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Stability of E-optimal Points for Sequences of Functions

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The main concern of this article is the stability of optimal points for sequences of functions under suitable assumptions about convergence of their epigraphs. The optimality is based on an improvement set given in the value space. Several theorems, obtained in two previous papers, about the existence and stability of optimal points related to sequences of sets in infinite dimensional spaces, are here used in the case of sequences of functions' images. Most part of the present results are new also in the case where the criteria of optimality is a cone or the spaces have finite dimension.

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1. Introduction

The introduction of the concept of improvement set as optimization criteria, made by Chicco et al in [2], has encouraged this line of research in recent years. Some authors have already made extensions of the known theory about optimality with respect to cones, making use of such new criteria. For example, in [4] a characterization of improvement sets, in infinite dimensional spaces, by means of sub-level sets of scalar functions is given and equivalent conditions of optimality through scalarization are presented. In [5], instead, some sufficient conditions, that ensure the stability of optimal points with respect to closed improvement sets having nonempty topological interior, are proved.

To this aim, in that article, the stronger "continuous convergence" is adopted on the sequence of functions defining the optimal control problems. Moreover, through scalarization, the stability of sequences of approximate optimal solu-

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tions is also investigated therein. In [13] and [14], in an infinite dimensional framework, some sufficient conditions about the existence of optimal points and their stability for converging set sequences have been developed. In particular, in [13], in order to obtain that an optimal point of the limit set is the limit of optimal points, an example shows that an assumption like $E + E \subseteq E$, where E denotes the improvement set, is essential, when the origin is a cluster point for E. In [14] another example shows that the "upper" part of the stability inclusion of sets of optimal points fails if both the improvement set and the limit set have empty topological interior. This also happens if the optimality criteria refers to the generalized quasi interior of E (see [1], definition 2.3), and E is a convex cone.

In this paper, after giving some generalization or suitable reformulation of existence results obtained in [14], a sequence of functions on a sequence of domains, with values in a partially ordered normed space Y is considered. The main convergence assumption is that the sequence of epigraphs converges to the epigraph of a limit function in the sense of Mosco in the product space $X \times Y$. An analogous convergence, which reduces to the well known epi-convergence (or gamma-convergence) in the scalar case, has been recently introduced in [19] for similar purposes, but in a finite dimensional context and with cones as optimal criteria. With such kind of convergence both the lower and the upper part of stability inclusions for the sequence of optimal point sets are here obtained. To this aim it has been necessary the help of the recessive compactness property (see Definition 5.1), which is always fulfilled in the finite dimensional case.

Furthermore, by an example, it is proved that the only convergence of epigraphs is not enough to ensure the "upper" part of the optimal points stability. On this fact is based our "errata-corrige" at the end of this paper, linked to the our previous note [12]. This remark can also be applied to a result given in [19]. Finally, by a theorem which make use of the "lower" part of the Attouch-Wets convergence, the upper part of the optimal points stability is stated in the case where the cone or the improvement set is allowed to have empty topological interior. As far as we know, the only existing similar result, but in the case of minima for set sequences, is contained in [11], Theorem 4.5.

The outline of this paper is as follows. After the preliminaries of Section 2, in Section 3, some generalizations of the previous existence results, which permit to enlarge the class of the sets A having the domination property with respect to the improvement set, are given. In Section 4 are presented, with some improvement, the stability results obtained in [14], which are useful in the last section and make use of the existence results of Section 3 (see Remark 4.4(c)). In Section 5, the main results of this article about the stability of sequences of optimal points is given. The main theorems are formulated with somewhat general assumptions, then some sufficient conditions in order to guarantee their validity are given. Moreover, some examples and remarks, pointing out that certain assumptions are essential, are also provided.

2. Preliminaries and definitions

Definition 2.1. Let Y be a topological vector space and $C \subseteq Y$

- C is a *cone* iff $\lambda C \subseteq C$, $\forall \lambda > 0$;
- it is a *pointed cone* iff, additionally, $(cl\, C) \cap (-cl\, C) = \{0\}$, where $cl\, C$ denotes the closure of C;
- C is a *convex cone* iff it is a cone and $C + C \subseteq C$;
- C is a *proper cone* iff it is a cone and $\{0\} \neq C \neq Y$;
- ([16], Definition 3.1) Let C be a convex cone. A subset B of C is a *base* for C iff B is convex and each nonzero element $y \in C$ has a unique representation of the form $y = \lambda b$ for some $\lambda > 0$, $b \in B$.

