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Cauchy Metrizability of Bornological Universes

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We call a bornology on a metric space (X, d) d-Cauchy metrizable if there exists a metric ρ on X, Cauchy equivalent to d, such that the family of ρ-bounded subsets coincides with the bornology. Recall that two metrics on a set are said to be Cauchy equivalent if the collections of Cauchy sequences with respect to both the metrics are same. In this paper we give necessary and sufficient conditions for a bornology on a metric space (X, d) to be d-Cauchy metrizable. We solve this problem for two different approaches, one given by Hu [7, 8] and the other given by Beer [2]. Furthermore, we investigate the same for some most common bornologies.

Keywords: Cauchy continuous function, Cauchy equivalent metrics, bornology, bounded set, totally bounded, metric mode of convergence to infinity

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1. Introduction

The concept of boundedness in metric spaces is well known to everyone having some basic knowledge of metric spaces. By seeing the properties possessed by the family of bounded sets in a metric space, the concept of bornology was introduced with an objective to have a notion of boundedness in a topological space. So a *bornology* $\mathbf{B}$ on a nonempty set X is a family of subsets of X which is a cover of X and closed under finite unions and subsets. Moreover, if X is a nonempty set equipped with a topology τ and a bornology $\mathbf{B}$ then the triple $(X, \tau, \mathbf{B})$ is called a *bornological universe*. The collection of all bounded

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subsets of a metric space is a well known example of a bornology on a metric space. In 1949, Hu [7] characterized those bornologies on a metrizable space X which coincide with the families of bounded sets for some admissible metric for the space. Later in 1999, Beer addressed a similar problem in [2] for the existence of a uniformly equivalent metric, where two metrics are called *uniformly equivalent* if the identity maps between the two metric spaces are uniformly continuous. Furthermore given a metric space (X, d) and a bornology of subsets $\mathbf{B}$, in [6] Garrido and Meroño gave necessary and sufficient conditions for the existence of some metric ρ uniformly equivalent to d such that $\mathbf{B}$ agreed with the collection of ρ-bounded subsets with the same approach as that of Hu. Since we have another equivalence of metrics that lies strictly in between the uniform equivalence and the topological equivalence of metrics, now the question arises: given a bornology on a metric space (X, d), when does there exist a metric ρ on X, Cauchy equivalent to d, such that the family of ρ-bounded subsets coincides with the given bornology? We call such bornologies *d-Cauchy metrizable*. In fact, our objective is to give necessary and sufficient conditions for a bornology on a metric space (X, d) to be d-Cauchy metrizable (see Theorem 2.8). Moreover, we look into the characterizations of a metric space (X, d) such that some of the most common bornologies on (X, d), namely the collection of finite subsets of X, the collection of subsets of X with compact closures and the collection of totally bounded subsets of (X, d), are d-Cauchy metrizable.

In [2], Beer called a family of subsets $\mathbf{B}$ of a metric space (X, d) *nontrivial metric boundedness structure* if it satisfies the following conditions:

(a) It is a bornology of proper subsets of X with a countable base.
(b) For every $A \in \mathbf{B}$, there exists $B \in \mathbf{B}$ such that $\overline{A} \subseteq \operatorname{int}(B)$.

Furthermore, the set of all subsets of X is called the *trivial metric boundedness structure* on X. In the same paper, Beer adopted a dual approach to the study of boundedness structures, based on the notion of *metric mode of convergence to infinity.* In Theorem 3.2 of [2], he proved one-to-one correspondence between the bounded sets with respect to a metric mode of convergence to infinity (see Definition 3.1) and a nontrivial metric boundedness structure for a regular Hausdorff space. Thereby he recasted Hu's theorem of metrizability of a bornology on a metric space using his dual point of view. Based on the same approach, he gave equivalent conditions for the existence of a uniformly equivalent metric ρ on a metric space (X, d) such that the collections of bounded sets with respect to the metric ρ and with respect to a given metric mode of convergence to infinity, $\mathbf{B}(\langle F_n\rangle)$, coincide. Thus, our next goal is to answer the following question: given a metric mode of convergence to infinity $\langle F_n\rangle$ on a metric space (X, d), what are the necessary and sufficient conditions under which there exists a Cauchy equivalent metric ρ on X such that $\mathbf{B}(\langle F_n\rangle)$ coincides with the set of ρ-bounded subsets of X? Here, we would like to point out that using this dual approach, Beer proved the existence of uncountably many distinct metric boundedness structures in a noncompact metrizable space (see Theorem 3.5 of [2]) which might not be apparent from the Hu's approach. Thus, its significance motivated us to look into our major objective through the Beer's perspective.

The symbols $\mathbb{R}, \mathbb{N}$ and $\mathbb{Q}$ denote the sets of real numbers, natural numbers and rational numbers respectively. Unless mentioned otherwise, $\mathbb{R}$ and its subsets carry the usual distance metric. If (X, d) is a metric space, $x \in X$ and $\delta > 0$, then $B_d(x, \delta)$ denotes the open ball in (X, d), centered at x with radius δ (we omit the subscript d if there is no confusion of the metric involved). Also, $(\widetilde{X}, \widetilde{d})$ denotes the completion of the metric space (X, d). We call a Cauchy sequence in a metric space (X, d) d-Cauchy, similarly a totally bounded subset of (X, d) is called d-totally bounded.

2. Cauchy metrizable bornologies

Recall that two metrics on a set X are said to be *equivalent* if they generate the same topology, whereas two metrics are called *uniformly equivalent* if the identity maps between the two metric spaces are uniformly continuous. In [11], Williamson and Janos have looked for those metrics that have the same collections of Cauchy sequences.

Definitions 2.1. Metrics d and ρ on a set X are said to be *Cauchy equivalent* if every d-Cauchy sequence in X is ρ-Cauchy and vice versa.

Clearly, Cauchy equivalence is an equivalence relation on the collection of all metrics on a set X. Moreover, metrics d and ρ on a set X are Cauchy equivalent if and only if the identity functions between the two metric spaces, (X, d) and (X, ρ), are Cauchy-continuous, where a function $f : (X, d) \to (Y, \rho)$ is said to be *Cauchy-continuous* if for any Cauchy sequence (x_n) in (X, d), $(f(x_n))$ is a Cauchy sequence in (Y, ρ). Since a uniformly continuous function is Cauchy-continuous and a Cauchy-continuous function is continuous, all uniformly equivalent metrics on a set are Cauchy equivalent, whereas all Cauchy equivalent metrics on a set are equivalent. But the converses of these statements fail. For example, consider $\mathbb{N}$ with the metrics d and ρ defined as: $d(n, m) = |n - m|$ and $\rho(n, m) = |\frac{1}{n} - \frac{1}{m}|$ and consider the set $[0, \infty)$ with the metrics d_1 and ρ_1 defined as: $d_1(x, y) = |x - y|$ and $\rho_1(x, y) = |x^2 - y^2|$.

