon\|(v/n,v/n)\|,$$

or, equivalently,

$$\langle (\zeta,\theta),(v,v)\rangle \le \varepsilon\|(v,v)\|.$$

Letting $\varepsilon \to 0+$ in the latter inequality, we get $\langle (\zeta,\theta),(v,v)\rangle \le 0$ for all $v \in \mathbb{R}^n$. This yields $\zeta = -\theta$.

Let $w \in F(\alpha)$ be such that $\langle w,\zeta\rangle = h(\alpha,\zeta)$. By Theorem 2.1, there is a C^1 trajectory $x(\cdot)$ of $-F$ such that $x(0) = \alpha$ and $\dot{x}(0) = -w$. For $t > 0$ sufficiently small, we have $x(t) \in B(\alpha,\sigma)$ and, of course, $T(x(t),\alpha) \le t$. Taking $x = x(t), y = \alpha$ and $\lambda = t$ in (33), we have that

$$\langle \zeta, x(t)-\alpha\rangle \le \varepsilon(\|x(t)-\alpha\|+t),$$

and then

$$\langle \zeta, x(t)-x(0)\rangle \le \varepsilon(Mt+t).$$

Dividing the latter inequality by $t > 0$ then letting $t \to 0+$ and keeping in mind that $\dot{x}(0) = -w$, one gets $\langle \zeta,-w\rangle \le \varepsilon(M+1)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $h(\alpha,\zeta) = \langle \zeta,w\rangle \ge 0$.

Now let $(\alpha,\zeta) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that $h(\alpha,\zeta) \ge 0$. Assume that $(\zeta,-\zeta) \notin \widehat{\partial}^\infty T(\alpha,\alpha)$, then there exist a constant $C > 0$, sequences $\{(\alpha_n,\beta_n)\} \subset \mathbb{R}^n \times \mathbb{R}^n, \{\lambda_n\} \subset \mathbb{R}$ such that $(\alpha_n,\beta_n) \to (\alpha,\alpha)$, $(\alpha_n,\beta_n) \ne (\alpha,\alpha)$, $\lambda_n \ge T(\alpha_n,\beta_n)$ and

$$\langle (\zeta,-\zeta),(\alpha_n,\beta_n)-(\alpha,\alpha)\rangle > C(\|(\alpha_n,\beta_n)-(\alpha,\alpha)\|+\lambda_n), \quad \forall n.$$

The latter implies that

$$\langle \zeta, \alpha_n-\beta_n\rangle > C(\|(\alpha_n,\beta_n)-(\alpha,\alpha)\|+T(\alpha_n,\beta_n)), \quad \forall n. \tag{34}$$

Set $T_n := T(\alpha_n,\beta_n)$. It follows from (34) that $T_n \le 2\|\zeta\|.\|\alpha_n-\beta_n\|/C < \infty$ for all n. Thus, for each n, there exists a trajectory $x_n(\cdot)$ of F such that $x_n(0) = \alpha_n, x_n(T_n) = \beta_n$. By Gronwall's Lemma, for all $t \in [0,T_n]$,

$$\|x_n(t)-\alpha\| \le \|x_n(t)-\alpha_n\| + \|\alpha_n-\alpha\| \le MT_n + \|(\alpha_n,\beta_n)-(\alpha,\alpha)\|.$$

Let $y_n(\cdot) := \mathrm{Proj}_{F(\alpha)}(\dot{x}_n(\cdot))$ on $[0,T_n]$. By the Lipschitz continuity of F,

$$\|y_n(t)-\dot{x}_n(t)\| \le L\|x_n(t)-\alpha\| \le LMT_n + L\|(\alpha_n,\beta_n)-(\alpha,\alpha)\|.$$

Now

$$\begin{aligned}
&C(\|(\alpha_n,\beta_n)-(\alpha,\alpha)\|+T_n)\\
\leq&\langle\zeta,\alpha_n-\beta_n\rangle\\
\leq&\int_0^{T_n}\langle\zeta,\dot{x}_n(t)\rangle dt-\int_0^{T_n}\langle\zeta,y_n(t)\rangle dt\quad(\text{since } h(\alpha,\zeta)\geq 0)\\
\leq&\|\zeta\|\int_0^{T_n}\|\dot{x}_n(t)-y_n(t)\|dt\\
\leq&L\|\zeta\|(MT_n^2+\|(\alpha_n,\beta_n)-(\alpha,\alpha)\|T_n)\\
\leq&C_1(T_n+\|(\alpha_n,\beta_n)-(\alpha,\alpha)\|)^2 \qquad (35)
\end{aligned}$$

for some constant $C_1>0$ depending only on ζ, M and L.

