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Separation Properties in some Idempotent and Symmetrical Convex Structure

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$\mathbb{B}$-convexity was defined in [7] as a suitable Kuratowski-Painlevé upper limit of linear convexities over a finite dimensional Euclidean vector space. Recently, an alternative formulation over the whole Euclidean vector space was proposed in [6]. In this paper a convex separation framework is proposed as well as some extension of known results established over posets. We first analyze the algebraic properties of some class of subsets characterized by a suitable notion of dual form. Along this line some extended separation results are established by considering the Kuratowski-Painlevé limit of a sequence of linear halfspaces.

Keywords: Idempotence, semilattices, generalized convexity, $\mathbb{B}$-convexity.

1991 Mathematics Subject Classification: 06D50, 32F17.

Introduction

$\mathbb{B}$-convexity was defined in [7] as a suitable upper Kuratowski-Painlevé limit of generalized convexities inspired from Ben-Tal [5] and Avriel [3, 4]. In the case where convex sets are subsets of $\mathbb{R}^n_+$, $\mathbb{B}$-convexity is defined making the formal transformation $+ \to \max$ over the nonnegative Euclidean orthant $\mathbb{R}^n_+$. Hence, it is linked to Max-Plus algebra via a suitable homeomorphism and, a proposition that is true in the framework of $\mathbb{B}$-convexity holds, with obvious lexical modifications, in Max-Plus convexity [12]. All these notions are related to general concepts of abstract convexity developed in [14] and [16]. $\mathbb{B}$-convex functions were analyzed in [1]. Hahn-Banach like separation properties [10] as well as fixed point results [8] (see also [13]) have been established. $\mathbb{B}^{-1}$-convex sets were introduced in [2] and [15]. A comparisons to $\mathbb{B}$-convexity is proposed in [11]. A suitable algebraic structure extending $\mathbb{B}$-convexity to the whole Euclidean vector space was recently proposed in [6] where a special class of idempotent magma is considered in which associativity is relaxed to preserve both symmetry and idempotence. It has been shown that such a notion of convexity is equivalently characterized from the Kuratowski-Painlevé limit of the generalized convex hull of two points defined in [7].

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In this paper we propose a convex separation framework over $\mathbb{R}^n$. We first analyze the algebraic properties of some class of subsets whose the properties are very similar to those of linear halfspaces. Their restriction to $\mathbb{R}^n_+$ yields the $\mathbb{B}$-halfspaces defined in [10]. To do that, a notion of dual $\mathbb{B}$-forms is defined and one consider their lower and upper semi-continuous regularized respectively termed lower and upper $\mathbb{B}$-forms. Their relations to the extended definition proposed in [6] are analyzed. Along this line, we propose some results over the whole Euclidean vector space based on the Kuratowski-Painlevé limit of a suitable sequence of linear halfspaces. It is shown that such a sequence of linear halfpaces has a Kuratowski-Painlevé limit which coincides with the elements of this class of "pseudo" halfspaces. Finally, we deduce from these properties a separation result for some class of $\mathbb{B}$-convex sets defined over $\mathbb{R}^n$. It is also proved that these sets are the intersection of all the lower pseudo halfspaces which contain them.

The paper unfolds as follows. Section 1 presents the framework of $\mathbb{B}$-convexity. Section 2 focusses on the notion of dual $\mathbb{B}$-form. Section 3 considers the limit of a sequence of linear half-spaces and establishes some separation results for a class of $\mathbb{B}$-convex sets.

1. Generalized means and convexity

The generalized means were used in the field of optimization theory by Avriel [3] and Ben Tal [5]. In particular Avriel introduced the r-convex functions that generalize log-convexity. This notion was extended from an algebraic point of view by Ben Tal [5] who showed that transposing an algebraic structure via an isomorphism, a particular form of convexity can be derived over a vector space. In [7] the $\mathbb{B}$-convexity is introduced as a limit of linear convexities. More precisely, for all $p \in \mathbb{N}$, let us consider a bijection $\varphi_p \colon \mathbb{R} \to \mathbb{R}$ defined by:

$$\varphi_p \colon x \to x^{2p+1} \tag{1}$$

and $\Phi_p(x_1, ..., x_n) = (\varphi_p(x_1), ..., \varphi_p(x_n))$; this is basically the approach of Ben-Tal [5] and Avriel [3]. One can induce a field structure on $\mathbb{R}$ for which φ_p becomes a field isomorphism. Given this change of notation via φ_p and Φ_p we can define an $\mathbb{R}$-vector space structure on $\mathbb{R}^n$ by:

$$\lambda \overset{\varphi_p}{\cdot} x = \Phi_p^{-1}(\varphi_p(\lambda).\Phi_p(x)) = \lambda.x \quad \text{and} \quad x \overset{\varphi_p}{+} y = \Phi_p^{-1}(\Phi_p(x) + \Phi_p(y));$$

we call these two operations the *indexed scalar product* and the *indexed sum* (indexed by φ).

The φ_p-*sum* denoted $\overset{\varphi_p}{\sum}$ of $(x^{(1)}, ..., x^{(m)}) \in \mathbb{R}^{n\times m}$ is defined by[1]

$$\sum_{i\in[m]}^{\varphi_p} x^{(i)} = \Phi_p^{-1}\Big(\sum_{i\in[m]} \Phi_p(x^{(i)})\Big), \tag{2}$$

[1]For all $n \in \mathbb{N}$, $[n] = \{1, ..., n\}$.

and the Φ_p-*convex hull* of a finite set $A \subset \mathbb{R}^n$ is defined by:

$$Co^{\Phi_p}(A) = \left\{ \sum_{x\in A}^{\varphi} t_x \overset{\varphi_p}{\cdot} x : \sum_{x\in A}^{\varphi_p} t_x = \varphi_p^{-1}(1),\, \varphi_p(t_x) \geq 0 \right\}, \tag{3}$$

which can be rewritten as

$$Co^{\Phi_p}(A) = \Big\{ \Phi_p^{-1}\big(\sum_{x\in A} t_x^{2p+1}.\Phi_p(x) \big) : \big(\sum_{x\in A} t_x^{2p+1} \big)^{\frac{1}{2p+1}} = 1, t_x \geq 0 \Big\}. \tag{4}$$

This definition can be made more transparent showing that

$$Co^{\Phi_p}(A) = \Phi_p^{-1}\big(Co\,(\Phi_p(A)) \big). \tag{5}$$

For simplicity, throughout the paper we denote for all $x, y \in \mathbb{R}^n$

$$x \overset{p}{+} y = x \overset{\varphi_p}{+} y. \tag{6}$$

Moreover for all $S \subset \mathbb{R}^n$ we simplify the notations denoting

$$Co^p(S) = Co^{\Phi_p}(S). \tag{7}$$

1.1. Kuratowski-Painlevé limit and $\mathbb{B}$-convexity.

Recall that Kuratowski-Painlevé lower limit of the sequence of sets $\{A_n\}_{n\in\mathbb{N}}$, denoted $Li_{n\to\infty}A_n$, is the set of points x for which there exists a sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ of points such that $x^{(n)} \in A_n$ for all n and $x = \lim_{n\to\infty} x^{(n)}$. The *Kuratowski-Painlevé upper limit* of the sequence of sets $\{A_n\}_{n\in\mathbb{N}}$, denoted $Ls_{n\to\infty}A_n$, is the set of points x for which there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of points such that $x^{(n_k)} \in A_{n_k}$ for all k and $x = \lim_{k\to\infty} x^{(n_k)}$. A sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ is said to converge, in the Kuratowski-Painlevé sense, to a set A if $Ls_{n\to\infty}A_n = A = Li_{n\to\infty}A_n$, in which case we write $A = Lim_{n\to\infty}A_n$. We now define $\mathbb{B}$-convexity and its characterization in term of Kuratowski-Painlevé limit Φ_p-convex sets. Let $A \subset \mathbb{R}^n_+$ be a finite part. Definition of $\mathbb{B}$-convex sets is based on the definition of the Kuratowski-Painlevé limit of a sequence of Φ_p-convex sets. For a non-empty finite subset $A \subset \mathbb{R}^n$ we let $Co^\infty(A) = Ls_{p\to\infty}Co^p(A)$. In [7] a subset L of $\mathbb{R}^n$ is said $\mathbb{B}$-convex if for all finite parts A of L one has $Co^\infty(A) \subset L$. When $A \subset \mathbb{R}^n_+$, we have[2]:

$$Co^\infty(A) = \left\{ \bigvee_{x\in A} t_x x : t_x \in [0,1],\ \max_{x\in A}\{t_x\} = 1 \right\}.$$

[2]We denote by $\bigvee_{k\in[m]} x^{(k)}$ the least upper bound of $x^{(1)}, \ldots, x^{(m)} \in \mathbb{R}^n$, that is:

$$\bigvee_{k\in[m]} x^{(k)} = \Big(\max\{x_1^{(1)}, \ldots, x_1^{(m)}\}, \ldots, \max\{x_n^{(1)}, \ldots, x_n^{(m)}\} \Big).$$

In [7] it is shown that in this case $Co^{\infty}(A) = Lim_{p\to\infty} Co^p(A)$, where Lim stands for the Kuratowski-Painlevé limit. Moreover it is also proved that the sequence is Haussdorf convergent.

$\mathbb{B}$-convex sets have a number of remarkable properties. In particular they are linked to the linear convex hull operator Co^p. If $L \subset \mathbb{R}^n_+$ is $\mathbb{B}$-convex, then:

$$L = Lim_{p\to\infty} Co^p(L) = \bigcap_{p=0}^{\infty} Co^p(L).$$

For all $S \subset \mathbb{R}^n_+$, let us denote $Co^{\infty}(S) = Lim_{p\to\infty} Co^p(S)$. Properties of this operation are developed in [7]. In particular for all $S \subset \mathbb{R}^n_+$, $Co^{\infty}(S)$ is $\mathbb{B}$-convex. In the case where S is a subset of $\mathbb{R}^n$, let us denote $Co^{\infty}(S) = Ls_{p\to\infty} Co^p(S)$. Along this line, an extended definition of $\mathbb{B}$-convexity was proposed over $\mathbb{R}^n$ in [7]. A subset C of $\mathbb{R}^n$ is $\mathbb{B}$-convex, if for all finite parts A of C, $Co^{\infty}(A) \subset C$.

1.2. An algebraic structure extended to the whole Euclidean vector space

In [6] it was shown that for all $x, y \in \mathbb{R}$ one has:

$$\lim_{p\to+\infty} x \overset{p}{+} y = \begin{cases} x & \text{if} \quad |x| > |y| \\ \frac{1}{2}(x+y) & \text{if} \quad |x| = |y| \\ y & \text{if} \quad |x| < |y|. \end{cases} \tag{8}$$

Along this line one can introduce the binary operation $\boxplus$ for all $x, y \in \mathbb{R}$ by:

$$x \boxplus y = \lim_{p\to+\infty} x \overset{p}{+} y, . \tag{9}$$

Though the operation $\boxplus$ does not satisfy associativity, it can be extended by constructing a non-associative algebraic structure which returns to a given n-tuple a real value. For all $x \in \mathbb{R}^n$ and all subsets I of $[n]$, let us consider the map $\xi_I[x] \colon \mathbb{R} \to \mathbb{Z}$ defined for all $\alpha \in \mathbb{R}$ by

$$\xi_I[x](\alpha) = \mathrm{Card}\{i \in I : x_i = \alpha\} - \mathrm{Card}\{i \in I : x_i = -\alpha\}. \tag{10}$$

This map measures the symmetry of the occurrences of a given value α in the components of a vector x.

For all $x \in \mathbb{R}^n$, let $\mathcal{J}_I(x)$ be a subset of I defined by

$$\mathcal{J}_I(x) = \Big\{j \in I : \xi_I[x](x_j) \neq 0\Big\} = I \backslash \big\{j : \xi_I[x](x_j) = 0\big\}. \tag{11}$$

$\mathcal{J}_I(x)$ is called *the residual index set* of x. It is obtained by dropping from I all the i's such that $\mathrm{Card}\{j \in I : x_j = x_i\} = \mathrm{Card}\{j \in I : x_j = -x_i\}$.