The next proposition gives a way to construct different Cauchy equivalent metrics on a set. In fact, it is mainly used in proving the main results of the paper.

Proposition 2.2. *Let (X, d) be a metric space and let f be a function on (X, d) with values in another metric space (Y, σ). Define a metric ρ on X by:*

$$\rho(a, b) = d(a, b) + \sigma(f(a), f(b)) \quad \text{for } a,\ b \text{ in } X$$

Then the metrics d and ρ are Cauchy equivalent on X if and only if f is Cauchy-continuous.

Before we talk about the idea behind our present work, we would like to recall a few relevant definitions.

Definitions 2.3. A family $\mathbf{B}$ of subsets of a nonempty set X is called a *bornology* on X provided the following conditions are satisfied:

(a) $\mathbf{B}$ forms a cover of X;
(b) whenever $B \in \mathbf{B}$ and $A \subseteq B$, then $A \in \mathbf{B}$;
(c) If $A, B \in \mathbf{B}$ then $A \cup B \in \mathbf{B}$.

A triple $(X, \tau, \mathbf{B})$ is called a *bornological universe* where (X, τ) is a topological space and $\mathbf{B}$ is a bornology on X. Note that a bornological universe on a metric space (X, d) is denoted by $(X, \tau_d, \mathbf{B})$.

Definition 2.4. Let $\mathbf{B}$ be a bornology on a nonempty set X. Then a family $\mathcal{C} \subseteq \mathbf{B}$ is said to be a *base* for $\mathbf{B}$ if every member of $\mathbf{B}$ is contained in some member of $\mathcal{C}$.

The first bornology that comes to mind on a metric space (X, d) is the family of all d-bounded subsets, denoted by $\mathbf{B}_d(X)$. Keeping this particular bornology in mind, in [7] S. -T. Hu gave necessary and sufficient conditions for a bornology on a metrizable space X to be the family of bounded sets for some admissible metric for the space. The precise statement of his result is as follows:

Theorem 2.5. *Let X be a metrizable space. Then a bornology $\boldsymbol{B}$ on X is same as the collection of subsets of X bounded with respect to some compatible metric d if and only if the following two conditions are satisfied:*

(a) *$\boldsymbol{B}$ has a countable base,*
(b) *For every $A \in \boldsymbol{B}$, there exists $B \in \boldsymbol{B}$ such that $\overline{A} \subseteq \operatorname{int}(B)$.*

Note that a bornology $\mathbf{B}$ on a metrizable space X is said to be *metrizable* if there exists an equivalent metric ρ on X such that $\mathbf{B} = \mathbf{B}_\rho(X)$. In 1999, Beer [2, pp. 201–203], showed that a bornology $\mathbf{B}$ on a metric space (X, d) coincides with $\mathbf{B}_\rho(X)$ for some uniformly equivalent metric ρ provided condition (a) above holds and condition (b) is replaced by this condition: There exists $\delta > 0$, such that for every $A \in \mathbf{B}$, $A^\delta = \{x : d(x, A) < \delta\} \in \mathbf{B}$. We call such a bornology a *d-uniformly metrizable bornology* in the sequel. Beer's approach was based on metric modes of convergence to infinity and can be viewed as dual to Hu's, whereas the subsequent study of Garrido and Meroño [6] used the same perspective as did Hu. Now the same question arises with respect to the existence of a Cauchy equivalent metric ρ whose bounded sets agree with $\mathbf{B}$. Our next result answers the question, but before that, we include a couple of definitions.

Definition 2.6 ([7]). A continuous non-negative function $\chi : X \to [0, \infty)$ on a bornological universe $(X, \tau, \mathbf{B})$ is said to be a *characteristic function* if: $A \in \mathbf{B}$ if and only if $\chi(A)$ is a bounded subset of $[0, \infty)$.

Definition 2.7. A bornology $\mathbf{B}$ on a metric space (X, d) is said to be *d-Cauchy metrizable* if there exists a metric ρ on X, Cauchy equivalent to d, such that $\mathbf{B} = \mathbf{B}_\rho(X)$.

Observe that if $\mathbf{B}$ is a bornology on a metric space (X, d) and $X \in \mathbf{B}$, then $\mathbf{B}$ is the power set of X and hence the metric d^* defined as: $d^*(x, y) = \min\{1, d(x, y)\}$ is Cauchy equivalent to d such that $\mathbf{B} = \mathbf{B}_{d^*}(X)$. Thus, from now onwards we assume that $X \notin \mathbf{B}$.

Theorem 2.8. *Let $\boldsymbol{B}$ be a bornology on a metric space (X, d). Then the following statements are equivalent:*

(a) *$\boldsymbol{B}$ is d-Cauchy metrizable.*

(b) *there exists a countable base $\{B_n : n \in \mathbb{N}\}$ for $\boldsymbol{B}$ such that $\overline{B}_n \subseteq B_{n+1}$ $\forall\, n \in \mathbb{N}$, and given a d-Cauchy sequence (x_m) in X, there exist $k,\ n_o \in \mathbb{N}$ such that $x_m \in B_k \setminus \overline{B}_{k-2}$ for all $m \geq n_o$, where we are assuming $B_0 = B_{-1} = \emptyset$.*

(c) *there exists a Cauchy continuous characteristic function for the bornological universe $(X, \tau_d, \boldsymbol{B})$.*