We denote by $int\, C$ the topological interior of C and by C^* the cone in the dual space Y^* of all the continuous functionals which are non-negative on C.

Throughout this paper we always consider proper cones.

Definition 2.2. (see [2, 4]) If $C \subset Y$ is a cone, a subset $E \subset Y$ is said to be an *improvement set* with respect to C, and write that E is a $C{-}i.s.$, iff

- $E = E + C$;
- $0 \notin E$.

Definition 2.3. If Y is a normed space and $E \subset Y$ is an improvement set, we say that E is *w-closely supported* iff

$$\forall\, \varepsilon > 0\ \exists \text{ a weakly closed set } K_\varepsilon \subset Y \setminus \{0\} \text{ s.t. } E \setminus B(0,\varepsilon) \subseteq K_\varepsilon\,, \tag{1}$$

where $B(0,\varepsilon)$ denotes the open ball of centre 0 and radius ε.

Remark 2.4. Let Y be a normed space. The w-closely supported property of an improvement set $E \subset Y$ is clearly true both when E is weakly closed and in the case where Y has finite dimension and $E \cup \{0\}$ is closed.

As it can be easily verified, if E is a C–$i.s.$ with $int\, E \neq \emptyset$, also $int\, E$ is a C–$i.s.$; so we introduce the following definition of E-optimal and weak E-optimal point.

Definition 2.5. (see [2, 4, 20]) Let Y be a topological vector space. If $a \in A \subseteq Y$ and $E \subset Y$ is a $C{-}i.s.$, we say that a is an *E–optimal point* of A ($a \in O^E(A)$), iff $(a - E) \cap A = \emptyset$. We say that a is a *weak E–optimal point* of A ($a \in O^{intE}(A)$), iff $(a - int\, E) \cap A = \emptyset$.

We recall the notion of Kuratowsky-Painlevé and Mosco sequential convergence for subsets of a topological space.

Definition 2.6. Let Z be a first countable topological space endowed with a topology τ and A_n, $A \subseteq Z$. Let

$$\tau - \liminf A_n = \{x \in Z : \exists\, x_n \xrightarrow{\tau} x, \exists\, \bar{n} \in \mathbb{N} \text{ such that } x_n \in A_n\ \forall\, n \geq \bar{n}\}$$

$$\tau - \limsup A_n = \{x \in Z : \exists\, (n_k)_{k\in\mathbb{N}}, \exists\, x_k \in A_{n_k} \text{ such that } x_k \xrightarrow{\tau} x\}.$$

We say that the sequence $(A_n)_{n\in\mathbb{N}}$ *converges to A in the sense of Kuratowsky-Painlevé* with respect to the topology τ, and we denote it by $A_n \overset{\tau-K}{\longrightarrow} A$ or $A = \tau - K\lim A_n$, if

$$\tau - \liminf A_n = \tau - \limsup A_n = A.$$

In a normed space Y we say that the sequence $(A_n)_{n\in\mathbb{N}}$ *converges to A in the sense of Mosco*, and we denote it as $A_n \overset{M}{\longrightarrow} A$, iff

$$s - \liminf A_n = w - \limsup A_n = A\,,$$

where s and w denote the strong and the weak topology, respectively.

3. Existence of E-optimal points and E-domination property

We define a property analogous to the *domination property* (see [11], [8], [9]), generalized to improvement sets.

Definition 3.1. If Y is a normed space, $C \subset Y$ a given cone and $E \subseteq Y$ a $C-i.s.$, we say that a set $A \subseteq Y$ satisfies the *E-domination property*, iff for every $a \in A$ the set $a - (E \cup \{0\}) \cap O^E(A)$ is nonempty.

Moreover, we say that A satisfies the *section-domination property*, iff for every $a \in A$ the set $O^E(A \cap a - (E \cup \{0\}))$ is nonempty.