For all positive natural numbers n and for all subsets I of $[n]$, let $\digamma_I \colon \mathbb{R}^n \to \mathbb{R}$ be the map defined for all $x \in \mathbb{R}^n$ by

$$\digamma_I(x) = \begin{cases} \max_{i\in\mathcal{J}_I(x)} x_i & \text{if} \quad \xi_I[x]\big(\max_{i\in\mathcal{J}_I(x)} |x_i|\big) > 0 \\ \min_{i\in\mathcal{J}_I(x)} x_i & \text{if} \quad \xi_I[x]\big(\max_{i\in\mathcal{J}_I(x)} |x_i|\big) < 0 \\ 0 & \text{if} \quad \xi_I[x]\big(\max_{i\in\mathcal{J}_I(x)} |x_i|\big) = 0. \end{cases} \tag{12}$$

where $\xi_I[x]$ is the map defined in (10) and $\mathcal{J}_I(x)$ is the residual index set of x. The operation that takes an n-tuple $(x_1, ..., x_n)$ of $\mathbb{R}^n$ and returns a single real element $\digamma_I(x_1, ..., x_n)$ is called a n-ary extension of the binary operation $\boxplus$ for all natural numbers $n \geq 1$ and all $x \in \mathbb{R}^n$, if I is a nonempty subset of $[n]$. Then, for all n-tuples $x = (x_1, ..., x_n)$, one can define the operation:

$$\boxplus_{i\in I} x_i = \lim_{p\to\infty} \sum_{i\in I}^{\varphi_p} x_i = \digamma_I(x). \tag{13}$$

This operation encompasses as a special case the binary operation defined in equation (8). Clearly, for all $(x_1, x_2) \in \mathbb{R}^2$

$$\boxplus_{i\in\{1,2\}} x_i = x_1 \boxplus x_2.$$

For example, if $x = (-3, -2, 3, 3, 1, -3)$, we have $F_{[6]}(-3, -2, 3, 3, 1, -3) = F_{[2]}(-2, 1) = -2 = \boxplus_{i\in[6]} x_i$. There are some basic properties that can be inherited from the above algebraic structure. We briefly summarize some basic properties: (i) If all the elements of the family $\{x_i\}_{i\in I}$ are mutually non symmetrical, then:

$$\boxplus_{i\in I} x_i = \arg\max_{\lambda} \big\{|\lambda| : \lambda \in \{x_i\}_{i\in I}\big\}; \tag{14}$$

(ii) For all $\alpha \in \mathbb{R}$, one has: $\alpha\Big(\boxplus_{i\in I} x_i\Big) = \boxplus_{i\in I}(\alpha x_i)$; (iii) Suppose that $x \in \epsilon\mathbb{R}^n_+$ where ϵ is $+1$ or -1. Then $\boxplus_{i\in I} x_i = \epsilon \max_{i\in I}\{\epsilon x_i\}$; (iv) We have $|\boxplus_{i\in I} x_i| \leq \boxplus_{i\in I} |x_i|$; (v) For all $x \in \mathbb{R}^n$:

$$\Big[x_i \boxplus \big(\boxplus_{j\in I\setminus\{i\}} x_j\big)\Big] \in \Big\{0, \boxplus_{j\in I} x_j\Big\} \quad \text{and} \quad \boxplus_{i\in I} x_i = \boxplus_{i\in I} \Big[x_i \boxplus \big(\boxplus_{j\in I\setminus\{i\}} x_j\big)\Big].$$

The algebraic structure $(\mathbb{R}, \boxplus, \cdot)$ can be extended to $\mathbb{R}^n$. Suppose that $x, y \in \mathbb{R}^n$, and let us denote $x \boxplus y = (x_1 \boxplus y_1, ..., x_n \boxplus y_n)$. Moreover, let us consider m vectors $x^{(1)}, ..., x^{(m)} \in \mathbb{R}^n$, and define

$$\boxplus_{j\in[m]} x^{(j)} = \Big(\boxplus_{j\in[m]} x_1^{(j)}, ..., \boxplus_{j\in[m]} x_n^{(j)}\Big). \tag{15}$$

Along this line an extended definition of $\mathbb{B}$-convexity other $\mathbb{R}^n$ was suggested in [6]. In the remainder, we say that a subset C of $\mathbb{R}^n$, is $\mathbb{B}^\sharp$-*convex* if for all $x, y \in C$ and all $t \in [0,1]$, $x \boxplus ty \in C$. Equivalently a set $C \subset \mathbb{R}^n$ is said to be a $\mathbb{B}^\sharp$-*convex set* if for all $x, y \in C$, $Co^\infty(\{x,y\}) \subset C$. This implies that a $\mathbb{B}$-convex subset of $\mathbb{R}^n$ is $\mathbb{B}^\sharp$-convex. Moreover, it was also established in [6] that C is $\mathbb{B}^\sharp$-convex if and only if for all $x^{(1)}, ..., x^{(m)} \in C$ and all $t \in [0,1]^m$ such that $\max_{i\in[m]} t_i = 1$, we have $\boxplus_{i\in[m]} t_i x^{(i)} \in C$. In the case where $C \subset \mathbb{R}^n_+$, we have

$$\boxplus_{i\in[m]} t_i x^{(i)} = \bigvee_{i\in[m]} t_i x^{(i)} \in C, \tag{16}$$

which is an equivalent characterization of $\mathbb{B}$-convexity over $\mathbb{R}^n_+$. Clearly, we retrieve the definition provided in [7] when $C \subset \mathbb{R}^n_+$. A subset C of $\mathbb{R}^n$ is $\mathbb{B}$-convex and *fixed* if for all subsets S of C, $Co^\infty(S) \subset C$. Moreover, a fixed $\mathbb{B}$-convex set is stable and $\mathbb{B}$-convex. This definition implies that $Co^\infty(C) = C$ and consequently a fixed $\mathbb{B}$-convex set is closed. This definition also implies that a fixed $\mathbb{B}$-convex set C is $\mathbb{B}^\sharp$-convex, since $x, y \in C$ implies that $\{x, y\} \in C$ and $Co^\infty(\{x,y\}) \subset C$.

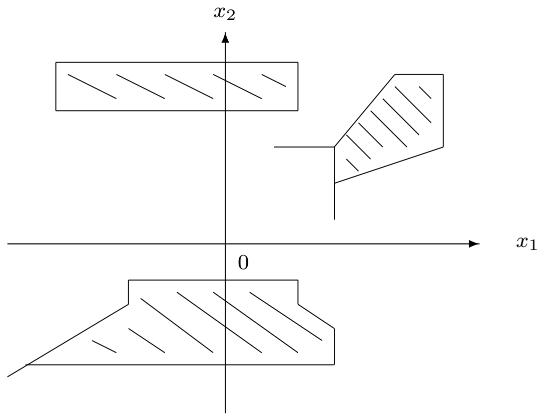

Figure 1.1: Examples of $\mathbb{B}^\sharp$-convex sets.

2. Dual forms and structure of half-spaces

2.1. Duality

This section presents the algebraic properties induced by an isomorphism between a scalar field and a set and their implications on convexity. Most of the results have been pointed in details by Avriel [4] and Ben Tal [5]. A norm $\|\cdot\|$ yields another norm induced by the algebraic operations $\overset{p}{+}$ and $\cdot$. The map $\|\cdot\|_{\varphi_p} : \mathbb{R}^n \to \mathbb{R}$ defined by $\|x\|_{\varphi_p} = \varphi_p^{-1}(\|\Phi_p(x)\|)$ is a norm over $\mathbb{R}^n$ endowed with the operations $\overset{p}{+}$ and $\cdot$. Since φ_p is continuous over $\mathbb{R}$, the topological structure is the same. Along this line it is natural to define a scalar product. If $\langle\cdot,\cdot\rangle$ is an inner product over $\mathbb{R}^n$, then there exists a symmetric bilinear form $\langle\cdot,\cdot\rangle_{\varphi_p} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by:

$$\langle x, y\rangle_{\varphi_p} = \varphi_p^{-1}\big(\langle\Phi_p(x), \Phi_p(y)\rangle\big) \tag{17}$$

Equivalently:

$$\langle x, y\rangle_{\varphi_p} = \big(\sum_{i\in[n]} x_i^{2p+1} y_i^{2p+1}\big)^{\frac{1}{2p+1}}. \tag{18}$$

The dual characterization of a Φ_p-convex set is now formulated using the support function, which is based upon the inner product $\langle\cdot,\cdot\rangle_{\varphi_p}$. Let $y \in \mathbb{R}^n$. Then the *functional support of a convex set* $C \subset \mathbb{R}^n$ is defined as the map $h_C^{\varphi_p} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ with

$$h_C^{\varphi_p}(y) = \sup \big\{\langle y, x\rangle_{\varphi_p} : x \in C\big\}. \tag{19}$$

Now, let us denote $\big[\langle y, \cdot\rangle_{\varphi_p} \le \lambda\big] = \big\{x \in \mathbb{R}^n : \langle y, x\rangle_{\varphi_p} \le \lambda\big\}$. The following properties are immediate:

(a) The functional support is $h_C^{\varphi_p}(y) = \varphi_p\big(h_{\Phi_p^{-1}(C)}(\Phi_p^{-1}(y))\big)$.

(b) If C is closed then:

$$C = \bigcap_{y\in\mathbb{R}^n} \big[\langle y, \cdot\rangle_{\varphi_p} \le h_C^{\varphi_p}(y)\big], \tag{20}$$

Again to simplify the notations used in the paper let us denote

$$\langle\cdot,\cdot\rangle_{\varphi_p} = \langle\cdot,\cdot\rangle_p \ \text{ and } \ h^p = h^{\varphi_p}. \tag{21}$$

2.2. On some idempotent and non-associative dual forms

In the following the operation $\langle\cdot,\cdot\rangle_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined for all $x, y \in \mathbb{R}^n$ by $\langle x, y\rangle_\infty = \boxplus_{i\in[n]} x_i y_i$. Let $\|\cdot\|_\infty$ be the Tchebychev norm defined by $\|x\|_\infty = \max_{i\in[n]} |x_i|$. It is established in [6] that for all $x, y \in \mathbb{R}^n$, we have: (i) $\sqrt{\langle x, x\rangle_\infty} = \|x\|_\infty$; ($ii$) $|\langle x, y\rangle_\infty| \le \|x\|_\infty \|y\|_\infty$; ($iii$) for all $\alpha \in \mathbb{R}$, $\alpha\langle x, y\rangle_\infty = \langle \alpha x, y\rangle_\infty = \langle x, \alpha y\rangle_\infty$.

In the following, we say that a map $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathbb{B}$-*form* if there exists some $a \in \mathbb{R}^n$ such that:

$$f(x) = \boxplus_{i\in[n]} a_i x_i = \langle a, x\rangle_\infty = \lim_{p\to\infty} \langle a, x\rangle_p. \tag{22}$$

In the following the dual relation between $\mathbb{B}^\sharp$-convex sets and the maps of the form $x \mapsto \langle a, x\rangle_\infty$, where $a \in \mathbb{R}^n$, is analyzed.

The function above is depicted in Figure 2.2. The points satisfying $\langle a, x\rangle_\infty = 0$ are represented by the thick line. In the following, for all subsets A of $\mathbb{R}^n$, cl(A) and int(A) stand for the closure and the interior of A.

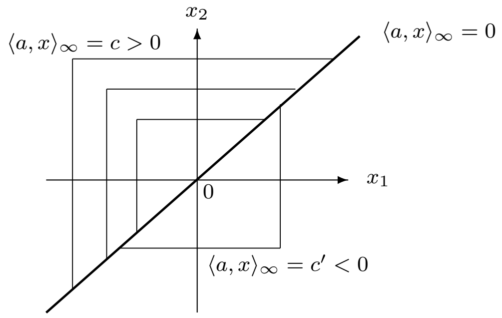

Figure 2.1:
The level lines of the form $\langle a, .\rangle_\infty$

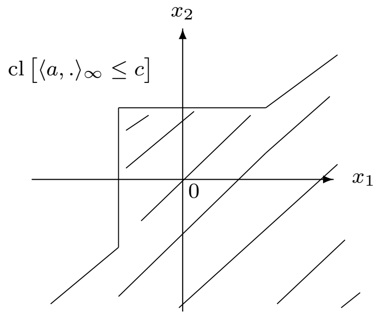

Figure 2.2:
The domain cl $\big[\langle a, .\rangle_\infty \leq c\big]$

In the following, for all maps $f\colon \mathbb{R}^n \to \mathbb{R}$ and all real numbers c, the notation $[\,f \leq c\,]$ stands for the set $f^{-1}(\,]-\infty, c]\,)$. Similarly, $[\,f < c\,]$ stands for $f^{-1}(\,]-\infty, c[\,)$ and $[\,f \geq c\,] = [\,-f \leq -c\,]$. The next result analyze some important topological properties of these sets.

Proposition 2.1. *For all $\mathbb{B}$-forms $f\colon \mathbb{R}^n \to \mathbb{R}$ and all $c \in \mathbb{R}$, the following properties hold:*

(a) $\;\mathrm{cl}\big[f \leq c\big] = \mathrm{cl}\big[f < c\big]$

(b) $\;\mathbb{R}^n \backslash \mathrm{int}\,[f \geq c\,] = \mathrm{cl}\big[f \leq c\big]$

(c) $\;\mathbb{R}^n \backslash \mathrm{cl}\,[f \leq c\,] = \mathrm{int}\,[f \geq c\big]$.