Proof. $(a) \Rightarrow (b)$: Let ρ be a metric on X, Cauchy equivalent to d such that $\mathbf{B} = \mathbf{B}_\rho(X)$. Let $a \in X$ be fixed. For every $n \in \mathbb{N}$, let $B_n = \{x \in X : \rho(x, a) < n\}$. Without loss of generality, assume that $B_n \neq B_{n+1}$ $\forall\, n \in \mathbb{N}$. Clearly, $\overline{B}_n \subseteq B_{n+1}$. Let (x_n) be a d-Cauchy sequence in X and hence it is ρ-Cauchy. Then the sequence (x_n) is contained in some B_k. Let k be the smallest natural number such that (x_n) is eventually in B_k. If $k = 1$ or 2, then we are done. Suppose that $k > 2$. Now we claim that (x_n) cannot frequently intersect $\overline{B}_{k-2}$. Suppose that $(x_{n_p})_{p\in\mathbb{N}}$ is a subsequence of (x_n) which is contained in $\overline{B}_{k-2}$. Hence, there exists a sequence $(y_{n_p})_{p\in\mathbb{N}}$ in B_{k-2} with $\rho(x_{n_p}, y_{n_p}) < \frac{1}{p}$ $\forall\, p \in \mathbb{N}$. Consequently, $\rho(x_{n_p}, a) < \frac{1}{p} + k - 2$. Moreover, by the choice of k, there exists a subsequence $(x_{m_q})_{q\in\mathbb{N}}$ of (x_n) which is contained in $B_k \setminus B_{k-1}$. Thus, $k - 1 \leq \rho(x_{m_q}, a) < k$. Since (x_n) is a Cauchy sequence, $\exists\, N \in \mathbb{N}$ such that $\rho(x_{m_q}, x_{n_p}) < \frac{1}{2}$ $\forall\, q, p \geq N$. Then, eventually we can have, $k - 1 \leq \rho(x_{m_q}, a) \leq \rho(x_{m_q}, x_{n_p}) + \rho(x_{n_p}, a) < \frac{1}{2} + k - 2 + \frac{1}{p}$. By taking p sufficiently large, we will get a contradiction. Thus, (x_n) is eventually contained in $B_k \setminus \overline{B}_{k-2}$.

$(b) \Rightarrow (c)$: For $n \in \mathbb{N}$, let A_n be the disjoint union of $\overline{B}_n$ and $X \setminus B_{n+1}$. Define a function, $\psi_n : A_n \to [0, 1]$ such that $\psi_n(\overline{B}_n) = 0$ and $\psi_n(X \setminus B_{n+1}) = 1$. We claim that ψ_n is a Cauchy continuous function. Let (x_m) be a d-Cauchy sequence in A_n. By (b), there exist $k,\ n_o \in \mathbb{N}$ such that $x_m \in B_k \setminus \overline{B}_{k-2}$ for all $m \geq n_o$. Suppose (x_m) intersects both $\overline{B}_n$ and $X \setminus B_{n+1}$ frequently. Then $k \leq n + 1$ as (x_m) intersects $\overline{B}_n$ frequently. But since (x_m) intersects $X \setminus B_{n+1}$ frequently, $k > n + 1$. We get a contradiction. Consequently, (x_m) is eventually in $\overline{B}_n$ or else eventually in $X \setminus B_{n+1}$. So $(\psi_n(x_m))_{m\in\mathbb{N}}$ is eventually a constant sequence. Since (x_m) was arbitrary Cauchy sequence in A_n, ψ_n is

Cauchy-continuous. Then by Lemma 3 of [4], there exists a Cauchy continuous function, $\phi_n : X \to [0,1]$ such that $\phi_n(a) = \psi_n(a) \;\; \forall\, a \in A_n$. Now define a function $\chi : X \to \mathbb{R}$ as:

$$\chi(x) = n - 1 + \phi_n(x)$$

for every $x \in B_{n+1} \setminus B_n$ and for every $n \in \mathbb{N}$, and $\chi(B_1) = 0$.
We now prove that χ is Cauchy continuous. Let (x_n) be a d-Cauchy sequence in X. Then there exist $k,\ n_o \in \mathbb{N}$ such that $\{x_n : n \geq n_o\} \subseteq B_k \setminus \overline{B}_{k-2}$. Assuming ϕ_0 and ϕ_{-1} to be identically zero functions, we claim that for $p,\ q \geq n_o$,

$$|\chi(x_p) - \chi(x_q)| \leq |\phi_{k-2}(x_p) - \phi_{k-2}(x_q)| + |\phi_{k-1}(x_p) - \phi_{k-1}(x_q)|.$$

Firstly suppose $k \geq 3$. Then three cases may arise: (1) $x_p,\ x_q \in B_k \setminus B_{k-1}$, (2) $x_p,\ x_q \in B_{k-1} \setminus \overline{B}_{k-2}$, or (3) $x_p \in B_k \setminus B_{k-1}$ and $x_q \in B_{k-1} \setminus \overline{B}_{k-2}$. Only the third case requires some explanation, while the rest are easy to see. If $x_p \in B_k \setminus B_{k-1}$ and $x_q \in B_{k-1} \setminus \overline{B}_{k-2}$, then

$$\begin{aligned} |\chi(x_p) - \chi(x_q)| &= |\phi_{k-1}(x_p) + 1 - \phi_{k-2}(x_q)| \\ &= |\phi_{k-1}(x_p) - \phi_{k-1}(x_q) + \phi_{k-2}(x_p) - \phi_{k-2}(x_q)|, \\ &\qquad \text{as } \phi_{k-1}(x_q) = 0 \text{ and } \phi_{k-2}(x_p) = 1 \\ &\leq |\phi_{k-1}(x_p) - \phi_{k-1}(x_q)| + |\phi_{k-2}(x_p) - \phi_{k-2}(x_q)|. \end{aligned}$$

Hence the claim is proved for $k \geq 3$, while it can be similarly proved for $k = 1,\ 2$. Now by the Cauchy continuity of ϕ_{k-2} and ϕ_{k-1}, $\chi(x_n)$ is a Cauchy sequence. Consequently, χ is Cauchy-continuous.

Moreover, if $C \in \mathbf{B}$, then $C \subseteq B_n$ for some $n \in \mathbb{N}$ and hence $\chi(C)$ is bounded, whereas if $A \subseteq X$ and $\chi(A)$ is bounded, then $A \subseteq B_n$ for some n and hence $A \in \mathbf{B}$. Thus, χ is the required characteristic function.

$(c) \Rightarrow (a)$: Let χ be a Cauchy continuous characteristic function for the bornological universe $(X, \tau_d, \mathbf{B})$. Define a metric ρ on X as:

$$\rho(x,y) = d^*(x,y) + |\chi(x) - \chi(y)|, \quad \text{x, y} \in X$$

where $d^*(x,y) = \min\{1, d(x,y)\}$. By Proposition 2.2, the metric ρ is Cauchy equivalent to d^* and hence Cauchy equivalent to d. Also, $A \in \mathbf{B}_\rho(X) \Leftrightarrow \chi(A)$ is bounded $\Leftrightarrow A \in \mathbf{B}$. Thus, $\mathbf{B}$ is d-Cauchy metrizable. □

Remark 2.9. By carefully observing the proof of the previous theorem, one can see that condition (b) can be replaced by the following condition: there exists a countable base $\{B_n : n \in \mathbb{N}\}$ for $\mathbf{B}$ such that $B_n \subseteq B_{n+1} \;\; \forall\, n \in \mathbb{N}$, and given a d-Cauchy sequence (x_m) in X, there exist $k,\ n_o \in \mathbb{N}$ such that $x_m \in B_k \setminus B_{k-2}$ for all $m \geq n_o$, where we are assuming $B_0 = B_{-1} = \emptyset$.