Since $T_n+\|(\alpha_n,\beta_n)-(\alpha,\alpha)\|>0$ for all n, it follows from (35) that $C\leq C_1(T_n+\|(\alpha_n,\beta_n)-(\alpha,\alpha)\|)$ for all n. Letting $n\to\infty$ in the latter inequality we get $C\leq 0$. This is a contradiction. Thus $(\zeta,-\zeta)\in\widehat{\partial}^\infty T(\alpha,\alpha)$. □

Theorem 3.5. *For $(\alpha,\beta)\in\mathcal{R}$ with $0<r:=T(\alpha,\beta)$, we have*

$$\widehat{\partial}^\infty T(\alpha,\beta)=\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)\cap\{(\zeta,\theta)\in\mathbb{R}^n\times\mathbb{R}^n: h(\alpha,\zeta)=h(\beta,-\theta)=0\}.$$

Proof. Let $(\zeta,\theta)\in\widehat{\partial}^\infty T(\alpha,\beta)$. Then for any $\varepsilon>0$, there exists $\eta>0$ such that

$$\langle(\zeta,\theta),(x,y)-(\alpha,\beta)\rangle\leq\varepsilon\left(\|(x,y)-(\alpha,\beta)\|+|\lambda-T(\alpha,\beta)|\right), \qquad (36)$$

for all $(x,y)\in B((\alpha,\beta),\eta)$ and $\lambda\geq T(x,y)$.
It deduces from (36) that

$$\langle(\zeta,\theta),(x,y)-(\alpha,\beta)\rangle\leq\varepsilon\|(x,y)-(\alpha,\beta)\|,$$

for all $(x,y)\in B((\alpha,\beta),\eta)\cap\mathcal{R}(r)$. This means that $(\zeta,\theta)\in\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$.

Let $z(\cdot)$ be as in the proof of Proposition 3.2. By the triangle inequality, we have

$$T(z(t),\beta)\leq T(z(t),\alpha)+T(\alpha,\beta)\leq t+r,\quad\forall t\in[0,T].$$

In (36), taking $x=z(t), y=\beta$ and $\lambda=t+r$ with $t>0$ sufficiently small, we obtain

$$\langle\zeta,z(t)-\alpha\rangle\leq\varepsilon(\|z(t)-\alpha\|+t)\leq\varepsilon(M+1)t.$$

Dividing the latter inequality by $t>0$ then letting $t\to 0$, we have

$$-h(\alpha,\zeta)=\langle\zeta,-w\rangle=\langle\zeta,\dot{z}(0)\rangle\leq\varepsilon(M+1).$$

Since $\varepsilon>0$ is arbitrary, we conclude that $h(\alpha,\zeta)\geq 0$. Combining with Proposition 3.2, we obtain $h(\alpha,\zeta)=h(\beta,-\theta)=0$.

Now let $(\zeta,\theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ with $h(\alpha,\zeta) = h(\beta,-\theta) = 0$. We will show that $(\zeta,\theta) \in \widehat{\partial}^\infty T(\alpha,\beta)$. Assume to the contrary that $(\zeta,\theta) \notin \widehat{\partial}^\infty T(\alpha,\beta)$, then there exist a constant $C > 0$ and sequences $\{(\alpha_n,\beta_n)\} \subset \mathbb{R}^n \times \mathbb{R}^n$, $\{\lambda_n\} \subset \mathbb{R}$ such that $(\alpha_n,\beta_n) \to (\alpha,\beta)$, $(\alpha_n,\beta_n) \neq (\alpha,\beta)$, $\lambda_n \geq T_n := T(\alpha_n,\beta_n)$ and for all n

$$\langle(\zeta,\theta),(\alpha_n,\beta_n)-(\alpha,\beta)\rangle > C\left(\|(\alpha_n,\beta_n)-(\alpha,\beta)\| + |\lambda_n - T(\alpha,\beta)|\right). \tag{37}$$

There are two cases.