Proof: (a) If $[f \leq c]$ is an empty set, this property is immediate. Suppose that it is a nonempty set. Since it is obvious that $\big[f < c\big] \subset \big[f \leq c\big]$, we have $\mathrm{cl}\big[f < c\big] \subset \mathrm{cl}\big[f \leq c\big]$. Let us show the converse. Let $a \in \mathbb{R}^n$ such that $f(x) = \langle a, x\rangle_\infty$ and assume that $x \in \mathrm{cl}\big[f \leq c\big]$. By definition there exists a sequence $\big\{x^{(k)}\big\}_{k\in\mathbb{N}}$ such that for all k $f(x^{(k)}) = \langle a, x^{(k)}\rangle_\infty \leq c$ and $\lim_{k\to\infty} x^{(k)} = x$. Let us construct the sequence $\big\{y^{(k)}\big\}_{k\in\mathbb{N}}$ defined for $i = 1, ..., n$ by:

$$y_i^{(k)} = \begin{cases} x_i^{(k)} - \frac{1}{k} & \text{if } a_i > 0 \\ x_i^{(k)} + \frac{1}{k} & \text{if } a_i < 0 \\ \quad x_i^{(k)} & \text{if } a_i = 0. \end{cases}$$

It is easy to see that $f(y^{(k)}) = \langle a, y^{(k)}\rangle_\infty < c$ for any $k \in \mathbb{N}$. But $x = \lim_{k\to\infty} x^{(k)} = \lim_{k\to\infty} y^{(k)}$, and consequently $x \in \mathrm{cl}\big[f < c\big]$ which proves (a).
(b) We have, using (a), $\mathbb{R}^n \backslash \mathrm{int}\big[f \geq c\big] = \mathrm{cl}\big(\mathbb{R}^n \backslash [f \geq c]\big) = \mathrm{cl}\big[f < c\big] = \mathrm{cl}\,[f \leq c\big]$ and (b) is proved. The proof of (c) is similar. $\square$

For all real numbers α and all $x \in \mathbb{R}^n$, $B_\infty(x, \alpha[$ and $B_\infty(x, \alpha]$ respectively denote the open and closed ball centered at x of radius α, with respect to the Tchebychev norm $\|\cdot\|_\infty$. The next results are concerned with the topological regularity of the subsets $\big[f \leq c\big]$.

Proposition 2.2. *For all $\mathbb{B}$-forms $f\colon \mathbb{R}^n \to \mathbb{R}$, and all $c \in \mathbb{R}$, if the subset $\big[f \le c\big]$ is nonempty then it has a nonempty interior. Moreover*

$$\big[f \le c\big] \subset \mathrm{cl}\big(\mathrm{int}[f \le c]\big) \text{ and } \mathrm{int}\big(\mathrm{cl}[f \le c\,]\big) \subset \big[f \le c\big].$$

Proof: Suppose that $x \in \big[f \le c\big]$. All we need to prove is that there is a sequence $\{x^{(k)}\}_{k\in\mathbb{N}} \subset \mathrm{int}\big[f \le c\big]$ such that $\lim_{k\to\infty} x^{(k)} = x$. Let $a \in \mathbb{R}^n$ such that $f(x) = \langle a, x\rangle_\infty$.

Let us consider the sequence $\big\{x^{(k)}\big\}_{k\in\mathbb{N}}$ defined for $i = 1, ..., n$ by:

$$x_i^{(k)} = \begin{cases} x_i^{(k)} - \frac{2}{k} & \text{if } a_i \ge 0 \\ x_i + \frac{2}{k} & \text{if } a_i < 0. \end{cases} \tag{23}$$

Moreover, for all natural numbers k, let us fix $\bar{x}^{(k)} \in \mathbb{R}^n$ such that

$$\bar{x}_i^{(k)} = \begin{cases} x_i - \frac{1}{k} & \text{if } a_i \ge 0 \\ x_i + \frac{1}{k} & \text{if } a_i < 0. \end{cases} \tag{24}$$

Clearly, we have both $\langle a, x^{(k)}\rangle_\infty < c$ and $\langle a, \bar{x}^{(k)}\rangle_\infty < c$. By construction, for all positive natural numbers k, $\|x^{(k)} - \bar{x}^{(k)}\|_\infty = \frac{1}{k}$. Let K_a be the cone of all the vectors u in $\mathbb{R}^n$ such that $u_i \ge 0$ if $a_i \ge 0$ and $u_i \le 0$ otherwise. The cone K_a defines a partial order over $\mathbb{R}^n$. From (23) and (24), $\bar{x}^{(k)}$ is the supremum element of the cube $B_\infty\big(x^{(k)}, \frac{1}{k}\big]$. It follows that,

$$B_\infty\big(x^{(k)}, \frac{1}{k}\big] \subset \bar{x}^{(k)} - K_a.$$

Moreover for all $u \in \bar{x}^{(k)} - K_a$, $\langle a, u\rangle_\infty \le \langle a, \bar{x}^{(k)}\rangle_\infty < c$. However, this implies that, for all k, $x^{(k)} \in B_\infty\big(x^{(k)}, \frac{1}{2k}\big[\subset \mathrm{int}\big[f \le c\big]$ and since $\lim_{k\to\infty} x^{(k)} = x$, this ends the proof. To prove the second inclusion, note that using the same arguments yields $\big[f < c\big] \subset \mathrm{cl}\big(\mathrm{int}[f < c]$. Since this property is independent of the sign of f, we have for all c, $\big[f > c\big] \subset \mathrm{cl}\big(\mathrm{int}[f > c]\big)$. Therefore $\mathbb{R}^n \backslash \big[f \le c\big] \subset \mathrm{cl}\big(\mathrm{int}\big(\mathbb{R}^n\backslash[f \le c]\big)\big)$. Hence $\mathbb{R}^n\backslash\big[f \le c\big] \subset \mathbb{R}^n\backslash\, \mathrm{int}\big(\mathrm{cl}\big([f \le c]\big)\big)$. Consequently, $\mathrm{int}\big(\mathrm{cl}\big([f \le c]\big) \subset \big[f \le c\big]$, which proves the second statement. □

For all $u, v \in \mathbb{R}$, let us define the binary operation

$$u \,\bar{\smile}\, v = \begin{cases} u & \text{if } |u| > |v| \\ \min\{u, v\} & \text{if } |u| = |v| \\ v & \text{if } |u| < |v|. \end{cases}$$

An elementary calculus shows that $u \boxplus v = \frac{1}{2}\Big[u \,\bar{\smile}\, v - \big[(-u) \,\bar{\smile}\, (-v)\big]\Big]$.

Similarly one can introduce a symmetrical binary operation defined for all $u, v \in \mathbb{R}$, let us define the binary operation

$$u \overset{+}{\smile} v = \begin{cases} u & \text{if} \quad |u| \ > \ |v| \\ \max\{u, v\} & \text{if} \quad |u| \ = \ |v| \\ v & \text{if} \quad |u| \ < \ |v|. \end{cases}$$

Equivalently, one has: $u \overset{+}{\smile} v = -\big[(-u) \overset{-}{\smile} (-v)\big]$. This means that we equivalently have $u \boxplus v = \frac{1}{2}\Big[(u \overset{-}{\smile} v) + (u \overset{+}{\smile} v)\Big]$

Given m elements $u_1, ..., u_m$ of $\mathbb{R}$, not all of which are 0, let I_+, respectively I_-, be the set of indices for which $0 < u_i$, respectively $u_i < 0$. One can then write $u_1 \overset{-}{\smile} \cdots \overset{-}{\smile} u_m = (\overset{-}{\smile}_{i \in I_+} u_i) \overset{-}{\smile} (\overset{-}{\smile}_{i \in I_-} u_i) = (\max_{i \in I_+} u_i) \overset{-}{\smile} (\min_{i \in I_-} u_i)$ from which we have

$$u_1 \overset{-}{\smile} \cdots \overset{-}{\smile} u_m = \begin{cases} \max_{i \in I_+} u_i & \text{if} \quad I_- = \emptyset \text{ or } \max_{i \in I_-} |u_i| < \max_{i \in I_+} u_i \\ \min_{i \in I_-} u_i & \text{if} \quad I_- = \emptyset \text{ or } \max_{i \in I_+} u_i < \max_{i \in I_-} |u_i| \\ \min_{i \in I_-} u_i & \text{if} \qquad \max_{i \in I_-} |u_i| = \max_{i \in I_+} u_i. \end{cases} \tag{25}$$

Let us define a *lower* $\mathbb{B}$-*form* on $\mathbb{R}^n$ as a map $g : \mathbb{R}^n \to \mathbb{R}$ such that for all $(x_1, ..., x_n) \in \mathbb{R}^n_+$,

$$g(x_1, ..., x_n) = \langle a, x \rangle^-_\infty = a_1 x_1 \overset{-}{\smile} \cdots \overset{-}{\smile} a_n x_n. \tag{26}$$

It was established in [6] that for all $c \in \mathbb{R}$, $g^{-1}\left(\,]-\infty, c]\right) = \{x \in \mathbb{R}^n : g(x) \le c\}$ is closed. It follows that a $\mathbb{B}$-form is lower semi-continuous. It was established in [10] that $g^{-1}\left(\,]-\infty, c]\right) \cap \mathbb{R}^n_+$ is a $\mathbb{B}$-halfspace, that is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ whose the complement in $\mathbb{R}^n_+$ is also $\mathbb{B}$-convex.

Similarly, one can define an *upper* $\mathbb{B}$-*form* as a map $h \colon \mathbb{R}^n \to \mathbb{R}$ such that, for all $(x_1, ..., x_n) \in \mathbb{R}^n$,

$$h(x_1, ..., x_n) = \langle a, x \rangle^+_\infty = a_1 x_1 \overset{+}{\smile} \cdots \overset{+}{\smile} a_n x_n. \tag{27}$$

For all $x \in \mathbb{R}^n$, we clearly, have the following identities

$$\langle a, x \rangle^+_\infty = -\langle a, -x \rangle^-_\infty \text{ and } \langle a, x \rangle^-_\infty = -\langle a, -x \rangle^+_\infty. \tag{28}$$

The largest (smallest) lower (upper) semi-continuous minorant (majorant) of a map f is said to be the lower (upper) semi-continuous regularization of f. In the next statements it is shown that the lower (upper) $\mathbb{B}$-forms are the lower (upper) semi-continuous regularized of the $\mathbb{B}$-form.

Proposition 2.3. *[6] Let g be a lower $\mathbb{B}$-form defined by $g(x_1, ..., x_n) = a_1 x_1 \overset{-}{\smile} \cdots \overset{-}{\smile} a_n x_n$, for some $a \in \mathbb{R}^n$. Then g is the lower semi-continuous regularization of the map $x \mapsto \langle a, x \rangle_\infty = \boxplus_{i \in [n]} a_i x_i$.*

The following corollary is then immediate.

Corollary 2.4. *Let h be an upper $\mathbb{B}$-form defined by $h(x_1,...,x_n) = a_1x_1 \overset{+}{\smile} \cdots \overset{+}{\smile} a_nx_n$, for some $a \in \mathbb{R}^n$. Then h is the upper semi-continuous regularization of the map $x \mapsto \langle a, x\rangle_\infty = \boxplus_{i\in[n]} a_ix_i$.*

The main objective of this paper is to analyze dual properties of $\mathbb{B}$-convex sets over $\mathbb{R}^n$. To do that the $\mathbb{B}$-forms play a crucial role. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a $\mathbb{B}$-form.

Proposition 2.5. *Let f^- and f^+ be respectively the lower and upper semi-continuous regularized of f over $\mathbb{R}^n$. For all $c \in \mathbb{R}$,*

$$\mathrm{cl}\,\big[f \le c\big] = \big[f^- \le c\big] \text{ and } \mathrm{cl}\,\big[f \ge c\big] = \big[f^+ \ge c\big].$$

Proof: By definition $f^- \le f$. Therefore, $[f \le c] \subset [f^- \le c]$. Since f^- is lower semi-continuous, we obtain $\mathrm{cl}\,[f \le c] \subset [f^- \le c]$. Let us prove the converse inclusion. Suppose that $x \in [f^- \le c]$. Let $a \in \mathbb{R}^n$ be a vector such that for all $x \in \mathbb{R}^n$ $f^-(x) = a_1x_1 \overset{-}{\smile} \cdots \overset{-}{\smile} a_nx_n$. Suppose that $x \in [f^- \le c]$. All we need to prove is that $x \in \mathrm{cl}\,[f \le c]$. We consider three cases:

(i) $f^-(x) = \max_{i\in[n]} a_ix_i$. From the definition of f^-, $f^-(x) = f(x)$. Therefore $f(x) \le c$, and $x \in \mathrm{cl}\,[f \le c]$.