Other than $\mathbf{B}_d(X)$, there are various bornologies on a metric space (X,d) which are well known. We list a few of them here:

(a) $\mathbf{F}(X)$ = the collection of finite subsets of X.
(b) $\mathbf{C}(X)$ = the collection of subsets of X which are contained in some compact subset of (X, d).
(c) $\mathbf{TB}_d(X)$ = the collection of totally bounded subsets of (X, d).

Now we would like to look into more specific equivalent conditions under which these particular bornologies are d-Cauchy metrizable. Let us first observe that the collections of totally bounded subsets with respect to Cauchy equivalent metrics on a set are same. It can be easily seen by the following sequential characterization of totally bounded subsets in a metric space: a subset A of (X, d) is totally bounded if and only if every sequence in A has a Cauchy subsequence. But note that if the metric spaces (X, d) and (X, ρ) have the same collections of totally bounded subsets, then the metrics d and ρ need not be even equivalent. For example, consider $X = \mathbb{R}$ and d be the usual metric and let ρ be the metric defined as: $\rho(x, y) = |f(x) - f(y)|$ where $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} 0 & : \quad x = 1 \\ 1 & : \quad x = 0 \\ x & : \quad \text{else.} \end{cases}$$

For proving our next result we need the following definition from [4].

Definition 2.10. A subset A of a metric space (X, d) is said to be *F-discrete* if every totally bounded subset of A is finite.

Theorem 2.11. *Let (X, d) be a metric space. Then $\boldsymbol{F}(X)$ is d-Cauchy metrizable if and only if (X, d) is countable and F-discrete.*

Proof. Let ρ be a metric on X, Cauchy equivalent to d such that $\mathbf{F}(X) = \mathbf{B}_\rho(X)$. Let a be a fixed element of X. Since $X = \bigcup_{n\in\mathbb{N}} B_\rho(a, n)$, where $B_\rho(a, n)$ denotes the open ball with respect to the metric ρ, X is countable. Let A be a d-totally bounded subset of X. Since d and ρ are Cauchy equivalent, A is ρ-totally bounded as well. Hence, $A \in \mathbf{B}_\rho(X)$, which implies A is finite.

Conversely, let $X = \{x_n : n \in \mathbb{N}\}$. Define $B_n = \{x_1, x_2, ..., x_n\}$ for every $n \in \mathbb{N}$. Then these B_n's are satisfying the conditions given in Theorem 2.8(b) and hence $\mathbf{F}(X)$ is d-Cauchy metrizable. Note that since every d-totally bounded subset of X is finite, every d-Cauchy sequence in X is eventually constant. □

By Theorem 2.11, one can easily see that if we consider $\mathbb{N}$ with the discrete metric : $\rho(n, m) = 1$ for $n \neq m$, $\rho(n, n) = 0$, n, $m \in \mathbb{N}$, then $\mathbf{F}(\mathbb{N})$ is d-Cauchy metrizable. In fact, the usual metric d is Cauchy equivalent to the discrete metric such that $\mathbf{F}(\mathbb{N}) = \mathbf{B}_d(\mathbb{N})$.

Before moving towards the Cauchy metrizability of $\mathbf{C}(X)$, let us recall that a metric space satisfies the *Heine-Borel property* if every closed and bounded subset of the metric space is compact. Moreover, a metric space is locally compact if every point has a compact neighbourhood. In [9], Vaughan proved that a metric space admits an equivalent metric with the Heine-Borel property if and only if it is locally compact and separable. In our next result, we prove that the addition of completeness property will be equivalent to the existence of a Cauchy equivalent metric with the Heine-Borel property.

Theorem 2.12. *Let (X, d) be a metric space. Then the following statements are equivalent:*

(a) $\boldsymbol{C}(X)$ *is d-Cauchy metrizable.*

(b) *there exists a metric ρ on X, Cauchy equivalent to d, with the Heine-Borel property.*

(c) (X, d) *is complete, separable and locally compact.*

Proof. $(a) \Rightarrow (b)$: Let ρ be a metric on X Cauchy equivalent to d such that $\mathbf{C}(X) = \mathbf{B}_\rho(X)$ and hence (X, ρ) satisfies the Heine-Borel property.

$(b) \Rightarrow (c)$: Let $a \in X$. Since $X = \bigcup_{n\in\mathbb{N}} B_\rho[a, n]$, where $B_\rho[a, n]$ denotes the closed ball with centre a and radius n with respect to ρ, X is separable and locally compact (topological properties with respect to both the metrics are same). Now we just need to show the completeness of (X, d). Let (x_n) be a d-Cauchy sequence in X. Then it is ρ-Cauchy and hence (x_n) is contained in $B_\rho[a, m]$ for some $m \in \mathbb{N}$, which is compact. Thus, the sequence (x_n) clusters, which proves our claim.

$(c) \Rightarrow (a)$: Suppose there exists a subsequence (y_{n_k}) of (y_n) contained in B_{m-1}. Since B_{m-1} is compact, $y \in B_{m-1}$, which is a contradiction. Hence, by Theorem 2.8, $\mathbf{C}(X)$ is d-Cauchy metrizable. Since (X, d) is separable and locally compact, (X, d) admits an equivalent metric ρ such that (X, ρ) has the Heine-Borel property. Since (X, d) and (X, ρ) are complete, the identity functions between the two metric spaces are Cauchy continuous and hence ρ is Cauchy equivalent to d such that $\mathbf{C}(X) = \mathbf{B}_\rho(X)$. □

Remark 2.13. By Theorem 2′ of [11] and the previous theorem, $\mathbf{C}(X)$ is d-Cauchy metrizable if and only if there exists a metric ρ on X which is Cauchy equivalent and locally identical to d with the Heine-Borel property, where two metrics d and ρ on X are said to be *locally identical* if every point $x \in X$ has a neighbourhood V such that $d(y, z) = \rho(y, z)$ for $y,\ z \in V$.

Note that if $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ then $\mathbf{C}(X)$ is not d-Cauchy metrizable as it is not complete, where d is the usual metric. But it is metrizable by the aforesaid result of Vaughan [9].