Case 1. $T_n \leq r$ for infinitely many n. In this case $(\alpha_n,\beta_n) \in \mathcal{R}(r)$. Since $(\zeta,\theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$, for any $\varepsilon > 0$, there is a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\langle(\zeta,\theta),(\alpha_n,\beta_n)-(\alpha,\beta)\rangle \leq \varepsilon\|(\alpha_n,\beta_n)-(\alpha,\beta)\|.$$

Combining with (37) we have, for n sufficiently large, that

$$\begin{aligned} C\|(\alpha_n,\beta_n)-(\alpha,\beta)\| &\leq C\left(\|(\alpha_n,\beta_n)-(\alpha,\beta)\| + |\lambda_n - T(\alpha,\beta)|\right) \\ &< \varepsilon\|(\alpha_n,\beta_n)-(\alpha,\beta)\|. \end{aligned}$$

Since $\|(\alpha_n,\beta_n)-(\alpha,\beta)\| > 0$, it follows from the latter inequalities that $C < \varepsilon$. This is a contradiction since $\varepsilon > 0$ is arbitrary.

Case 2. $T_n > r$ for infinitely many n. Set $h_n := (T_n - r)/2$. Let $x_n(\cdot)$ be a trajectory of F such that $x_n(0) = \alpha_n$ and $x_n(T_n) = \beta_n$. Set $\gamma_n := x_n(h_n), w_n := x_n(h_n + r) = x_n(T_n - h_n)$. Then $T(\gamma_n, w_n) = r$ and therefore $(\gamma_n, w_n) \in \mathcal{R}(r)$. For $t \in [0, T_n]$, we have

$$\|x_n(t) - \alpha\| \leq \|x_n(t) - \alpha_n\| + \|\alpha_n - \alpha\| \leq Mt + \|\alpha_n - \alpha\|, \tag{38}$$

and

$$\|x_n(T_n - t) - \beta\| \leq \|x_n(T_n - t) - \beta_n\| + \|\beta_n - \beta\| \leq Mt + \|\beta_n - \beta\|. \tag{39}$$

Let $p_n(\cdot)$ and $q_n(\cdot)$ be the projections of $\dot{x}_n(\cdot)$ on $F(\alpha)$ and $F(\beta)$, respectively, restricted to $[0, T_n]$. From (38), (39) and by Lipschitz continuity of F we have

$$\|\dot{x}_n(t) - p_n(t)\| \leq L\|x_n(t) - \alpha\| \leq LMt + L\|\alpha_n - \alpha\|, \tag{40}$$

and

$$\|\dot{x}_n(T_n - t) - q_n(T_n - t)\| \leq L\|x_n(T_n - t) - \beta\| \leq LMt + L\|\beta_n - \beta\|, \tag{41}$$

for all $t \in [0, T_n]$.

Now

$$\begin{aligned} &\langle(\zeta,\theta),(\alpha_n,\beta_n)-(\alpha,\beta)\rangle \\ &= \langle(\zeta,\theta),(\alpha_n,\beta_n)-(\gamma_n,w_n)\rangle + \langle(\zeta,\theta),(\gamma_n,w_n)-(\alpha,\beta)\rangle. \end{aligned} \tag{42}$$

We first estimate the second term on the right-hand side of (42). Since $(\gamma_n, w_n) \in \mathcal{R}(r)$ and $(\zeta, \theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha, \beta)$, for any $\varepsilon > 0$ and for all n sufficiently large, one has

$$\langle (\zeta, \theta), (\gamma_n, w_n) - (\alpha, \beta) \rangle \leq \varepsilon \|(\gamma_n, w_n) - (\alpha, \beta)\|.$$

Using (38)–(39),

$$\begin{aligned}\langle (\zeta, \theta), (\gamma_n, w_n) - (\alpha, \beta) \rangle &\leq \varepsilon\{(Mh_n + \|\alpha_n - \alpha\|) + (Mh_n + \|\beta_n - \beta\|)\} \\ &\leq 2(M+1)\varepsilon(\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + 2h_n). \qquad (43)\end{aligned}$$

Using (40)–(41) and the fact $h(\alpha, \zeta) = h(\beta, -\theta) = 0$, we can estimate the first term on the right-hand side of (42) as follows