It is easy to check that when $E \cup \{0\}$ is closed with respect to the sum of its elements the section-domination property is equivalent to the E-domination property.

Definition 3.2. Let (Y, s) be a locally convex topological vector space and $E \subset Y \setminus \{0\}$ be such that $E + E \subseteq E \cup \{0\}$. We consider in Y the reflexive and transitive binary relation defined by $y \preceq_E y'$ iff $y' - y \in E \cup \{0\}$ and we denote $y \prec_E y'$ when $y' - y \in E$. We say that

(a) a net $(y_\tau)_{\tau\in\mathcal{T}}$ is
   - *E-decreasing* [*E-increasing*] iff $y_\tau \preceq_E y_{\tau'}$ [$y_{\tau'} \preceq_E y_\tau$] whenever $\tau' < \tau$;
   - *E-monotone* iff it is E-increasing or E-decreasing;
   - *strictly E-decreasing* iff $y_\tau \prec_E y_{\tau'}$, whenever $\tau' < \tau$.

(b) A set $A \subseteq Y$ is E-complete if it has no covers of the form $\bigcup_{\tau\in\mathcal{T}} (a_\tau - E \cup \{0\})^c$, where $(a_\tau)_{\tau\in\mathcal{T}}$ is a strictly E-decreasing net in A.

(c) We say that Y has the $(w-s)$ *property* with respect to E, iff every weakly convergent net, which is E-monotone, is also strongly convergent.

(d) If $S \subset Y$ is a given set, we say that $y \in Y$ belongs to the *monotone lower E-closure* of S iff it belongs to

$$cl_{s,E}(S) = \{y \in Y : \exists \text{ an } E\text{-decreasing net } (y_\sigma)_{\sigma\in\Sigma} \text{ in } S, y_\sigma \overset{s}{\to} y\}.$$

We say that S is $s.\ell.-E$ closed, iff $cl_{s,E}(S) = S$.

Remark 3.3. As already seen in [14], Proposition 20, the Lebesgue spaces L^p, $1 \leq p < \infty$, have the $(w{-}s)$ property with respect to the cone $E \cup \{0\}$ of non-negative functions.

In Theorem 19 of [14] the assumption of closedness of A can be weakened with the $s.\ell.{-}E$ closure, obtaining the following proposition.

Proposition 3.4. *Let Y be a Banach space, $C \subset Y$ a pointed closed convex cone, $E \subset Y$ a $C{-}i.s.$ such that*

(i) $E \cup \{0\}$ *is weakly closed and* $E + E \subseteq E \cup \{0\}$*;*

(ii) Y *has the* $(w{-}s)$ *property with respect to* E*.*

Then every $A \subseteq Y$, $A \neq \emptyset$, which is weakly relatively compact and $s.\ell.{-}E$ closed, is E-complete.

Proof. The proof is the same of Theorem 19 of [14] □

The following two corollaries are analogous to Theorems 21 and 22 of [14]. They follow directly from Proposition 3.4, but the closure assumption is in the $s.\ell.{-}E$ sense. The proof is the same as in the aforesaid theorems of [14].

Corollary 3.5. *Let Y be a Banach space, $C \subset Y$ a given cone, $E \subset Y$ a $C{-}i.s.$ We consider a set $A \subseteq Y$ such that A is $s.\ell.{-}E$ closed and such that*

(i) $E \cup \{0\}$ *is weakly closed and* $E + E \subseteq E \cup \{0\}$*;*

(ii) Y *has the* $(w{-}s)$ *property with respect to* E*.*

Then for every $a \in A$ such that $a - (E \cup \{0\}) \cap A$ is weakly relatively compact, it results

$$(a - (E \cup \{0\})) \cap O^E(A) \neq \emptyset \,. \tag{2}$$

Corollary 3.6. *Let Y be a Banach space, $C \subset Y$ a given cone, $E \subset Y$ a $C{-}i.s.$ We consider a set $A \subseteq Y$ such that $A + C$ is $s.\ell.{-}E$ closed, and*

(i) $E \cup \{0\}$ *is weakly closed, and* $E + E \subseteq E \cup \{0\}$*;*

(ii) Y *has the* $(w{-}s)$ *property with respect to* E*.*

Moreover, one of the following assumptions is also fulfilled:

(iii) Y *is a reflexive Banach space and* $a \in A$ *is such that* $a - (E \cup \{0\}) \cap A$ *is bounded; or*

(iii') $a \in A$ *is such that* $a - (E \cup \{0\}) \cap (A + C)$ *is weakly relatively compact.*

Then (2) *holds.*

The next definition is introduced in order to motivate, by the next remark, the notion of monotone E-closure in Corollary 3.6.