Theorem 2.14. *Let (X, d) be a metric space and let $(\widetilde{X}, \widetilde{d})$ be its completion. Then the following statements are equivalent:*

(a) $\boldsymbol{TB}_d(X)$ *has a countable base.*

(b) $\boldsymbol{TB}_d(X)$ *is d-Cauchy metrizable.*

(*c*) *$\mathbf{TB}_d(X)$ has a countable base $\{B_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, $B_n^{\delta_n} \subseteq B_{n+1}$ for some $\delta_n > 0$, where $B_n^{\delta_n} = \{x \in X : d(x, B_n) < \delta_n\}$.*

(*d*) *(X, d) is separable and for every totally bounded subset A of (X, d), there exists a $\delta > 0$ such that A^δ is totally bounded.*

(*e*) *$(\widetilde{X}, \widetilde{d})$ is separable and locally compact.*

(*f*) *$\mathbf{TB}_{\widetilde{d}}(\widetilde{X})$ is d-Cauchy metrizable.*

Proof. $(a) \Rightarrow (b)$: Let $\mathbf{TB}_d(X)$ has a countable base, say $\{B_n : n \in \mathbb{N}\}$. Then by Lemma 3.3 of [3], $\{cl_{\widetilde{X}}(B_n) : n \in \mathbb{N}\}$ is a countable base of $\mathbf{C}(\widetilde{X})$. Since every first countable hemicompact space is locally compact [1], $\widetilde{X}$ is locally compact. Moreover, $\widetilde{X} = \bigcup_{n\in\mathbb{N}} cl_{\widetilde{X}}(B_n)$, which implies $(\widetilde{X}, \widetilde{d})$ is $\sigma-$compact and hence it is separable. Now by Theorem 2.12, $\mathbf{C}(\widetilde{X})$ is d-Cauchy metrizable. Let $\widetilde{\rho}$ be a metric on $\widetilde{X}$, Cauchy equivalent to $\widetilde{d}$, such that $\mathbf{C}(\widetilde{X}) = \mathbf{B}_{\widetilde{\rho}}(\widetilde{X})$. Then the metric $\rho = \widetilde{\rho}\mid_{X\times X}$ is Cauchy equivalent to d on X such that $\mathbf{TB}_d(X) = \mathbf{B}_\rho(X)$, which implies $\mathbf{TB}_d(X)$ is d-Cauchy metrizable.

$(b) \Rightarrow (c)$: Let $\mathbf{TB}_d(X)$ be d-Cauchy metrizable. Let ρ be a metric on X, Cauchy equivalent to d, such that $\mathbf{TB}_\rho(X) = \mathbf{TB}_d(X) = \mathbf{B}_\rho(X)$. For every $n \in \mathbb{N}$, let $B_n = B_\rho(a, n)$, where a is some fixed element of X. Then $\{B_n : n \in \mathbb{N}\}$ is a countable base of $\mathbf{TB}_d(X)$. Now we claim that for every $n \in \mathbb{N}$, $B_n^{\delta_n} \subseteq B_{n+1}$ for some $\delta_n > 0$. Suppose for every $m \in \mathbb{N}$, $\exists\ x_m \in B_n^{\frac{1}{m}} \setminus B_{n+1}$. Then there exists a sequence (y_m) in B_n such that $d(x_m, y_m) < \frac{1}{m}$. Since B_n is totally bounded, $\exists$ a Cauchy subsequence (y_{m_k}) of (y_m). Then (z_{m_k}) is a Cauchy sequence in (X, d) and hence in (X, ρ), where $z_{m_{2k}} = y_{m_k}$ and $z_{m_{2k-1}} = x_{m_k}$ $\forall\ k \in \mathbb{N}$. Since $\rho(x_{m_k}, a) \leq \rho(x_{m_k}, y_{m_k}) + \rho(y_{m_k}, a) < 1 + n \ \ \forall\ k \in \mathbb{N}$. Hence, $(x_{m_k}) \subseteq B_{n+1}$, which is a contradiction to the choice of x_m. Thus, for every $n \in \mathbb{N}$, $B_n^{\delta_n} \subseteq B_{n+1}$ for some $\delta_n > 0$.

$(c) \Rightarrow (d)$: Since X is a countable union of totally bounded sets $\{B_n : n \in \mathbb{N}\}$, (X, d) is separable. Also, if A is a totally bounded subset of (X, d), then $A \subseteq B_m$ for some $m \in \mathbb{N}$ and hence $A^{\delta_m} \subseteq B_m^{\delta_m} \subseteq B_{m+1}$. Thus, A^{δ_m} is totally bounded.

$(d) \Rightarrow (e)$: $(\widetilde{X}, \widetilde{d})$ is separable as (X, d) is separable. Let $\widetilde{x} \in \widetilde{X}$. Then there exists a sequence (x_n) in X converging to $\widetilde{x}$. Thus, $A = \{x_n : n \in \mathbb{N}\}$ is totally bounded subset of (X, d) and hence there exists $\delta > 0$ such that A^δ is totally bounded. Now we claim that $B_{\widetilde{d}}[\widetilde{x}, \frac{\delta}{2}] = \{\widetilde{y} \in \widetilde{X} : \widetilde{d}(\widetilde{x}, \widetilde{y}) \leq \frac{\delta}{2}\}$ is totally bounded and hence compact in $(\widetilde{X}, \widetilde{d})$. Let $(\widetilde{y}_n)$ be a sequence in $B_{\widetilde{d}}[\widetilde{x}, \frac{\delta}{2}]$. Then $\exists$ a sequence (y_n) in X such that $\widetilde{d}(\widetilde{y}_n, y_n) < \frac{1}{n}$, which implies (y_n) is eventually in $B_d(x_k, \delta)$ for sufficiently large $k \in \mathbb{N}$. Thus, (y_n) has a Cauchy subsequence, as $B_d(x_k, \delta)$ is totally bounded. Consequently, $(\widetilde{y}_n)$ has a Cauchy subsequence. Hence the claim is proved and thus $(\widetilde{X}, \widetilde{d})$ is locally compact.

$(e) \Rightarrow (f)$: Since $\mathbf{TB}_{\widetilde{d}}(\widetilde{X}) = \mathbf{C}(\widetilde{X})$, by Theorem 2.12 $\mathbf{TB}_{\widetilde{d}}(\widetilde{X})$ is d-Cauchy metrizable.