(ii) $f^-(x) = \min_{i\in[n]} a_ix_i$, and $|\min_{i\in[n]} a_ix_i| > |\max_{i\in[n]} a_ix_i|$. From the definition of f^-, we also have $f^-(x) = f(x)$, $f(x) \le c$, and $x \in \mathrm{cl}\,[f \le c]$.

(iii) $f^-(x) = \min_{i\in[n]} a_ix_i$, and $|\min_{i\in[n]} a_ix_i| = |\max_{i\in[n]} a_ix_i|$. Let $I_+ = \{i \in [n] : a_ix_i \ge 0\}$. Let us construct the sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ such that for all $k \ge 1$,

$$x_i^{(k)} = \begin{cases} x_i - \frac{1}{k} & \text{if} \quad i \in I_+ \text{ and } a_i > 0 \\ x_i + \frac{1}{k} & \text{if} \quad i \in I_+ \text{ and } a_i < 0 \\ x_i & \text{otherwise.} \end{cases}$$

By construction, $f(x^{(k)}) = f^-(x^{(k)}) \le c$ for all k. Since $\lim_{k\to\infty} x^{(k)} = x$, we deduce that $x \in \mathrm{cl}\,[f \le c]$. Therefore, the converse inclusion is established, which ends the proof. $\square$

The next result is an immediate consequence of the characterization of the closed $\mathbb{B}$-halfspaces established in [10]. Recall that a $\mathbb{B}$-halfspace of $\mathbb{R}^n_+$ is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ whose the complement is also $\mathbb{B}$-convex.

Corollary 2.6. *For all closed $\mathbb{B}$-halfspaces $H \subset \mathbb{R}^n_+$ having a nonempty interior there exists a dual $\mathbb{B}$-form $f\colon \mathbb{R}^n \to \mathbb{R}$ and a real number c such that*

$$H = \mathrm{cl}\,[f \le c] \cap \mathbb{R}^n_+.$$

Moreover, for all $\mathbb{B}$-forms f and all $c \in \mathbb{R}$ the subsets $\mathrm{cl}\,\big[f \le c\big] \cap \mathbb{R}^n_+$ is $\mathbb{B}$-halfspace of $\mathbb{R}^n_+$ whose the complement is $[f > c] \cap \mathbb{R}^n_+ = \mathrm{int}\,[f \ge c] \cap \mathbb{R}^n_+$.

Proof: From [10], if H is a $\mathbb{B}$-halfspace of $\mathbb{R}^n_+$ having a nonempty interior, then there is a map $g\colon \mathbb{R}^n \to \mathbb{R}$ defined by $g(x_1, ..., x_n) = a_1 x_1 \overline{\smile} \cdots \overline{\smile} a_n x_n$, for some $a \in \mathbb{R}^n$ and some $c \in \mathbb{R}$ such that $H = [g \leq c] \cap \mathbb{R}^n_+$. However, g is the lower semi-continuous regularized of the map $f\colon x \mapsto \langle a, x\rangle_\infty$. From Proposition (2.5), $H = \mathrm{cl}[f \leq c] \cap \mathbb{R}^n_+$. Suppose now that f is a $\mathbb{B}$-form defined over $\mathbb{R}^n$. Let f^- be the lower semi-continuous regularized of f. From [10], $[f^- \leq c] \cap \mathbb{R}^n_+$ is a $\mathbb{B}$-halfspace of $\mathbb{R}^n_+$. Since $\mathrm{cl}\,[f \leq c] = [f^- \leq c]$, the result of the second statement follows. Finally, we have shown in Lemma 2.1.(b) that $\mathbb{R}^n \backslash [f \leq c] = \mathrm{int}[f \geq c]$. Hence $\mathbb{R}^n_+ \backslash [f \leq c] = \mathrm{int}[f \geq c] \cap \mathbb{R}^n_+$, which ends the proof. $\square$

Remark 2.7. In general, given a $\mathbb{B}$-form $f : x \mapsto \langle a, x\rangle_\infty$, the domain $[f \leq c] \cap \mathbb{R}^n_+$ is not a $\mathbb{B}$-convex set. Assume that $a = (1, -1, 1)$ and let us consider the vectors $x = (1, 1, 0)$ and $y = (0, 1, 1)$. We have $\langle a, x\rangle_\infty = 0$ and $\langle a, y\rangle_\infty = 0$. However, $x \vee y = (1, 1, 1)$ and $\langle a, x \vee y\rangle_\infty = 1$. Consequently the subset $[f \leq 0]$ is not a $\mathbb{B}$-convex set. However, it is easy to prove that $x \vee y \in \mathrm{cl}\,[f \leq c]$. This is shown by considering the sequence $\{z^{(k)}\}_{k\in\mathbb{N}}$ such that, for all natural numbers $k > 1$, $z^{(k)} = (1, 1+\frac{1}{k}, 1)$. This implies that $\langle a, z^{(k)}\rangle_\infty = -1-\frac{1}{k} < -1$. Thus, for all $k > 1$, $z^{(k)} \in [f \leq 0]$, and, consequently, since $\lim_{k\to\infty} z^{(k)} = x \vee y = (1, 1, 1)$, we deduce that $x \vee y \in \mathrm{cl}\,[f \leq 0]$.

Before establishing the next proposition, we derive some intermediary properties.

Lemma 2.8. *For all $x, y \in \mathbb{R}^n$,*

$$\text{if} \quad \|x \boxplus y\|_\infty < \Big|\mathop{\boxplus}\limits_{i\in[n]} x_i\Big|, \qquad \text{then} \quad \Big(\mathop{\boxplus}\limits_{i\in[n]} x_i\Big) \boxplus \Big(\mathop{\boxplus}\limits_{i\in[n]} y_i\Big) = 0.$$

Proof: We first establish that for all $\alpha, \beta \in \mathbb{R}$, $|\alpha \boxplus \beta| < |\alpha|$ implies that $\alpha \boxplus \beta = 0$. If $\alpha \neq -\beta$, then $\max\{|\alpha|, |\beta|\} = |\alpha \boxplus \beta|$. Consequently $|\alpha| > |\alpha \boxplus \beta|$, implies that $\max\{|\alpha|, |\beta|\} > |\alpha \boxplus \beta|$. Consequently, $\alpha = -\beta$. For all $u \in \mathbb{R}^n$, let us denote

$$I(u) = \{i \in [n] : |u_i| > \|x \boxplus y\|_\infty\}.$$

We first prove that $I(x) = I(y)$. By definition, for all $i \in I(x)$, $|x_i| > \|x \boxplus y\|_\infty = \max_{j\in[n]} |x_j \boxplus y_j|$, which implies that for all $i \in I(x)$, we have $|x_i| > |x_i \boxplus y_i|$. Therefore, for all $i \in I(x)$, $x_i \boxplus y_i = 0$ and $|x_i| = |y_i|$. Consequently, $I(x) \subset I(y)$. Moreover, if $|y_i| > \max_{j\in[n]} |x_j \boxplus y_j|$, then $x_i \boxplus y_i = 0$, and we have the converse inclusion $I(y) \subset I(x)$. Therefore, we have $I(x) = I(y)$ and $x_i = -y_i$ for all $i \in I(x) = I(y)$. Since $\big|\boxplus_{i\in[n]} x_i\big| > \|x \boxplus y\|_\infty$, one has $\boxplus_{i\in[n]} x_i \neq 0$. Thus, there is some $i_0 \in [n]$, such that $\boxplus_{i\in[n]} x_i = x_{i_0}$. Since by hypothesis $\|x \boxplus y\|_\infty < \big|\boxplus_{i\in[n]} x_i\big|$, we have $i_0 \in I(x)$. Since $x_i = -y_i$ for all $i \in I(x) = I(y)$, we have $y_{i_0} = -x_{i_0} = -\big(\mathop{\boxplus}_{i\in[n]} x_i\big) = \mathop{\boxplus}_{i\in[n]} y_i$, which ends the proof. $\square$

Proposition 2.9. *For all $\mathbb{B}$-forms $f\colon \mathbb{R}^n \to \mathbb{R}$, and all $c \in \mathbb{R}$ if $x, y \in [f \leq c]$ then, for all $t \in [0,1]$,*

$$x \boxplus ty \in \mathrm{cl}\,[f \leq c] = [f^- \leq c],$$

where f^- is the lower semi-continuous regularized of f.

Proof: Let $x, y \in [f \leq c]$. We need to prove that for all $t \in [0,1]$, $x \boxplus ty \in \mathrm{cl}\,[f \leq c]$. We first assume that $x, y \in [f \leq c]$. By definition, there is some $a \in \mathbb{R}^n$ such that $f(x) = \langle a, x\rangle_\infty$. Let us denote $z = x \boxplus ty$ and

$$I = \arg\max_{i\in[n]} |a_i z_i|.$$

We consider two cases:

(i) There exists some $i_0 \in I$ such that $a_{i_0} z_{i_0} \leq 0$.
Remark that if $a_{i_0} z_{i_0} = 0$, then $a_i z_i = 0$ for all i and $f(z) = 0$. Let us introduce the sequence $\{z^{(k)}\}_{k\in\mathbb{N}}$ of $\mathbb{R}^n$ defined for all positive natural numbers k by:

$$z_i^{(k)} = \begin{cases} z_i & \text{if } i \neq i_0 \\ z_{i_0} - \frac{a_{i_0}}{k} & \text{if } i = i_0. \end{cases}$$

Clearly, $\lim_{k\to\infty} z^{(k)} = z$. For k sufficiently large, we have $|a_{i_0} z_{i_0} - \frac{a_{i_0}{}^2}{k}| = -a_{i_0} z_{i_0} + \frac{a_{i_0}{}^2}{k}$. Thus, for all $i \in [n]\backslash\{i_0\}$

$$|a_{i_0} z_{i_0}^{(k)}| = |a_{i_0} z_{i_0} - \frac{a_{i_0}{}^2}{k}| = -a_{i_0} z_{i_0} + \frac{a_{i_0}{}^2}{k} = |a_{i_0} z_{i_0}| + \frac{a_{i_0}{}^2}{k} > |a_i z_i|.$$

Thus, by definition:

$$\langle a, z^{(k)}\rangle_\infty = a_{i_0}(z_{i_0} - \frac{a_{i_0}}{k}) = a_{i_0} z_{i_0} - \frac{a_{i_0}{}^2}{k} < \langle a, z\rangle_\infty = \langle a, x \boxplus ty\rangle_\infty.$$

In the following, we prove that $\langle a, x\rangle_\infty \boxplus \langle a, ty\rangle_\infty \geq a_{i_0} z_{i_0}$. If $\langle a, x\rangle_\infty < a_{i_0} z_{i_0} = -\max_{i\in[n]} |(a_i x_i) \boxplus (t a_i y_i)|$, then $|\langle a, x\rangle_\infty| > \max_{i\in[n]} |(a_i x_i) \boxplus (t a y_i)|$. From Lemma 2.8, this implies that $\langle a, x\rangle_\infty \boxplus \langle a, ty\rangle_\infty = 0 \geq a_{i_0} z_{i_0}$. Similarly, if $\langle a, ty\rangle_\infty < a_{i_0} z_{i_0}$, then $|\langle a, ty\rangle_\infty| > \max_{i\in[n]} |(a_i x_i) \boxplus (t a y_i)|$ and from Lemma 2.8, one has $\langle a, x\rangle_\infty \boxplus \langle a, ty\rangle_\infty = 0 \geq a_{i_0} z_{i_0}$. Therefore, in any case we have $\langle a, x\rangle_\infty \geq a_{i_0} z_{i_0}$ and $\langle a, ty\rangle_\infty \geq a_{i_0} z_{i_0}$, which implies that $\langle a, x\rangle_\infty \boxplus \langle a, ty\rangle_\infty \geq a_{i_0} z_{i_0}$. Consequently:

$$\langle a, z^{(k)}\rangle_\infty < a_{i_0} z_{i_0} \leq \langle a, x\rangle_\infty \boxplus \langle a, ty\rangle_\infty \leq c \quad \text{for all } k \geq 1.$$

Thus $z^{(k)} \in [f \leq c]$ for all k and since $\lim_{k\to\infty} z^{(k)} = z = x \boxplus ty$ we deduce that $x \boxplus ty \in \mathrm{cl}\,[f \leq c]$.

(ii) $\forall i \in I$, $a_i z_i > 0$. In that case $\langle a, x \boxplus ty\rangle_\infty > 0$, thus

$$\langle a, x \boxplus ty\rangle_\infty = \max\{\langle a, x\rangle_\infty, \langle a, ty\rangle_\infty\}.$$

Hence $x \boxplus ty \in \mathrm{cl}\,[f \leq c]$. Consequently

$$x, y \in [f \leq c] \Longrightarrow x \boxplus ty \in \mathrm{cl}\,[f \leq c],$$

which ends the proof. □

The next result is then an immediate consequence of Proposition 2.2.