Definition 3.7. (See [1]) Let Y be a Hausdorff topological vector space and $C \subset Y$ a convex set. A point $u \in C$ is said to belong to the *quasi relative interior* of C, denoted by qri$C$, iff $cl(cone(C - u))$ is a subspace in Y, where $cone(C - u) := \{\lambda v : \, v \in C - u, \, \lambda \geq 0\}$.

Remark 3.8. If Y has finite dimension and $int\,C = \emptyset$, also when $E = C \setminus \{0\}$, there are simple examples showing that $cl_{s,E}(A+C) \subsetneqq cl(A+C)$. It happens, for example, if $A = \{(x,y) \in \mathbb{R}^2 : y \leq -\frac{1}{x}, x \in]0,\infty[\}$, and $C = \{0\} \times [0,\infty[$.

If C is closed and convex with $int\,C \neq \emptyset$ and E is a C–*i.s.* satisfying the condition $E+E \subseteq E \cup \{0\}$, it is possible to prove that $cl_{s,E}(A+C) = cl(A+C)$ in any dimension.

The following example shows that it is possible to find an infinite dimensional space Y, a closed convex and pointed cone C, with non-empty quasi relative interior and a set $A \subset Y$ such that $cl_{s,E}(A+C) \subsetneqq cl(A+C)$.

Let $Y = L^2([0,\infty[)$, $A = \{u_k,\, k \in \mathbb{Z}_+\}$, $u_k(t) = \begin{cases} 1 & \text{if } t \in [k, k+\frac{1}{k^2}[\\ 0 & \text{otherwise for } t \in [0,\infty[\end{cases}$, $C = \{u \in L^2([0,\infty[: u \geq 0,\ u \text{ is decreasing on } [0,\infty[\}$, $E = C \setminus \{0\}$. Clearly $0 \in cl(A+C)$, however there exists no E-decreasing sequence $(y_n)_{n\in\mathbb{N}}$ in $A+C$ such that $y_n \to 0$. In fact, if $y_n = u_{k(n)} + c_n$, where $u_{k(n)} \in A$, $c_n \in C$ and $y_n \to 0$, since $y_n \geq u_{k(n)} \geq 0$, it results $u_{k(n)} \to 0$. Up to a subsequence, we can assume that $k(n) < k(n+1)$ for each $n \in \mathbb{N}$. Therefore, if $(y_n)_{n\in\mathbb{N}}$ were E-decreasing, we would get $c_n(t) \geq c_{n+j}(t) + 1 \geq 1$ on $[k(n+j), k(n+j) + \frac{1}{k^2(n+j)}[$ for every $j \geq 1$. Hence $c_n(t) \geq 1$ on $[0,\infty[$ and $c_n \notin L^2([0,\infty[)$. It is also clear that the cone C is pointed, convex and closed.

Moreover C has non empty quasi relative interior, because, as we now prove, all the points in $\tilde{C} = \{u \in C\ : u \in C^1([0,\infty[), u(t) > 0,\ u'(t) < 0\ \forall t \in [0,\infty[\}$ are in qri$C$.