$(f) \Rightarrow (a)$: Let $\widetilde{\rho}$ be a metric on $\widetilde{X}$, Cauchy equivalent to $\widetilde{d}$, such that $\mathbf{TB}_{\widetilde{d}}(\widetilde{X}) = \mathbf{B}_{\widetilde{\rho}}(\widetilde{X})$. Let $\rho = \widetilde{\rho}\mid_{X\times X}$. Then ρ is Cauchy equivalent to d on X, such that $\mathbf{TB}_d(X) = \mathbf{B}_\rho(X)$. Hence $\mathbf{TB}_d(X)$ is d-Cauchy metrizable and by Theorem 2.8, $\mathbf{TB}_d(X)$ has a countable base. □

Remark 2.15. By Theorem 3.4 of [3] and the previous theorem, $\mathbf{TB}_d(X)$ is metrizable if and only if $\mathbf{TB}_d(X)$ is d-Cauchy metrizable.

Now we give an example of a d-Cauchy metrizable bornology which is not d-uniformly metrizable.

Example 2.16. As a metric subspace of the Hilbert space l_2 of square summable sequences, let $X = \{e_n + \frac{1}{n}e_k : n,\ k \in \mathbb{N}\}$, where e_n is a sequence with 1 at n^{th} place and 0 at all other places. Then X itself is countable and hence separable. Since every Cauchy sequence in (X, d) is eventually constant, (X, d) is complete. Moreover, (X, d) is locally compact being a discrete space. Hence, by Theorem 2.14, $\mathbf{TB}_d(X)$ is d-Cauchy metrizable. On the other hand, (X, d) is not uniformly locally compact as for every $n \in \mathbb{N}$, the closed ball in X with center $e_n + \frac{1}{n}e_1$ and radius $\frac{\sqrt{2}}{n}$ contains a non-clustering sequence. Consequently, by Theorem 4.2 of [6], $\mathbf{TB}_d(X)$ is not d-uniformly metrizable.

3. Similar problem for metric mode of convergence to infinity

It is well known that a metric space (X, d) is noncompact if and only if there exists a sequence $\langle F_n \rangle$ of nonempty closed subsets of X with $F_{n+1} \subseteq \mathrm{int}(F_n)$ and $\bigcap F_n = \emptyset$. If $\langle F_n \rangle$ and $\langle E_n \rangle$ are two such sequences of nonempty closed subsets of X and if for each k, there exist $n,\ j \in \mathbb{N}$ such that $E_k \supset F_j$ and $F_k \supset E_n$, then $\langle F_n \rangle$ and $\langle E_n \rangle$ are said to be *equivalent* in [2], denoted by $\langle F_n \rangle \equiv \langle E_n \rangle$. Beer declared the equivalence classes for this equivalence relation to be *metric modes of convergence to infinity* (note that in the sequel, we identify an equivalence class with one of its representatives). In the same paper, Beer defined bounded sets with respect to $\langle F_n \rangle$. We mention its precise definition for further use.

Definition 3.1. A subset B of a metric space (X, d) is said to be *bounded* with respect to a metric mode of convergence to infinity $\langle F_n \rangle$ if for some n, $B \cap F_n = \emptyset$. Moreover, the collection of bounded subsets of X with respect to $\langle F_n \rangle$ is denoted by $\mathbf{B}(\langle F_n \rangle)$.

Beer noted that $\mathbf{B}(\langle F_n \rangle)$ satisfies the following conditions:

(a) It is a bornology of proper subsets of X with a countable base.
(b) For every $A \in \mathbf{B}(\langle F_n \rangle)$, there exists $B \in \mathbf{B}(\langle F_n \rangle)$ such that $\overline{A} \subseteq \mathrm{int}(B)$.

Beer defined a family of subsets of (X, d) satisfying (a) and (b) to be *nontrivial metric boundedness structure*. Hence by Hu's theorem, if $\langle F_n \rangle$ is a metric mode of convergence to infinity for (X, d), then there exists an (unbounded) metric ρ

equivalent to d such that $\mathbf{B}(\langle F_n\rangle) = \mathbf{B}_\rho(X)$. We intend to present alternative necessary and sufficient conditions on $\langle F_n\rangle$ that guarantee that $\mathbf{B}(\langle F_n\rangle)$ is d-Cauchy metrizable. But firstly let us recall the following definition from [2].

Definition 3.2. Let (X, d) be a metric space. A continuous function $f : X \to [0,\infty)$ is called a *forcing function* for a metric mode of convergence to infinity $\langle F_n\rangle$ if $\sup\{f(x) : x \in A\} < \infty$ if and only if $A \in \mathbf{B}(\langle F_n\rangle)$.

Observe that forcing functions are similar to characteristic functions which we defined earlier.

Theorem 3.3. *Let (X, d) be a metric space and let $\langle F_n\rangle$ be a metric mode of convergence to infinity. Then the following are equivalent:*

(a) *there exists a metric ρ Cauchy equivalent to d such that $\boldsymbol{B}(\langle F_n\rangle) = \boldsymbol{B}_\rho(X)$.*

(b) *there exists a subsequence $\langle F_{n_k}\rangle$ of $\langle F_n\rangle$ such that every d-Cauchy sequence is contained in $F^c_{n_j}$ for some $j \in \mathbb{N}$ and it cannot intersect frequently both $F^c_{n_k}$ and $F_{n_{k+1}}$ $\forall\, k \in \mathbb{N}$.*

(c) *$\langle F_n\rangle$ admits a Cauchy continuous forcing function.*

Proof. $(a) \Rightarrow (b)$: Let a be a fixed element of X. For every $n \in \mathbb{N}$, let $B_n = B_\rho(a, n)$. Since $B_1 \in \mathbf{B}_\rho(X) = \mathbf{B}(\langle F_n\rangle)$, $B_1 \subseteq F^c_{n_1}$ for some $n_1 \in \mathbb{N}$. Similarly, since $F^c_{n_1} \in \mathbf{B}(\langle F_n\rangle)$, $F^c_{n_1} \subseteq B_{j_2}$ for some $j_2 > j_1$, where $j_1 = 1$. Now choose $F^c_{n_2}$ such that $B_{j_2+1} \subseteq F^c_{n_2}$ and $n_2 > n_1$. Proceeding in this way, we get a subsequence $\langle F_{n_k}\rangle$ of $\langle F_n\rangle$. Now we claim that this is the required subsequence. Let (x_n) be a d-Cauchy sequence in X. Since ρ is Cauchy equivalent to d, (x_n) is ρ-Cauchy and hence $(x_n) \in \mathbf{B}_\rho(X)$. Thus, $(x_n) \subseteq F^c_{n_j}$ for some $j \in \mathbb{N}$. Now suppose for some $k \in \mathbb{N}$, there exist subsequences (x_{p_n}) and (x_{q_n}) of (x_n) such that $(x_{p_n}) \subseteq F^c_{n_k}$ and $(x_{q_n}) \subseteq F_{n_{k+1}}$. Then $(x_{p_n}) \subseteq B_{j_{k+1}}$ and $(x_{q_n}) \subseteq B^c_{j_{k+1}+1}$. Consequently, $\rho(x_{p_n}, a) < j_{k+1}$ and $\rho(x_{q_n}, a) \geq j_{k+1}+1$, which is a contradiction as (x_n) is a Cauchy sequence.