Corollary 2.10. *For all $\mathbb{B}$-forms $f\colon \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$, if $x, y \in \mathrm{int}[f^- \leq c]$, then, for all $t \in [0,1]$, $x \boxplus ty \in [f^- \leq c]$. Moreover if $x, y \in \mathrm{int}[f^+ \geq c]$ then $x \boxplus ty \in [f^+ \geq c]$.*

Remark 2.11. In general the subset $[f^- \leq c]$ is not $\mathbb{B}^\sharp$-convex. Suppose for example that $f^-(x) = \langle a, x\rangle_\infty^-$ where $a, x, y \in \mathbb{R}^3$, $a = (1,1,1)$, $x = (3,2,-3)$ and $y = (-3,2,3)$. We have $\langle a, x\rangle_\infty^- = -3$, $\langle a, y\rangle_\infty^- = -3$. It follows that $x, y \in [f^- \leq 1]$. However, $x \boxplus y = (0,2,0)$ and $\langle a, x \boxplus y\rangle_\infty^- = 2$ which implies that $x \boxplus y \notin [f^- \leq 1]$. It is interesting to note that $x \notin \mathrm{int}[f^- \leq 1]$. For example, for all $\epsilon > 0$, let us denote $x_\epsilon = (3,2,-3+\epsilon)$. We have $f^-(x_\epsilon) = 3$ and therefore $x_\epsilon \notin [f^- \leq 1]$. Since $\lim_{\epsilon \to 0} x_\epsilon = x$, we deduce that $x \notin \mathrm{int}[f^- \leq 1]$. It is also easy to check that $y \notin \mathrm{int}[f^- \leq 1]$. Similarly, in general, the subset $[f^+ \geq c]$ is not $\mathbb{B}^\sharp$-convex. For example in the case above $\langle a, x\rangle_\infty^+ = 3$, $\langle a, y\rangle_\infty^+ = 3$ and $x, y \in [f^+ \geq 3]$ but $\langle a, x \boxplus y\rangle_\infty^+ = 2$.

It is shown below that reversing the sense of the inequalities both in the lower and upper case yields a convex set. In the following we will say that a map $g\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathbb{B}^\sharp$-*quasi concave* if $g^{-1}(\,[c, +\infty[\,)$ is $\mathbb{B}^\sharp$-convex for all $c \in \mathbb{R}$. A map $h : \mathbb{R}^n \to \mathbb{R}$ is $\mathbb{B}^\sharp$-*quasi convex* if $h^{-1}(\,]-\infty, c\,])$ is $\mathbb{B}^\sharp$-convex.

Proposition 2.12. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a $\mathbb{B}$-form and let f^- and f^+ be respectively the lower and upper semi-continuous regularized of f over $\mathbb{R}^n$. Then f^- and f^+ are respectively $\mathbb{B}^\sharp$-quasi concave and $\mathbb{B}^\sharp$-quasi convex.*

Proof: By definition f^- is a lower $\mathbb{B}$-form and there is some $a \in \mathbb{R}^n$ such that $f^-(x) = \langle a, x\rangle_\infty^-$. Let $x, y \in [f^- \geq c]$. We need to prove that for all $t \in [0,1]$, $f^-(x \boxplus ty) \geq c$. Since the case $t = 0$ is obvious, we shall assume that $t > 0$. By construction, there is some $i_\circ, j_\circ, k_\circ \in [n]$ such that

$$a_{i_\circ} x_{i_\circ} = f^-(x), \quad a_{j_\circ} y_{j_\circ} = f^-(y) \quad \text{and} \quad a_{k_\circ}\big(x_{k_\circ} \boxplus t y_{k_\circ}\big) = f^-(x \boxplus ty).$$

This implies that $|a_{i_\circ} x_{i_\circ}| \geq |a_i x_i|$ for all $i \in [n]$ and $|a_{j_\circ} y_{j_\circ}| \geq |a_j y_j|$ for all $j \in [n]$. Let us consider two cases:

(i) $c > 0$. In such a case, $a_{i_\circ} x_{i_\circ} > 0$ and $t a_{j_\circ} y_{j_\circ} > 0$. Therefore, if $a_i x_i < 0$ and $t a_j y_j < 0$ for some $(i,j) \in [n] \times [n]$ then $|a_i x_i| < a_{i_\circ} x_{i_\circ}$ and $|t a_j y_j| < t a_{j_\circ} y_{j_\circ}$. It follows that

$$f^-(x \boxplus ty) = \max\{a_{i_\circ} x_{i_\circ}, t a_{j_\circ} y_{j_\circ}\} \geq \max\{c, tc\} = c.$$

(ii) $c \leq 0$. By construction there is some $k_\circ \in [n]$ such that $a_{k_\circ}(x_{k_\circ} \boxplus t y_{k_\circ}) = f^-(x \boxplus ty)$. If $a_{k_\circ}(x_{k_\circ} \boxplus t y_{k_\circ}) \geq 0$, then, since $c \leq 0$, the result is immediate.

Set $u = ty$. We need to prove that $f^-(x \boxplus u) \geq c$. Since $c \leq 0$, for all $t \in [0,1]$ we have $ct \geq c$. Hence $f^-(u) \geq ct \geq c$. Suppose that

$$f^-(x \boxplus u) = a_{k_\circ}(x_{k_\circ} \boxplus u_{k_\circ}) = (a_{k_\circ}x_{k_\circ}) \boxplus (a_{k_\circ}u_{k_\circ}) = a_{k_\circ}x_{k_\circ} = f^-(x). \qquad (29)$$

Assume that $a_{k_\circ}x_{k_\circ} < c$ and let us show a contradiction. This implies that $|a_{k_\circ}x_{k_\circ}| = -a_{k_\circ}x_{k_\circ} > |c| = -c$. Since $f^-(x) = a_{i_\circ}x_{i_\circ}$, we have $|a_{k_\circ}x_{k_\circ}| \leq |a_{i_\circ}x_{i_\circ}|$. Consequently, $|a_{i_\circ}x_{i_\circ}| > -c$. Hence, since $a_{i_\circ}x_{i_\circ} = f^-(x) \geq c$, we also have $a_{i_\circ}x_{i_\circ} > 0$. Moreover $f^-(x) = a_{i_\circ}x_{i_\circ} > 0$ and $a_{k_\circ}x_{k_\circ} < 0$ implies that $|a_{k_\circ}x_{k_\circ}| \neq |a_{i_\circ}x_{i_\circ}|$. Therefore, $|a_{k_\circ}x_{k_\circ}| < |a_{i_\circ}x_{i_\circ}|$. Thus from (29)

$$|(a_{i_\circ}x_{i_\circ}) \boxplus (a_{i_\circ}u_{i_\circ})| \leq |(a_{k_\circ}x_{k_\circ}) \boxplus (a_{k_\circ}u_{k_\circ})| = |a_{k_\circ}x_{k_\circ}| < |a_{i_\circ}x_{i_\circ}|.$$

From Lemma 2.8, this implies that $(a_{i_\circ}x_{i_\circ}) \boxplus (a_{i_\circ}u_{i_\circ}) = 0$. Hence, $a_{i_\circ}u_{i_\circ} = -a_{i_\circ}x_{i_\circ} < 0$ and $|a_{i_\circ}u_{i_\circ}| > |c| = -c$. Consequently $a_{i_\circ}u_{i_\circ} < c$. However, by hypothesis there is some $j_\circ$ such that $f^-(u) = a_{j_\circ}u_{j_\circ} \geq c$. This implies that $j_\circ \neq i_\circ$, $|a_{j_\circ}u_{j_\circ}| > |a_{i_\circ}u_{i_\circ}| = |a_{i_\circ}x_{i_\circ}|$. Since

$$|(a_{j_\circ}x_{j_\circ}) \boxplus (a_{j_\circ}u_{j_\circ})| \leq |(a_{k_\circ}x_{k_\circ}) \boxplus (a_{k_\circ}u_{k_\circ})| = |a_{k_\circ}x_{k_\circ}| < |a_{j_\circ}x_{j_\circ}|,$$

from Lemma 2.8, we have $(a_{j_\circ}x_{j_\circ}) \boxplus (a_{j_\circ}u_{j_\circ}) = 0$. However, this implies that $|a_{j_\circ}u_{j_\circ}| = |a_{j_\circ}x_{j_\circ}| > |a_{i_\circ}x_{i_\circ}| = |f^-(x)|$, which is a contradiction.

Suppose now that that $f^-(x \boxplus u) = (a_{k_\circ}x_{k_\circ}) \boxplus (a_{k_\circ}u_{k_\circ}) = a_{k_\circ}u_{k_\circ} = f^-(u)$. By symmetry, we then deduce that $f^-(x \boxplus u) \geq c$, which ends the first part of the proof.

Let f^+ be the upper semi-continuous regularized of f. It is an upper $\mathbb{B}$-form. Hence, the map $x \mapsto -f^+(-x)$ is a lower $\mathbb{B}$-form. Thus, for all $c \in \mathbb{R}$, $[-f^+(-x) \geq -c]$ is $\mathbb{B}^\sharp$-convex. Therefore $[f^+(-x) \leq c]$ is $\mathbb{B}^\sharp$-convex. Since the map $x \mapsto -x$ is bijective, the result follows. □

Corollary 2.13. *Let f be a $\mathbb{B}$-form defined over $\mathbb{R}^n$ and let f^- and f^+ be respectively the lower and upper regularized of f. Then for all $c \in \mathbb{R}$, $[f^- > c]$ and $[f^+ < c]$ are $\mathbb{B}^\sharp$-convex.*

Lemma 2.14. *Let A be a finite part of $\mathbb{R}^n$. Let f be a $\mathbb{B}$-form defined over $\mathbb{R}^n$ and let f^- and f^+ be respectively the lower and upper regularized of f. Let $\mathbb{B}^\sharp[A]$ be the smallest $\mathbb{B}^\sharp$-convex set which contains A. Then:*

$$\min\{f^-(a) : a \in A\} = \min\{f^-(x) : x \in \mathbb{B}^\sharp[A]\}$$

and

$$\max\{f^+(a) : a \in A\} = \max\{f^+(x) : x \in \mathbb{B}^\sharp[A]\}.$$

Proof: Suppose that $c = \min\{f^-(a) : a \in A\}$. For all $a \in A$, $f^-(a) \geq c$. Since $[f^- \geq c]$ is $\mathbb{B}^\sharp$-convex, $\mathbb{B}^\sharp[A] \subset [f^- \geq c]$ which proves that $c = \min\{f^-(x) : x \in \mathbb{B}^\sharp[A]\}$. The second part of the proof is similar. □

This property has an immediate implication stated in the following proposition.

Proposition 2.15. *For all $\mathbb{B}$-forms f and all real number $c \in \mathbb{R}$, we have the following properties:*
(a) *The subset* $\operatorname{int}[f \leq c]$ *is* $\mathbb{B}^{\sharp}$*-convex.*
(b) *The subsets* $\operatorname{int}[f^- \leq c]$ *and* $\operatorname{int}[f^+ \geq c]$ *are* $\mathbb{B}^{\sharp}$*-convex.*
(c) *We have*

$$\operatorname{int}[f^- \leq c] = \mathbb{R}^n \backslash [f^+ \geq c] \quad \textit{and} \quad \operatorname{int}[f^+ \geq c] = \mathbb{R}^n \backslash [f^- \leq c].$$

Proof: (a) For all $a \in \mathbb{R}^n$, the map $f^- : x \mapsto \langle a, x \rangle_\infty^-$ is the lower semicontinuous regularized of the map $f : x \mapsto \langle a, x \rangle_\infty$, it follows that $\operatorname{cl}[f \leq c] = [f^- \leq c]$ for all $c \in \mathbb{R}$. From Proposition 2.1.(c), we have

$$\operatorname{int}[f \geq c] = \mathbb{R}^n \backslash \operatorname{cl}[f \leq c] = \mathbb{R}^n \backslash [f^- \leq c] = [f^- > c].$$

However, we have established that $[f^- > c]$ is $\mathbb{B}^{\sharp}$-convex. Therefore $\operatorname{int}[f \geq c]$ is a $\mathbb{B}^{\sharp}$-convex subset of $\mathbb{R}^n$. Since the map $x \mapsto \langle a, x \rangle_\infty$ is homogenous of degree one, the result follows. (b) From Proposition 2.1.(b), we have

$$\mathbb{R}^n \backslash \operatorname{int}[f \geq c] = \operatorname{cl}[f \leq c] = [f^- \leq c].$$

From Proposition 2.2, one has $\operatorname{int}[f \geq c] = \operatorname{int}\big(\operatorname{cl}[f \geq c]\big) = \operatorname{int}[f^+ \geq c]$. Hence, we deduce the first part of the statement. To prove the second part of the statement, let us introduce that map $g = -f$. By construction g is a $\mathbb{B}$-form. Moreover $[g \leq -c] = [f \geq c]$, $g^- = -f^+$ and $g^+ = -f^-$. From the first part of the statement:

$$\operatorname{int}[g^- \leq -c] = \mathbb{R}^n \backslash [g^+ \geq -c].$$

Since $[g^- \leq -c] = [f^+ \geq c]$ and $[g^+ \geq -c] = [f^- \leq c]$, we deduce the second part of the statement. □

In view of the above properties, we say in the following that for all $\mathbb{B}$-forms f, the subset of $\mathbb{R}^n$ $[f \leq c]$ is a *pseudo-halfspace*; the subsets $[f^- \leq c]$ and $[f^+ \leq c]$ are respectively called *lower pseudo-halfspace* and *upper pseudo-halfspace*. Clearly, the restriction to $\mathbb{R}^n_+$ of a lower (upper) pseudo halfspace is an halfspace.