$$\begin{aligned}&\langle (\zeta, \theta), (\alpha_n, \beta_n) - (\gamma_n, w_n) \rangle \\ &= \langle \zeta, \alpha_n - \gamma_n \rangle + \langle \theta, \beta_n - w_n \rangle \\ &= -\int_0^{h_n} \langle \zeta, \dot{x}_n(s) \rangle ds + \int_{T_n - h_n}^{T_n} \langle \theta, \dot{x}_n(s) \rangle ds \\ &= -\int_0^{h_n} \langle \zeta, p_n(s) \rangle ds + \int_0^{h_n} \langle \zeta, p_n(s) - \dot{x}_n(s) \rangle ds \\ &\qquad + \int_{T_n - h_n}^{T_n} \langle \theta, q_n(s) \rangle ds + \int_{T_n - h_n}^{T_n} \langle \theta, \dot{x}_n(s) - q_n(s) \rangle ds \\ &\leq -h(\alpha, \zeta) h_n + \|\zeta\| \int_0^{h_n} \|p_n(s) - \dot{x}_n(s)\| ds \\ &\qquad - h(\beta, -\theta) h_n + \|\theta\| \int_{T_n - h_n}^{T_n} \|\dot{x}_n(s) - q_n(s)\| ds \\ &\leq \|\zeta\| (LMh_n^2 + Lh_n \|\alpha_n - \alpha\|) + \|\theta\| (LMh_n^2 + Lh_n \|\beta_n - \beta\|) \\ &\leq C_0 (\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + 2h_n)^2, \qquad (44)\end{aligned}$$

for some suitable constant $C_0 > 0$.
From (42)–(44), we have

$$\begin{aligned}&\langle (\zeta, \theta), (\alpha_n, \beta_n) - (\alpha, \beta) \rangle \\ &\leq 2(M+1)\varepsilon(\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + 2h_n) + C_0(\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + 2h_n)^2.\end{aligned}$$

Combining with (37), we have that

$$\begin{aligned}&C\left(\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + 2h_n\right) \\ &\leq C\left(\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + |\lambda_n - T(\alpha, \beta)|\right) \\ &< 2(M+1)\varepsilon(\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + 2h_n) + C_0(\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + 2h_n)^2.\end{aligned}$$

It follows that

$$C < 2(M+1)\varepsilon + C_0(\|(\alpha_n, \beta_n) - (\alpha, \beta)\| + 2h_n).$$

Letting $n \to \infty$ and then letting $\varepsilon \to 0+$ in the latter inequality, we get $C \leq 0$. This contradiction ends the proof. $\square$

The next theorem gives a connection between Fréchet normals to sub-level sets and to the epigraph of the bilateral minimal time function.

Theorem 3.6. *Let $(\alpha,\beta)\in\mathcal{R}$ be such that $0<r:=T(\alpha,\beta)$.*

(*i*) *If $(\zeta,\theta)\in\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ then $h(\alpha,\zeta)=h(\beta,-\theta)$ and $((\zeta,\theta),h(\alpha,\zeta))\in\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta),r)$.*

(*ii*) *If $(\alpha,\beta)\in\mathbb{R}^n\times\mathbb{R}^n$ and $\lambda\in\mathbb{R}$ satisfy $((\zeta,\theta),\lambda)\in\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta),r)$, then $\lambda\le 0$, $h(\alpha,\zeta)=h(\beta,-\theta)=\lambda$ and $(\zeta,\theta)\in\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$.*

Proof. (*i*) Since $(\zeta,\theta)\in\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$, it follows from Proposition 3.2 that $h(\alpha,\zeta)=h(\beta,-\theta)\le 0$. We have two cases.

Case 1. $h(\alpha,\zeta)=h(\beta,-\theta)=0$. Then by Theorem 3.5, $(\zeta,\theta)\in\widehat{\partial}^{\infty}T(\alpha,\beta)$. Equivalently, $((\zeta,\theta),0)\in\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta),r)$. Thus $((\zeta,\theta),h(\alpha,\zeta))\in\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta),r)$.

Case 2. $\lambda:=h(\alpha,\zeta)=h(\beta,-\theta)<0$. We set

$$\zeta_1=-\frac{\zeta}{\lambda},\quad\text{and}\quad\theta_1=-\frac{\theta}{\lambda}.$$

Then $(\zeta_1,\theta_1)\in\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ and $h(\alpha,\zeta_1)=h(\beta,-\theta_1)=-1$. Then by Theorem 3.3, we have $(\zeta_1,\theta_1)\in\widehat{\partial}T(\alpha,\beta)$. Equivalently, $((\zeta_1,\theta_1),-1)\in\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta),r)$. Therefore

$$((\zeta,\theta),h(\alpha,\zeta))=-\lambda((\zeta_1,\theta_1),-1)\in\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta),r).$$

(*ii*) By the nature of an epigraph, it follows from $((\zeta,\theta),\lambda)\in\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta),r)$ that $\lambda\le 0$. We also have two possible cases.