$(b) \Rightarrow (c)$: For $k \in \mathbb{N}$, let A_k be the disjoint union of $F^c_{n_k}$ and $F_{n_{k+1}}$. Define a function $g_k : A_k \to [0,1]$ as: $g_k(F^c_{n_k}) = 0$ and $g_k(F_{n_{k+1}}) = 1$. Then by (b), g_k is Cauchy continuous and hence by Lemma 3 of [4], there exists a Cauchy continuous function, $f_k : (X, d) \to [0, 1]$ such that $f_k(a) = g_k(a)$ $\forall\, a \in A_k$. Now define a function $f : (X, d) \to [0,\infty)$ as: $f(x) = \sum_{k=1}^{\infty} f_k(x)$ for $x \in X$. Since $\bigcap F_n = \emptyset$, every $x \in X$ is contained in $F^c_{n_k}$ eventually and hence f is finite valued. We claim that f is the required Cauchy continuous forcing function. Let (x_n) be a d-Cauchy sequence in X. Then (x_n) is contained in $F^c_{n_j}$ for some $j \in \mathbb{N}$ and hence for $n \in \mathbb{N}$, $f_k(x_n) = 0$ $\forall\, k \geq j$. This implies $f(x_n) = f_1(x_n)+f_2(x_n)+...+f_{j-1}(x_n)$ $\forall\, n \in \mathbb{N}$. Since f_k is Cauchy continuous, $(f_k(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence and so is $(f(x_n))$. Moreover, if $A \in \mathbf{B}(\langle F_n\rangle)$, then for some $m \in \mathbb{N}$, A is contained in $F^c_{n_k}$ $\forall\, k \geq m$ and hence $f(a) \leq m - 1$ $\forall\, a \in A$. Conversely, suppose $\sup\{f(x) : x \in A\} < \infty$. Let $x_n \in A \cap F_n$

$\forall\ n \in \mathbb{N}$. Thus, $f(x_{n_k}) \geq k-1$ for every $k \in \mathbb{N}$, which is a contradiction. So, $A \in \mathbf{B}(\langle F_n \rangle)$. Hence it follows that f is a required Cauchy continuous forcing function for $\langle F_n \rangle$.

$(c) \Rightarrow (a)$: Let f be a Cauchy continuous forcing function for $\langle F_n \rangle$. Define a metric ρ on X as:

$$\rho(x,y) = d^*(x,y) + |f(x) - f(y)|, \quad \text{x, y} \in X$$

where $d^*(x,y) = \min\{1, d(x,y)\}$. By Proposition 2.2, the metric ρ is Cauchy equivalent to d^* and hence Cauchy equivalent to d. Now let $A \in \mathbf{B}(\langle F_n \rangle)$. Then $\sup\{f(x) : x \in A\} = r$, for some $r \geq 0$. Let $a \in A$. Then, $\rho(x,a) \leq 1 + 2r$ $\forall\ x \in A$, which implies $A \in \mathbf{B}_\rho(X)$. On the other hand, let $a \in A$ and $r > 0$ such that $\rho(x,a) < r\ \ \forall\ x \in A$. Then $\ \forall\ x \in A$, $|f(x) - f(a)| < r$ and hence $f(x) < f(a) + r$. Thus, $\sup\{f(x) : x \in A\} < \infty$ and hence $A \in \mathbf{B}(\langle F_n \rangle)$, which completes the proof. $\square$

Now we give an example of a metric space (X,d) and a metric mode of convergence to infinity $\langle F_n \rangle$ such that there exists a metric ρ Cauchy equivalent to d with $\mathbf{B}(\langle F_n \rangle) = \mathbf{B}_\rho(X)$, but there does not exist any such metric which is uniformly equivalent to d.

Example 3.4. Consider Example 2.16. For $k \in \mathbb{N}$, let $F_k = \bigcup_{n=k}^{\infty}\{e_m + \frac{1}{m}e_n : m \in \mathbb{N}\}$. Then clearly, $\langle F_n \rangle$ is a sequence of nonempty closed subsets of X with $F_{n+1} \subseteq \operatorname{int}(F_n)$ and $\bigcap F_n = \emptyset$ and hence $\langle F_n \rangle$ is a metric mode of convergence to infinity. Since every d-Cauchy sequence in X is eventually constant, every Cauchy sequence is contained in $F_{n_j}^c$ for some $j \in \mathbb{N}$ and it can't intersect frequently both $F_{n_k}^c$ and $F_{n_{k+1}}$ $\ \forall\ k \in \mathbb{N}$ and hence by Theorem 3.3, there exists a metric ρ Cauchy equivalent to d such that $\mathbf{B}(\langle F_n \rangle) = \mathbf{B}_\rho(X)$. But note that since $(e_n + \frac{1}{n}e_k)_{k\in\mathbb{N}} \subseteq B_d(e_n + \frac{1}{n}e_m, \frac{\sqrt{2}}{n-1})$, $\nexists\ \delta > 0$ such that for each $n \in \mathbb{N}$, there exists $j > n$ such that $F_n \supseteq B_d(F_j, \delta)$, where $B_d(F_j,\delta) = \{x \in X : d(x, F_j) < \delta\}$. Then by Theorem 4.2 of [2], there does not exist any metric ρ on X, uniformly equivalent to d, such that ρ-bounded sets are precisely the bounded sets with respect to $\langle F_n \rangle$.