3. Kuratowski-Painlevé limit of linear halfspaces

This section is devoted to analyze the Kuratowski-Painlevé limit of a sequence of half-spaces defined with respect to the algebraic structure involved by the map Φ_p (say a Φ_p-half-space). It is shown that these limits correspond to the lower and upper pseudo halfspaces above defined.

3.1. Limit of linear halfspaces

This section analyzes the geometric deformation of a sequence of Φ_p-halfspace expressed as $H_p = [f_p \leq c_p]$ where, for each natural number p, $f_p\colon \mathbb{R}^n \to \mathbb{R}$ is a Φ_p-linear form defined by $f_p(x) = \langle a^{(p)}, x\rangle_p$, with $(a^{(p)}, c_p) \in \mathbb{R}^n \times \mathbb{R}$. It is shown that the Kuratowski-Painlevé limit of this sequence can be expressed with respect to the $\mathbb{B}$-forms.

Proposition 3.1. *Let f be a $\mathbb{B}$-form defined by $f(x) = \langle a, x\rangle_\infty$ for some point $a \in \mathbb{R}^n\backslash\{0\}$. For all natural numbers p let $f_p\colon \mathbb{R}^n \to \mathbb{R}$ be a map defined by $f_p(x) = \langle a^{(p)}, x\rangle_p$ where $\{a^{(p)}\}_{p\in\mathbb{N}}$ is a sequence of $\mathbb{R}^n\backslash\{0\}$. If there exists an increasing sequence of natural numbers $\{p_q\}_{q\in\mathbb{N}}$ and a sequence $\{c_p\}_{p\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{q\to\infty}(a^{(p_q)}, c_{p_q}) = (a, c)$, then:*

$$[f = c] \subset Li_{q\to\infty}\big[f_{p_q} = c_{p_q}\big] \quad \textit{and} \quad [f \leq c] \subset Li_{q\to\infty}\big[f_{p_q} \leq c_{p_q}\big].$$

Proof: We first assume that for some $x \in \mathbb{R}^n$ we have $f(x) = \langle a, x\rangle_\infty = c$. Since $a \neq 0$, the set $I = \{i \in [n] : a_i \neq 0\}$ is nonempty. Now, since $\lim_{q\to\infty} a^{(p_q)} = a$, there exists $q_0 \in \mathbb{N}$, such that $q \geq q_0 \Longrightarrow a_i^{(p_q)} \neq 0$, $\forall i \in I$. Let us pick some $i_0 \in I$. For $q \geq q_0$, we construct the sequence $\{x^{(p_q)}\}_{q\geq q_0}$, such that:

$$x_i^{(p_q)} = \begin{cases} \frac{1}{a_{i_0}^{(p_q)}} a_{i_0} x_{i_0} \overset{p_q}{-} \frac{1}{a_{i_0}^{(p_q)}} \overset{p_q}{\sum}_{i\notin I} a_i^{(p_q)} x_i & \text{if } i = i_0 \\ \frac{1}{a_i^{(p_q)}} a_i x_i & \text{if } i \in I \text{ and } i \neq i_0 \\ x_i & \text{if } i \notin I. \end{cases}$$

Since $a_i x_i = 0$ for all $i \notin I$, we have:

$$f_{p_q}(x^{(p_q)}) = \langle a^{(p_q)}, x^{(p_q)}\rangle_{p_q} = \overset{\varphi_{p_q}}{\sum_{i\in[n]}} a_i^{(p_q)} x_i^{(p_q)} = \overset{\varphi_{p_q}}{\sum_{i\in I}} a_i x_i = \langle a, x\rangle_{p_q}.$$

Moreover, since for all $i \notin I$, $\lim_{q\to\infty} a_i^{(p_q)} = 0$, and for all $i \in I$ (including i_0), $\lim_{q\to\infty} a_i^{(p_q)} = a_i \neq 0$, we have $\lim_{q\to\infty} \frac{1}{a_{i_0}^{(p_q)}} \overset{p_q}{\sum}_{i\notin I} a_i^{(p_q)} x_i = 0$. Thus, we deduce that $\lim_{q\to\infty} x^{(p_q)} = x$. Thus we can find a sequence $\{x^{(p_q)}\}_{q\in\mathbb{N}}$ such that $\langle a^{(p_q)}, x^{(p_q)}\rangle_{p_q} = \langle a, x\rangle_{p_q}$ and $\lim_{q\to\infty} x^{(p_q)} = x$.

We consider two cases:

(i) $c \neq 0$. We intend to construct a sequence $\{y^{(p_q)}\}_{q\in\mathbb{N}}$ such that for all natural numbers q, $y^{(p_q)} \in [f_{p_q} = c_{p_q}]$ and $\lim_{q\to\infty} y^{(p_q)} = x$. Since $c \neq 0$, and $\langle a, x\rangle_\infty = c$, there exists some $q_0 \in \mathbb{N}$ such that $q \geq q_0 \Longrightarrow c_{p_q} \neq 0$, $\frac{c}{\langle a,x\rangle_{p_q}} > 0$ and $\frac{c_{p_q}}{\langle a,x\rangle_{p_q}} > 0$. Let us denote:

$$y^{(p_q)} = \frac{c_{p_q}}{\langle a, x\rangle_{p_q}} x^{(p_q)}.$$

Thus:

$$f_{p_q}(y^{(p_q)}) = \langle a^{(p_q)}, y^{(p_q)} \rangle_{p_q} = \left\langle a^{(p_q)}, \frac{c_{p_q}}{\langle a, x \rangle_{p_q}} x^{(p_q)} \right\rangle_{p_q} = \frac{c_{p_q}}{\langle a, x \rangle_{p_q}} \langle a, x \rangle_{p_q} = c_{p_q}.$$

Therefore since $\lim_{q\to\infty} y^{(p_q)} = x$, we deduce that $x \in Li_{q\to\infty}[f_{p_q} = c_{p_q}]$ which proves the first statement.

(ii) $c = 0$. For all $q \geq q_0$, let us denote:

$$z^{(p_q)} = x^{(p_q)} \overset{p_q}{+} \Big(\frac{c_{p_q} \overset{p_q}{-} \langle a^{(p_q)}, x^{(p_q)} \rangle_{p_q}}{(\|a^{(p_q)}\|_2^{p_q})^2} \Big) a^{(p_q)},$$

where $\|\cdot\|_2$ is the usual Euclidean norm and $\|\cdot\|_2^{p_q} = \varphi_{p_q}^{-1} \|\Phi_{p_q}(\cdot)\|_2$ the corresponding norm according to the operations $\overset{p_q}{+}$ and $\cdot$. Since, we have $\langle a^{(p_q)}, x^{(p_q)} \rangle_{p_q} = \langle a, x \rangle_{p_q}$, one has

$$\lim_{q\to\infty} \langle a^{(p_q)}, x^{(p_q)} \rangle_{p_q} = \lim_{q\to\infty} \langle a, x \rangle_{p_q} = \langle a, x \rangle_\infty = c = 0 \text{ and } \lim_{q\to\infty} c_{p_q} = 0.$$

Thus, we deduce that

$$\lim_{q\to\infty} \frac{c_{p_q} \overset{p_q}{-} \langle a^{(p_q)}, x^{(p_q)} \rangle_{p_q}}{(\|a^{(p_q)}\|_2^{p_q})^2} = 0.$$

Hence, one has $\lim_{q\to\infty} z^{(p_q)} = \lim_{q\to\infty} x^{(p_q)} = x$. Notice that, it is easy to check that $\left(\|a^{(p_q)}\|_2^{p_q}\right)^2 = \langle a^{(p_q)}, a^{(p_q)} \rangle_{p_q}$. Moreover, we have shown that for all q, $\langle a^{(p_q)}, x^{(p_q)} \rangle_{p_q} = \langle a, x \rangle_{p_q}$. Therefore:

$$\begin{aligned} f_{p_q}(z^{(p_q)}) = \langle a^{(p_q)}, z^{(p_q)} \rangle_{p_q} &= \left\langle a^{(p_q)}, x^{(p_q)} \overset{p_q}{+} \Big(\frac{c_{p_q} \overset{p_q}{-} \langle a^{(p_q)}, x^{(p_q)} \rangle_{p_q}}{(\|a^{(p_q)}\|_2^{p_q})^2} \Big) a^{(p_q)} \right\rangle_{p_q} \\ &= \langle a^{(p_q)}, x^{(p_q)} \rangle_{p_q} \overset{p_q}{+} c_{p_q} \overset{p_q}{-} \langle a^{(p_q)}, x^{(p_q)} \rangle_{p_q} = c_{p_q}. \end{aligned}$$

Thus $x \in Li_{q\to\infty}[f_{p_q} = c_{p_q}]$, which completes the proof. □

Remark 3.2. The result above also implies that:

$$\langle a, x \rangle_\infty = c \implies x \in Ls_{p\to\infty}[f_p = c_p].$$

Moreover

$$\langle a, x \rangle_\infty \leq c \implies x \in Ls_{p\to\infty}[f_p \leq c_p].$$

Consequently, $[f \leq c]$ is a subset of the upper Painlevé-Kuratowski limit of the sequence $\{\, [f_p \leq c_p] \,\}_{p\in\mathbb{N}}$.

Proposition 3.3. *Let f be a $\mathbb{B}$-form defined by $f(x) = \langle a, x\rangle_\infty$ for some $a \in \mathbb{R}^n\backslash\{0\}$. For all natural numbers p let $f_p\colon \mathbb{R}^n \to \mathbb{R}$ be a map defined by $f_p(x) = \langle a^{(p)}, x\rangle_p$ where $\{a^{(p)}\}_{p\in\mathbb{N}}$ is a sequence of $\mathbb{R}^n\backslash\{0\}$. If there exists an increasing sequence of natural numbers $\{p_q\}_{q\in\mathbb{N}}$ and a sequence $\{c_p\}_{p\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{q\to\infty}(a^{(p_q)}, c_{p_q}) = (a, c)$, then:*

$$[f^- \leq c] = \mathrm{cl}\,[f \leq c] = Li_{q\to\infty}\big[f_{p_q} \leq c_{p_q}\big]$$

and

$$[f^+ \geq c] = \mathrm{cl}\,[f \geq c] = Li_{q\to\infty}\big[f_{p_q} \geq c_{p_q}\big],$$

where f^- and f^+ are respectively the lower and upper semi-continuous regularized of f.

Proof: Since $Li_{q\to\infty}[f_{p_q} \leq c_{p_q}]$ is a closed set, from Proposition 3.1, we immediately have $\mathrm{cl}\,[f \leq c] \subset Li_{q\to\infty}[f_{p_q} \leq c_{p_q}]$. Let us show the converse inclusion.