Case 1. $\lambda=0$. Then $(\zeta,\theta)\in\widehat{\partial}^{\infty}T(\alpha,\beta)$. By Theorem 3.5, we have

$$(\zeta,\theta)\in\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta),\quad\text{and}\quad h(\alpha,\zeta)=h(\beta,-\theta)=0=\lambda.$$

Case 2. $\lambda<0$. Set

$$\zeta_1=-\frac{\zeta}{\lambda},\quad\text{and}\quad\theta_1=-\frac{\theta}{\lambda}.$$

Then $((\zeta_1,\theta_1),-1)\in\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta),r)$. This implies $(\zeta_1,\theta_1)\in\widehat{\partial}T(\alpha,\beta)$. By Theorem 3.3,

$$(\zeta_1,\theta_1)\in\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta),\quad\text{and}\quad h(\alpha,\zeta_1)=h(\beta,-\theta_1)=-1.$$

Thus $(\zeta,\theta)=-\lambda(\zeta_1,\theta_1)\in\widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ and $h(\alpha,\zeta)=h(\beta,-\theta)=\lambda$. □

The result in Theorem 3.6 can be stated in the following way

Theorem 3.7. *Let $(\alpha,\beta) \in \mathcal{R}$ be such that $0 < r := T(\alpha,\beta)$. We have $(\zeta,\theta) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ if and only if*

$$((\zeta,\theta), h(\alpha,\zeta)) = ((\zeta,\theta), h(\beta,-\theta)) \in \widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta), r).$$

Using Theorem 3.7, one can easily prove the following Proposition

Proposition 3.8. *Let $(\alpha,\beta) \in \mathcal{R}$ with $0 < T(\alpha,\beta)$. One has*

$$\widehat{N}_{\mathcal{R}(T(\alpha,\beta))}(\alpha,\beta) = \{(0,0)\} \text{ if and only if } \widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta), T(\alpha,\beta)) = \{((0,0),0)\}.$$

The following result is a special feature of the bilateral minimal time function.

Theorem 3.9. *Let $(\alpha,\beta) \in \mathcal{R}$ with $\alpha \neq \beta$. We have*

$$\dim \widehat{N}_{\mathcal{R}(T(\alpha,\beta))}(\alpha,\beta) = \dim \widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta), T(\alpha,\beta)). \tag{45}$$

Proof. The argument is close to the one in [17] where an analogous result is proved for proximal normal cones in the context of the unilateral minimal time function. By Proposition 3.8, it is sufficient to consider the case when both $\widehat{N}_{\mathcal{R}(T(\alpha,\beta))}(\alpha,\beta)$ and $\widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta), T(\alpha,\beta))$ are nontrivial.

Set $r = T(\alpha,\beta)$. Let $\kappa = \dim \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ and $\ell = \dim \widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta), r)$. We first assume that $(\zeta_1,\theta_1),\cdots,(\zeta_\kappa,\theta_\kappa) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ are linearly independent. By Theorem 3.7, we have

$$((\zeta_1,\theta_1), h(\alpha,\zeta_1)),\cdots,((\zeta_\kappa,\theta_\kappa), h(\alpha,\zeta_\kappa)) \in \widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta), r).$$

Observe that $((\zeta_1,\theta_1), h(\alpha,\zeta_1)),\cdots,((\zeta_\kappa,\theta_\kappa), h(\alpha,\zeta_\kappa))$ are linearly independent. Thus $\kappa \leq \ell$.

Now assume that $((\zeta_1,\theta_1),\lambda_1),\cdots,(\zeta_\ell,\theta_\ell),\lambda_\ell) \in \widehat{N}_{\mathrm{epi}(T)}((\alpha,\beta), r)$ are linearly independent. It follows from Theorem 3.6 that $(\zeta_i,\theta_i) \in \widehat{N}_{\mathcal{R}(r)}(\alpha,\beta)$ and $h(\alpha,\zeta_i) = h(\beta,-\theta_i) = \lambda_i$ for all $i = 1,\cdots,\ell$. Observe that $(\zeta_i,\theta_i) \neq (0,0)$ for all $i = 1,\cdots,\ell$. Indeed, if $(\zeta_i,\theta_i) = (0,0)$ for some i, then $\lambda_i = 0$. This contradicts to the linear independence of $((\zeta_1,\theta_1),\lambda_1),\cdots,(\zeta_\ell,\theta_\ell),\lambda_\ell)$. We are going to show that $(\zeta_1,\theta_1),\cdots,(\zeta_\ell,\theta_\ell)$ are linearly independent. Consider