In Theorem 3.2 of [2], Beer established one-to-one correspondence between the bounded sets with respect to a metric mode of convergence to infinity and a nontrivial metric boundedness structure for a regular Hausdorff space. Recall that for a given nontrivial metric boundedness structure $\mathcal{S}$ for a regular Hausdorff space X, the unique metric mode of convergence to infinity $\langle F_n \rangle$ with $\mathcal{S} = \mathbf{B}(\langle F_n \rangle)$ was constructed as follows: let $\langle S_n \rangle$ be an increasing cofinal sequence in $\mathcal{S}$. Then choose a closed $T_1 \in \mathcal{S}$ such that $S_1 \subseteq \operatorname{int}(T_1)$ and inductively for each $n \geq 2$ choose a closed subset $T_n \in \mathcal{S}$ such that $T_{n-1} \cup S_n \subseteq \operatorname{int}(T_n)$. Now for each $n \in \mathbb{N}$, set $F_n = X - \operatorname{int}(T_n)$.
Let $B_n = \operatorname{int}(T_n)$ for each n. Then by construction (B_n) is an increasing cofinal sequence in $\mathcal{S}$ such that $\overline{B}_n \subseteq B_{n+1}$. Moreover, condition (b) of Theorem 3.3

implies that every Cauchy sequence is contained in B_{n_j} for some $j \in \mathbb{N}$ and it cannot intersect frequently both B_{n_k} and $B^c_{n_{k+1}}$ for all $k \in \mathbb{N}$. Let m be the least positive integer such that (x_n) is eventually contained in B_{n_m}. Certainly, (x_n) cannot frequently intersect $B_{n_{m-2}}$ (assume $B_{n_0} = B_{n_{-1}} = \emptyset$). As a result, the sequence (x_n) is eventually contained in $B_{n_m} \setminus B_{n_{m-2}}$. Thus, condition (b) of Theorem 3.3 implies that there exists an increasing cofinal sequence (B_n) in $\mathcal{S}$ such that every Cauchy sequence is contained in B_m for some $m \in \mathbb{N}$ and it cannot intersect frequently both B_n and B^c_{n+1} for all $n \in \mathbb{N}$.

Now we show that the converse is also true. So let $\langle F_n \rangle$ be a metric mode of convergence to infinity, then again by Theorem 3.2 of [2], $\mathbf{B}(\langle F_n \rangle)$ forms a nontrivial metric boundedness structure. Thus by the given condition there exists an increasing cofinal sequence (B_n) in $\mathbf{B}(\langle F_n \rangle)$ with certain properties. Now we use induction to construct a subsequence of $\langle F_n \rangle$ which satisfies condition (b). Since $B_1 \in \mathbf{B}(\langle F_n \rangle)$, there exists $n_1 \in \mathbb{N}$ such that $B_1 \subseteq F^c_{n_1}$. Since (B_n) is cofinal in $\mathbf{B}(\langle F_n \rangle)$, there exists $m_1 \in \mathbb{N}$ such that $F^c_{n_1} \subseteq B_{m_1}$. Now repeat the process for B_{m_1+1} (choose $n_2 > n_1$). Thus we get a subsequence $(F_{n_k})_{k \in \mathbb{N}}$ of (F_n) such that $B_{m_k+1} \subseteq F^c_{n_{k+1}} \subseteq B_{m_{k+1}}$ for all $k \in \mathbb{N}$. Now let (x_n) be a d-Cauchy sequence in X. Then (x_n) is contained in B_m for some m. Suppose $m < m_s$ for some $s \in \mathbb{N}$, then (x_n) is contained in $F^c_{n_{s+1}}$. Furthermore, if (x_n) frequently intersects both $F^c_{n_k}$ and $F_{n_{k+1}}$ for some $k \in \mathbb{N}$, then it frequently intersects both B_{m_k} and $B^c_{m_k+1}$, which is a contradiction. Finally it has been proved that condition (b) of Theorem 3.3 amounts to the following condition: there exists an increasing cofinal sequence (B_n) in $\mathcal{S}$ such that every Cauchy sequence is contained in B_m for some $m \in \mathbb{N}$ and it cannot intersect frequently both B_n and B^c_{n+1} for all $n \in \mathbb{N}$ (equivalently, there exists an increasing cofinal sequence (B_n) in $\mathcal{S}$ such that every Cauchy sequence is eventually contained in $B_k \setminus B_{k-2}$ for some $k \in \mathbb{N}$, where we are assuming $B_0 = B_{-1} = \emptyset$).

Recall that in [7, 8] Hu proved that if $\mathbf{B}$ is a nontrivial metric boundedness structure on a metric space (X, d), then there exists a compatible metric ρ on X such that $\mathbf{B}_\rho(X) = \mathbf{B}$. He mainly constructed a characteristic function f and using f, he defined ρ as : $\rho(x, y) = \min\{1, d(x, y)\} + |f(x) - f(y)|$. Now we know that if (X, d) is complete then f is Cauchy continuous and hence by Proposition 2.2, ρ is Cauchy equivalent to d. In our next result we show that the converse is also true, but before that we would like to give the following definition.

Definition 3.5. A family of subsets of a topological space (X, τ) is said to be *discrete* if every $x \in X$ has a neighbourhood that intersects at most one element of the family.

Theorem 3.6. *Let (X, d) be a metric space. Then the following statements are equivalent:*

(a) *(X, d) is complete.*

(b) *If $\langle F_n \rangle$ is a metric mode of convergence for X, then there exists a metric*

ρ, Cauchy equivalent to d, such that $\boldsymbol{B}(\langle F_n\rangle) = \boldsymbol{B}_\rho(X)$.

Proof. $(a) \Rightarrow (b)$: Since $\mathbf{B}(\langle F_n\rangle)$ forms a nontrivial metric boundedness structure, the implication follows from the discussion before Definition 3.5.

$(b) \Rightarrow (a)$: Suppose (X, d) is not complete. Let (x_n) be a d-Cauchy sequence in X with distinct points which does not cluster. Now we construct a metric mode of convergence which does not satisfy the condition (b). The construction is similar to that used by Beer in Theorem 4.4 of [2]. For every $n \in \mathbb{N}$, choose a closed neighbourhood $E(1, n)$ of x_n such that the family $\{E(1, n) : n \in \mathbb{N}\}$ is discrete and $E(1, n) \cap \{x_j : j \in \mathbb{N}\} = \{x_n\}\ \ \forall\ n \in \mathbb{N}$. Note that such a family exists because the sequence (x_n) does not cluster, which implies for each n, $r_n = \inf\{d(x_n, x_k) : k \neq n\} > 0$. Consequently, we can consider $E(1, n)$ to be the closed ball with center x_n and radius $\frac{r_n}{2^n}$. Now for each n, choose closed neighbourhoods $E(1, n),\ E(2, n), ...$ of x_n such that $E(k+1, n) \subseteq \operatorname{int} E(k, n)\ \ \forall\ k,\ n \in \mathbb{N}$. Let $F_k = \bigcup_{n=k}^{\infty} E(k, n)$. Then $\langle F_k\rangle$ is a metric mode of convergence to infinity. Since the Cauchy sequence (x_n) is eventually in F_k for every $k \in \mathbb{N}$, by Theorem 3.3 there does not exist any metric ρ which is Cauchy equivalent to d and $\mathbf{B}(\langle F_n\rangle) = \mathbf{B}_\rho(X)$, which is a contradiction. Hence (X, d) is complete. □

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