We need to prove that if $x \in Li_{p\to\infty}[f_{p_q} \leq c_{p_q}]$, then $x \in \mathrm{cl}\,[f \leq c]$. If $x \in Li_{p\to\infty}[f_{p_q} \leq c_{p_q}]$, then there is a sequence $\{x^{(p_q)}\}_{q\in\mathbb{N}}$ such that for all q, $\langle a^{(p_q)}, x^{(p_q)}\rangle_{p_q} \leq c_{p_q}$ and $\lim_{q\to\infty} x^{(p_q)} = x$. Let us denote

$$I = \arg\max_{i\in[n]} |a_i x_i|,$$

$I_+ = \{i \in I : a_i x_i > 0\}$ and $I_- = \{i \in I : a_i x_i < 0\}$. For all q, we have the inequalities

$$-n^{\frac{1}{2p_q+1}} \max_{i\in[n]} |a_i^{(p_q)} x_i^{(p_q)}| \leq \langle a^{(p_q)}, x^{(p_q)}\rangle_{p_q} \leq n^{\frac{1}{2p_q+1}} \max_{i\in[n]} |a_i^{(p_q)} x_i^{(p_q)}|. \tag{30}$$

We consider three cases:

(i) $I_- \neq \emptyset$. From (30), $\langle a^{(p_q)}, x^{(p_q)}\rangle_{p_q} \leq c_{p_q}$ for all q, implies that

$$c_{p_q} \geq -n^{\frac{1}{2p+1}} \max_{i\in[n]} |a_i^{(p_q)} x_i^{(p_q)}|.$$

Note that $\lim_{q\to\infty} n^{\frac{1}{2p_q+1}} = 1$. Consequently:

$$c = \lim_{q\to\infty} c_{p_q} \geq -\big(\lim_{q\to\infty} n^{\frac{1}{2p_q+1}}\big)\big(\lim_{q\to\infty} \max_{i\in[n]} |a_i^{(p_q)} x_i^{(p_q)}|\big) = -\max_{i\in[n]} |a_i x_i|.$$

Let $\{x^{(k)}\}_{k\in\mathbb{N}}$ be a sequence such that for all $k \geq 1$:

$$x_i^{(k)} = \begin{cases} x_i + \frac{1}{k} & \text{if } i \in I_- \text{ and } a_i < 0 \\ x_i - \frac{1}{k} & \text{if } i \in I_- \text{ and } a_i > 0 \\ x_i & \text{otherwise.} \end{cases}$$

It follows for all $k \geq 1$, $\langle a, x^{(k)}\rangle_\infty < -\max_{i\in[n]} |a_i x_i| \leq c$. Since $\lim_{k\to\infty} x^{(k)} = x$, we deduce that $x \in \mathrm{cl}\,[f \leq c]$.

(ii) $I_- = \emptyset$ and $I_+ \neq \emptyset$. In such a case $\langle a, x\rangle_\infty = \max_{i\in[n]} a_i x_i = \max_{i\in[n]} |a_i x_i|$. Moreover, $\lim_{q\to\infty}\langle a^{(p_q)}, x^{(p_q)}\rangle_{p_q} = \max_{i\in[n]} |a_i x_i| \leq \lim_{q\to\infty} c_{p_q} = c$. Let $\{x^{(k)}\}_{k\in\mathbb{N}}$ be the sequence such that such that for all $k \geq 1$:

$$x_i^{(k)} = \begin{cases} x_i + \frac{1}{k} & \text{if } i \in I_+ \text{ and } a_i < 0 \\ x_i - \frac{1}{k} & \text{if } i \in I_+ \text{ and } a_i > 0 \\ x_i & \text{otherwise.} \end{cases}$$

It follows that for all $k \geq 1$, $\langle a, x^{(k)}\rangle_\infty < \max_{i\in[n]} |a_i x_i| \leq c$. Since $\lim_{k\to\infty} x^{(k)} = x$, we deduce that $x \in \mathrm{cl}\,[f \leq c]$.

(iii) $I_- = I_+ = \emptyset$. In such a case, for all $i \in [n]$, $a_i x_i = 0$. From (30), $\lim_{q\to\infty}\langle a^{(p_q)}, x^{(p_q)}\rangle_{p_q} = 0$. Therefore $c = \lim_{q\to\infty} c_{p_q} \geq 0$. Let $\{x^{(k)}\}_{k\in\mathbb{N}}$ be the sequence such that for all $k \geq 1$:

$$x_i^{(k)} = \begin{cases} \frac{1}{k} & \text{if } a_i < 0 \\ -\frac{1}{k} & \text{if } a_i > 0 \\ x_i & \text{otherwise.} \end{cases}$$

It follows that for all $k \geq 1$, $\langle a, x^{(k)}\rangle_\infty < 0 \leq c$. Since $\lim_{k\to\infty} x^{(k)} = x$, we deduce that $x \in \mathrm{cl}\,[f \leq c]$ and $Li_{q\to\infty}[f_{p_q} \leq c_{p_q}] \subset \mathrm{cl}\ [f \leq c]$. From Proposition 3.1, we deduce that:

$$Li_{q\to\infty}[f_{p_q} \leq c_{p_q}] = \mathrm{cl}\ [f \leq c].$$

This ends the proof of the first part of the statement. The second part is then immediate, reversing the sense of the inequalities. □

Proposition 3.4. *Let f be a $\mathbb{B}$-form defined by $f(x) = \langle a, x\rangle_\infty$ for some $a \in \mathbb{R}^n\backslash\{0\}$. For all natural numbers p let $f_p\colon \mathbb{R}^n \to \mathbb{R}$ be a map defined by $f_p(x) = \langle a_p, x\rangle_p$ where $\{a^{(p)}\}_{p\in\mathbb{N}}$ is a sequence of $\mathbb{R}^n\backslash\{0\}$. If there exists a sequence $\{c_p\}_{p\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{q\to\infty}(a^{(p)}, c_p) = (a, c)$, then:*

$$Lim_{p\to\infty}\big[f_p \leq c_p\big] = \mathrm{cl}\,\big[f \leq c\big] = \big[f^- \leq c\big]$$

and

$$Lim_{p\to\infty}\big[f_p \geq c_p\big] = \mathrm{cl}\,\big[f \geq c\big] = \big[f^+ \geq c\big].$$

Proof: We need only prove that $Ls_{p\to\infty}[f_p \leq c_p] \subset Li_{p\to\infty}\big[f_p \leq c_p\big]$. If $x \in Ls_{p\to\infty}[f_p \leq c_p]$, then there is an increasing subsequence $\{p_q\}_{s\in\mathbb{N}}$ such that $x \in Li_{q\to\infty}[f_{p_q} \leq c_{p_q}]$. But from the proposition above we know

$$Li_{q\to\infty}[f_{p_q} \leq c_{p_q}] = Li_{p\to\infty}[f_p \leq c_p] = \mathrm{cl}\,[f \leq c].$$

Thus $x \in Li_{p\to\infty}\big[f_p \leq c_p\big]$, and this ends the proof. □

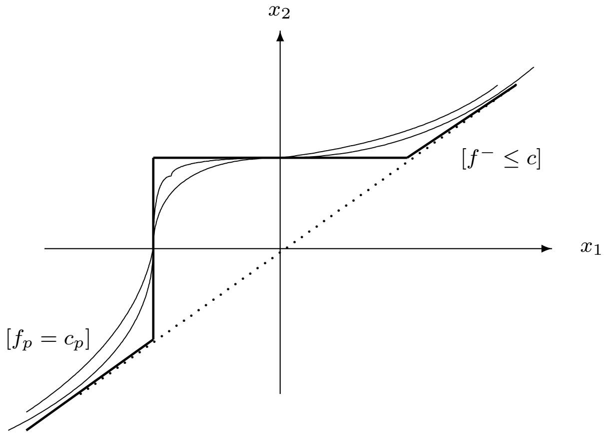

Figure 3.1: Geometric deformation and limit of Φ_p-halfspace.

3.2. Separation properties in limit

It was established in [10] and [9] that if $f\colon x \mapsto \langle a, x\rangle_\infty$ is a $\mathbb{B}$-form and if f^- is the lower semi-continuous regularized of f, then for all $x \in \mathbb{R}^n_+$ and all real numbers c :

$$\text{if } 0 \leq c \text{ then } f^-(x) \leq c \text{ if and only if } \max_{i\in I_+}\{a_i x_i\} \leq \max_{i\in I_-}\{-a_i x_i, c\} \tag{31}$$

and

$$\text{if } c \leq 0 \text{ then } f^-(x) \leq c \text{ if and only if } \max_{i\in I_+}\{a_i x_i, -c\} \leq \max_{i\in I_-}\{-a_i x_i\}, \tag{32}$$

where $I_- = \{i \in [n] : a_i < 0\}$ and $I_+ = \{i \in [n] : a_i > 0\}$.

Theorem 3.5. *If C and D are non-proximate $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ then there exists a $\mathbb{B}$-form $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that*

$$\sup_{x\in C} f^-(x) < \inf_{x\in D} f^-(x).$$

In the following, we use this property to establish a separation result over $\mathbb{R}^n$ which involve a finite collection of dual $\mathbb{B}$-forms.

For all $x, y \in \mathbb{R}^n$, let us denote the Hadamard product $x \boxdot y = (x_1y_1, ..., x_ny_n)$. In the following we say that two vectors $x, y \in \mathbb{R}^n$ are *copositive* if

$$x \boxdot y \in \mathbb{R}^n_+. \tag{33}$$

We say that a subset K of $\mathbb{R}^n$ is copositive if for all $x, y \in K$ one has $x \boxdot y \geq 0$. For all subsets L of $\mathbb{R}^n$, K is copositive and maximal in L if there does not exists a copositive subset $K' \subset L$ which contains K. A n-dimensional ortant in $\mathbb{R}^n$ is copositive and maximal in $\mathbb{R}^n$. Equivalently, a n-dimensional orthant in

$\mathbb{R}^n$ is a subset defined by a system of inequalities: $\epsilon_i x_i \geq 0$ for any $i \in [n]$, where each ϵ_i is $+1$ or -1. A n-dimensional closed orthant K of $\mathbb{R}^n$ can be written $K = \prod_{i=1}^n (\epsilon_i \mathbb{R}_+)$. For all subsets C of $\mathbb{R}^n$, we say that a collection $\mathcal{K}(C)$ of n-dimensional orthant is a *copositive recovering* of C if $C \subset \bigcup_{K \in \mathcal{K}(C)} K$. Moreover, $\mathcal{K}(C)$ is a minimal copositive recovering of C if there does not exists a copositive recovering of C, $\mathcal{K}'(C)$ such that $\mathcal{K}'(C) \subset \mathcal{K}(C)$. In the following $\mathbb{B}^\star$ denotes the set of all the dual $\mathbb{B}$-forms. The following statement comes from the fact that for all $i \in [n]$, the canonical projection $x \mapsto x_i$ is a special $\mathbb{B}$-form which is identical to its lower and upper semi-continuous regularized. Consequently, it is also both an upper and a lower $\mathbb{B}$-form.

Proposition 3.6. *Let C and D be two nonproximate $\mathbb{B}$-convex subsets of $\mathbb{R}^n$. There exist two subsets I and J of $\mathbb{N}$ and a finite subset $\{(f_{i,j}, c_{i,j}) : (i,j) \in I \times J\}$ of $\mathbb{B}^\star \times \mathbb{R}$ such that*

$$C \subset \bigcup_{i \in I} \bigcap_{j \in J} [f^-_{i,j} \leq c_{i,j}] \quad \textit{and} \quad D \subset \bigcap_{i \in I} \bigcup_{j \in J} [f^+_{i,j} \geq c_{i,j}].$$

Proof: Let $\mathcal{K}(C)$ a minimal collection of n-dimensional orthant which contain C. Fix some $K \in \mathcal{K}(C)$. If $K \in \mathcal{K}(D)$, from [10] there is some lower $\mathbb{B}$-form $f^-_{K,K}$ and some real number $c_{K,K}$ such that $C \cap K \subset [f^-_{K,K} \leq c_{K,K}]$ and $D \cap K \subset \mathbb{R}^n \backslash [f^-_{K,K} \leq c_{K,K}]$. However, from Proposition 2.1.(c),

$$\mathbb{R}^n \backslash [f^-_{K,K} \leq c_{K,K}] = \text{int}[f_{K,K} \geq c_{K,K}] \subset \text{cl}[f_{K,K} \geq c_{K,K}] = [f^+_{K,K} \geq c_{K,K}].$$

Moreover, for all $K' \in \mathcal{K}(D)$ there is a canonical dual $\mathbb{B}$-form such that $C \cap K \subset [f^-_{K,K'} \leq c_{K,K'}]$ and $D \cap K' \subset [f^+_{K,K'} \geq c_{K,K'}]$. It follows that

$$C \cap K \subset \bigcap_{K' \in \mathcal{K}(D)} [f^-_{K,K'} \leq c_{K,K'}]$$

and using the fact that $D = \bigcup_{K' \in \mathcal{K}(D)} D \cap K'$, we deduce that

$$D \subset \bigcup_{K' \in \mathcal{K}(D)} [f^+_{K,K'} \geq c_{K,K'}].) \tag{34}$$

However $C = \bigcup_{K \in \mathcal{K}(C)} C \cap K$. Therefore

$$C \subset \bigcup_{K \in \mathcal{K}(C)} \bigcap_{K' \in \mathcal{K}(D)} [f^-_{K,K'} \leq c_{K,K'}].$$

Moreover from (34)

$$D \subset \bigcap_{K \in \mathcal{K}(C)} \bigcup_{K' \in \mathcal{K}(D)} [f^+_{K,K'} \geq c_{K,K'}].$$

Since $\mathcal{K}(C)$ and $\mathcal{K}(D)$ are two finite collections, one can find two subsets I and J of $\mathbb{N}$ respectively isomorphic to $\mathcal{K}(C)$ and $\mathcal{K}(D)$. Thus, there exists a bijection $T\colon I \times J \to \mathcal{K}(C) \times \mathcal{K}(D)$. Now, identifying for all $(i,j) \in I \times J$ each $\mathbb{B}$-form $f_{T(i,j)}$ to $f_{i,j}$ and each $c_{T(i,j)}$ to $c_{i,j}$, we obtain the result. □

In the following, we extend Theorem 3.5 to some class of $\mathbb{B}$-convex sets defined over $\mathbb{R}^n$. Recall that we say that as subset S of $\mathbb{R}^n$ is a fixed $\mathbb{B}$-convex if $Co^\infty(S) = S$. From [7], this implies that all the closed $\mathbb{B}$-convex sets of $\mathbb{R}^n_+$ are fixed, and the result holds for "usual" $\mathbb{B}$-convex sets.