$$\sum_{i=1}^{\ell} a_i(\zeta_i,\theta_i) = 0, \tag{46}$$

for some $a_1,\cdots,a_\ell \in \mathbb{R}$. We claim that $a_1 = \cdots = a_\ell = 0$. Indeed, set

$$\mathcal{I} = \{i : a_i \geq 0, 1 \leq i \leq \ell\},$$

and

$$\mathcal{J} = \{i : a_i < 0, 1 \leq i \leq \ell\}.$$

Then $\mathcal{I} \cup \mathcal{J} = \{1, \cdots, \ell\}$.
Assume $\mathcal{I} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$. From (46), we have

$$\sum_{i \in \mathcal{I}} a_i(\zeta_i, \theta_i) = -\sum_{j \in \mathcal{J}} a_j(\zeta_j, \theta_j),$$

i.e.,

$$\sum_{i \in \mathcal{I}} a_i \zeta_i = -\sum_{j \in \mathcal{J}} a_j \zeta_j \quad \text{and} \quad \sum_{i \in \mathcal{I}} a_i \theta_i = -\sum_{j \in \mathcal{J}} a_j \theta_j \tag{47}$$

Since $((\zeta_i, \theta_i), \lambda_i) \in \widehat{N}_{\mathrm{epi}(T)}((\alpha, \beta), r)$, and $a_i \geq 0$ for all $i \in \mathcal{I}$, we have

$$\left(\left(\sum_{i \in \mathcal{I}} a_i \zeta_i, \sum_{i \in \mathcal{I}} a_i \theta_i \right), \sum_{i \in \mathcal{I}} a_i \lambda_i \right) = \sum_{i \in \mathcal{I}} a_i((\zeta_i, \theta_i), \lambda_i) \in \widehat{N}_{\mathrm{epi}(T)}((\alpha, \beta), r).$$

Then by Theorem 3.6,

$$h\left(\alpha, \sum_{i \in \mathcal{I}} a_i \zeta_i\right) = \sum_{i \in \mathcal{I}} a_i \lambda_i. \tag{48}$$

Similarly, since $((\zeta_j, \theta_j), \lambda_j) \in \widehat{N}_{\mathrm{epi}(T)}((\alpha, \beta), r)$, and $-a_j \geq 0$ for all $j \in \mathcal{J}$, we have

$$\begin{aligned} &\left(\left(-\sum_{j \in \mathcal{J}} a_j \zeta_j, -\sum_{j \in \mathcal{J}} a_j \theta_j \right), -\sum_{j \in \mathcal{J}} a_j \lambda_j \right) \\ = \; &-\sum_{j \in \mathcal{J}} a_j((\zeta_j, \theta_j), \lambda_j) \in \widehat{N}_{\mathrm{epi}(T)}((\alpha, \beta), r). \end{aligned}$$

Then

$$h\left(\alpha, -\sum_{j \in \mathcal{J}} a_j \zeta_j\right) = -\sum_{j \in \mathcal{J}} a_j \lambda_j. \tag{49}$$

It follows from (47)–(49) that

$$\sum_{i \in \mathcal{I}} a_i \lambda_i = -\sum_{j \in \mathcal{J}} a_j \lambda_j,$$

i.e.,

$$\sum_{i=1}^{\ell} a_i \lambda_i = 0. \tag{50}$$

We have from (46) and (50) that

$$\sum_{i=1}^{\ell} a_i((\zeta_i, \theta_i), \lambda_i) = ((0,0), 0).$$

Since $((\zeta_1, \theta_1), \lambda_1), \cdots, ((\zeta_\ell, \theta_\ell), \lambda_\ell)$ are linearly independent, the latter equality implies that $a_1 = \cdots = a_\ell = 0$. This contradicts to $\mathcal{J} \neq \emptyset$.
Similarly, it cannot happen that $\mathcal{I} = \emptyset$ and $\mathcal{J} \neq \emptyset$. In the case, $\mathcal{I} \neq \emptyset$ and $\mathcal{J} = \emptyset$, we get $a_1 = \cdots = a_\ell = 0$. This means that $(\zeta_1, \theta_1), \cdots, (\zeta_\ell, \theta_\ell)$ are linearly independent. Thus $\ell \leq \kappa$. This ends the proof. □

Remark 3.10. It is worth remarking that all results in this paper still hold true if we replace Fréchet normal cones, Fréchet (singular) subdifferentials by proximal normal cones, proximal (singular) subdifferentials, respectively.

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