Lemma 3.7. *Let C and D be two closed subsets of $\mathbb{R}^n$. Suppose that D is compact. If $Co^\infty(C) \cap Co^\infty(D) = \emptyset$, then there is some natural number p_0 such that for all $p \geq p_0$, $Co^p(C) \cap Co^p(D) = \emptyset$.*

Proof: Assume that this is not the case and let us show a contradiction. In such a case there is an increasing subsequence $\{p_q\}_{q\in\mathbb{N}}$ such that $Co^{p_q}(C) \cap Co^{p_q}(D) \neq \emptyset$. Just pick some x_{p_q} in $Co^{p_q}(C) \cap Co^{p_q}(D)$ for all $q \in \mathbb{N}$. Since D is compact one can extract a subsequence $\{p_{q_r}\}_{r\in\mathbb{N}}$ such that $\{x_{p_{q_r}}\}_{r\in\mathbb{N}}$ converges to some $\bar{x}$. Therefore $\bar{x} \in Li_{r\to\infty} Co^{p_{q_r}}(C) \cap Li_{r\to\infty} Co^{p_{q_r}}(D) \subset Co^\infty(C) \cap Co^\infty(D)$. But since $Co^\infty(C) \cap Co^\infty(D) \neq \emptyset$ this is a contradiction. □

For all $x \in \mathbb{R}^n$, let $B_\infty(x,d]$ and $C_\infty(x,d)$ respectively denote the ball and the circle of center x and of radius d, with respect to the norm $\|\cdot\|_\infty$.

Lemma 3.8. *Let $\{p_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers. For all nonempty closed subsets C of $\mathbb{R}^n$, and for all $x_0 \in C$, we have*

$$C = Lim_{k\to\infty}\big(B_\infty(x_0, p_k\,] \cap C\big).$$

Proof: Let us show that $Ls_{k\to\infty}\big(B_\infty(x_0,p_k\,] \cap C\big) \subset C$. Let $\{p_{k_l}\}_{l\in\mathbb{N}}$ be an increasing sequence of natural numbers and let $\{x^{(p_{k_l})}\}_{l\in\mathbb{N}}$ be a sequence of $\mathbb{R}^n$ such that $x^{(p_{k_l})} \in B_\infty(x_0, p_{k_l}\,] \cap C$ for all l. Suppose moreover that $\lim_{l\to\infty} x^{(p_{k_l})} = x$. Since C is closed, $x \in C$. Thus $Ls_{k\to\infty}\big(B_\infty(x_0,p_k\,] \cap C\big) \subset C$. Conversely, for all $x \in C$, there is some $k_0 \in \mathbb{N}$ such that for all $k > k_0$, $x \in B_\infty(x_0, p_k\,]$. Let us consider the sequence $\{x^{(p_k)}\}_{k\in\mathbb{N}}$ such that, for all $k > k_0$, $x^{(p_k)} = x$ and $x^{(p_k)} = x_0$ otherwise. This sequence converges to x and, for all k, $x^{(p_k)} \in B_\infty(x_0,p_k\,] \cap C$. Thus $x \in Li_{k\to\infty}\big(B_\infty(x_0,p_k\,] \cap C\big)$. Therefore $Ls_{k\to\infty}\big(B_\infty(x_0,p_k\,] \cap C\big) \subset C \subset Li_{k\to\infty}\big(B_\infty(x_0,p_k\,] \cap C\big) = Lim_{k\to\infty}\big(B_\infty(x_0,p_k\,] \cap C\big) = C$, which completes the proof. □

This property is useful to establish the following separation result.

Proposition 3.9. *Let C and D be two nonempty subsets of $\mathbb{R}^n$. Assume that $Co^\infty(C) \cap Co^\infty(D) = \emptyset$. There exists a $\mathbb{B}$-form $f\colon \mathbb{R}^n \to \mathbb{R}$ and a real number c such that*

$$C \subset [f^- \leq c] \quad \textit{and} \quad D \subset [f^+ \geq c],$$

where f^- and f^+ are respectively the lower and upper semi-continuous regularized of f.

Proof: Let $y_0 \in D$. For all $q \in \mathbb{N}$ the subset $\tilde{D}_q = B_\infty(y_0, q] \cap D$ is compact. By hypothesis $Co^\infty(C) \cap Co^\infty(D) = \emptyset$. Therefore $Co^\infty(C) \cap Co^\infty(\tilde{D}_q) = \emptyset$. From Lemma 3.7, for all q, there exists a natural number η_q such that for all $p \geq \eta_q$ we have

$$Co^p(C) \cap Co^p(\tilde{D}_q) = \emptyset.$$

Consequently, one can find an increasing sequence $\{p_q\}_{q\in\mathbb{N}}$ of natural numbers such that $p_q \geq \eta_q$ for all q. Therefore, for any q, there is also some $(a^{(p_q)}, c_{p_q}) \in C_\infty(0,1) \times \mathbb{R}$ such that

$$\tilde{D}_q \subset Co^{p_q}(\tilde{D}_q) \subset [f_{p_q} \geq c_{p_q}] \quad \text{and} \quad Co^{p_q}(C) \subset [f_{p_q} \leq c_{p_q}],$$

where $f_{p_q}\colon x \mapsto \langle a^{(p_q)}, x\rangle_{p_q}$ for all $x \in \mathbb{R}^n$. Let us show that there is a compact $K \subset C_\infty(0,1) \times \mathbb{R}$ such that for all q, $(a^{(p_q)}, c_{p_q}) \in K$. Let $x_0 \in C$ and $y_0 \in D$, by definition, we have $\langle a^{(p_q)}, x_0\rangle_p \leq c_{p_q}$ and $\langle a^{(p_q)}, y_0\rangle_p \geq c_{p_q}$, therefore since $a^{(p_q)} \in C_\infty(0,1)$ for all q:

$$-n\,\|x_0\|_\infty \leq \langle a^{(p_q)}, x_0\rangle_p \leq c_{p_q} \leq \langle a^{(p_q)}, y_0\rangle_p \leq n\,\|y_0\|_\infty.$$

Set $K = C_\infty(0,1) \times [-n\,\|x_0\|_\infty, n\,\|y_0\|_\infty]$. Thus we deduce that, for all natural numbers q, $(a^{(p_q)}, c_{p_q}) \in K$. Hence, one can extract from $\{(a^{(p_q)}, c_{p_q})\}_{p\in\mathbb{N}} \subset K$, an increasing subsequence $\{p_{q_k}\}_{k\in\mathbb{N}}$, such that $\lim_{k\to\infty} a^{(p_{q_k})} = a$ and $\lim_{k\to\infty} c_{p_{q_k}} = c$. Thus, from Proposition 3.3:

$$Li_{k\to\infty}\tilde{D}_{q_k} \subset Li_{k\to\infty}[f_{p_{q_k}} \geq c_{p_{q_k}}] = \mathrm{cl}\,[f \geq c] = [f^+ \geq c]$$

and

$$C \subset Li_{k\to\infty}[f_{p_{q_k}} \leq c_{p_{q_k}}] = \mathrm{cl}\,[f \leq c] = [f^- \leq c].$$

But, from Lemma 3.8

$$D = Li_{k\to\infty}\tilde{D}_{q_k},$$

thus $D \subset \mathrm{cl}\,[f \geq c]$ and we obtain the result. □

In the following, we use the fact that for all fixed $\mathbb{B}$-convex subsets C of $\mathbb{R}^n$ we have $Co^\infty(C) = C$. The following result is then immediate and is a slight extension of the separation result obtained in [10] where the convex sets were assumed to be non proximal.

Proposition 3.10. *Let C and D be two non empty fixed $\mathbb{B}$-convex sets in $\mathbb{R}^n$ such that $C \cap D = \emptyset$. There exists a $\mathbb{B}$-form $f\colon \mathbb{R}^n \to \mathbb{R}$ and a real number c such that:*

$$C \subset [f^- \leq c] \text{ and } D \subset [f^+ \geq c].$$

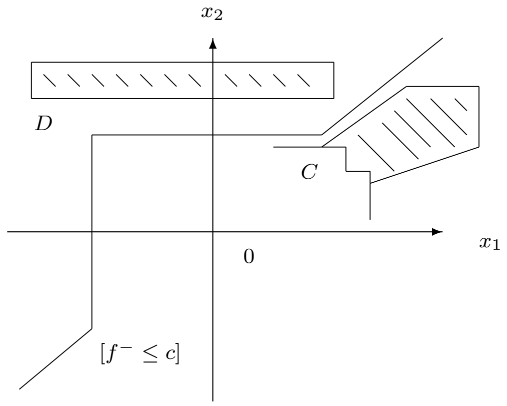

Figure 3.2: Separation property.

In the following we explore the dual structure of fixed $\mathbb{B}$-convex sets. To that end the next result is useful. A subset B of $\mathbb{R}^n$ is a *box* with respect to the usual partial order if there exist $u, v \in \mathbb{R}^n$ such that $u \leq v$ and $B = \{x \in \mathbb{R}^n : u \leq x \leq v\}$.

Lemma 3.11. *A box of $\mathbb{R}^n$ is a fixed $\mathbb{B}$-convex set.*

Proof: All we need to prove is that for all $p \in \mathbb{N}$, $Co^p(B) = B$. Let B be a box of $\mathbb{R}^n$. By definition, there exist $u, v \in \mathbb{R}^n$ such that:

$$B = \big\{x \in \mathbb{R}^n : u \leq x \leq v\big\}.$$

For all $p \in \mathbb{N}$, the map $\varphi_p \colon \lambda \mapsto \lambda^{2p+1}$ is an increasing map. Moreover the reciprocal map $(\varphi_p)^{-1} \colon \lambda \mapsto \lambda^{\frac{1}{2p+1}}$ is also an increasing map. It follows that for all $x, y \in \mathbb{R}^n$, if $x \leq y$, then $\Phi_p(x) \leq \Phi_p(y)$ and ${\Phi_p}^{-1}(x) \leq {\Phi_p}^{-1}(y)$. Therefore, we have

$$\Phi_p(B) = \big\{x \in \mathbb{R}^n : \Phi_p(u) \leq x \leq \Phi_p(v)\big\}.$$

By definition, for all p

$$Co^p(B) = {\Phi_p}^{-1}\, Co\big(\Phi_p(B)\big). \tag{35}$$

However, $\Phi_p(B)$ is a box and, consequently, it is convex which ensures that $Co\big(\Phi_p(B)\big) = \Phi_p(B)$. Hence, from (35) we have $Co^p(B) = {\Phi_p}^{-1}\big(\Phi_p(B)\big) = B$. Therefore $Co^\infty(B) = Ls_{p\to\infty}\, Co^p(B) = B$, which proves that B is fixed. $\square$

Note that it follows that for all $x_0 \in \mathbb{R}^n$, and all $d > 0$, since $B_\infty(x_0, d]$ is a box, it is a fixed $\mathbb{B}$-convex set.

Proposition 3.12. *A nonempty fixed $\mathbb{B}$-convex set is the intersection of all the lower pseudo-halpfspaces which contain it.*

Proof: Let C be a fixed $\mathbb{B}$-convex set. We need to prove that if $y \notin C$, then there exists a lower $\mathbb{B}$-form f^- and a real number c such that

$$C \subset [f^- \leq c] \quad \text{and } f^-(y) > c.$$

Since C is closed if $y \notin C$, then there is some $\epsilon > 0$ such that $B_\infty(y, \epsilon] \cap C = \emptyset$. However, $B_\infty(y, \epsilon]$ is a fixed $\mathbb{B}$-convex set. From Proposition 3.10, there is a dual $\mathbb{B}$-form $f \colon \mathbb{R}^n \to \mathbb{R}$ and some $c \in \mathbb{R}$ such that

$$C \subset [f^- \leq c] \quad \text{ and } B_\infty(y, \epsilon] \subset [f^+ \geq c].$$

This implies that $y \in \operatorname{int}[f^+ \geq c]$. From the topological regularity of $[f \geq c]$ we deduce that $y \in \operatorname{int}[f \geq c] = \mathbb{R}^n \setminus \operatorname{cl}[f \leq c] = \mathbb{R}^n \setminus [f^- \leq c]$. Hence, $f^-(y) > c$ which ends the proof. □

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