To prove this, it is enough to show that, if $u \in \tilde{C}$ is given, and $\tau \in \mathbb{R}$, v_1, $v_2 \in cl(cone(C-u))$, then $v_1 + v_2, \tau v_1 \in cl(cone(C-u))$. About the sum, it is enough to verify that $v_1 + v_2 \in cone(C-u)$, when v_1, $v_2 \in cone(C-u)$. Let $v_1 = \lambda_1(w_1 - u)$, $v_2 = \lambda_2(w_2 - u)$, where w_1, $w_2 \in C$, $\lambda_1, \lambda_2 \geq 0$. Since we can suppose λ_1, $\lambda_2 > 0$, we can write

$$v_1 + v_2 = (\lambda_1 + \lambda_2)\big(\frac{\lambda_1}{\lambda_1 + \lambda_2} w_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} w_2 - u\big),$$

which clearly belongs to $cone(C-u)$. Analogously, it is obvious that $\tau v \in cone(C-u)$, if $v \in cone(C-u)$ and $\tau \geq 0$. We consider now the case where $\tau < 0$. If $u \in \tilde{C}$, $\lambda > 0$ are given, let us first assume that $v \in C^1([0,\infty[) \cap C$ be such that $v(t) = 0$, when $t \geq T > 0$ and we show that

$$\tau\lambda(v - u) \in cone(C-u),$$

by proving that there exist $\alpha > 0$ and $w \in C$ such that $\tau\lambda(v-u) = \alpha(w-u)$. In fact, by letting $\delta := \max\{u'(t) : t \in [0,T]\}$, with a simple calculation, since $\delta < 0$, it suffices to choose $\alpha > 0$ great enough to satisfy

$$(\alpha - \tau\lambda)\delta - \tau\lambda \min\{v'(t) : t \in [0,T]\} \leq 0 \quad \text{and} \quad w = u + \frac{\tau\lambda}{\alpha}(v-u).$$

In the case where $\tau < 0$, $\zeta \in cl(cone(C-u))$, we get $\zeta = s - \lim_{j\to\infty} \lambda_j(v_j - u)$, where $\lambda_j > 0$, $v_j \in C$. For any fixed $j \in \mathbb{N}$, we can approximate v_j by a bounded function $w_j : \mathbb{R} \to \mathbb{R}$, which is zero outside of $[-1, T_j]$ for some $T_j > 0$, decreasing on $[-1, \infty[$, $w_{j|[0,\infty[} \in C$, and $\|\lambda_j(v_j - u) - \lambda_j(w_j - u)\|_{L^2([0,\infty[)} < \frac{1}{2j}$. Moreover, by standard arguments, it is possible to approximate w_j with a function $z_j \in C^1(\mathbb{R})$ having compact support, such that $z_{j|[0,\infty[} \in C$ and $\|\lambda_j(w_j - u) - \lambda_j(z_j - u)\|_{L^2([0,\infty[)} < \frac{1}{2j}$. Hence $\zeta = s - \lim_{j\to\infty} \lambda_j(z_{j|[0,\infty[} - u)$ in the $L^2([0,\infty[)$ norm. But, from the first case considered, $\tau\lambda_j(z_{j|[0,\infty[} - u) \in cone(C-u)$, so $\tau\zeta \in cl(cone(C-u))$.

4. Convergence of E-optimal points

Lemma 29 of [14] also holds in the following slightly different form, with a similar proof, which we present for the sake of completeness.

Lemma 4.1. *Let Y be a reflexive Banach space, $C \subset Y$ a given cone, $E \subset Y$ a $C-$i.s. and $A_n, A \subseteq Y$ satisfying the following hypotheses*

(i) $E \cup \{0\}$ *is weakly closed and E is w-closely supported;*

(ii) $A \subseteq s - lim\,inf A_n$ *and* (ii$'$) $w - lim\,sup A_n \subseteq A + C$;

(iii) *if* $\big((a_{n_k} - E) \cap A_{n_k}\big) \setminus B(a_{n_k}, \varepsilon) \neq \emptyset$ *for some* $\varepsilon > 0$ *and a subsequence* $(n_k)_{k\in\mathbb{N}}$, *where* $a_{n_k} \in A_{n_k}$ *and* $(a_{n_k})_{k\in\mathbb{N}}$ *is bounded, then there exists* $y_k \in \big((a_{n_k} - E) \cap A_{n_k}\big) \setminus B(a_{n_k}, \varepsilon)$ *for every* $k \in \mathbb{N}$, *such that* $(y_k)_{k\in\mathbb{N}}$ *is bounded.*

Then, if $a \in O^E(A)$, $a_n \in A_n$ and $a_n \xrightarrow{s} a$, the following properties hold

$$\forall\, \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } (a_n - E) \cap A_n \subseteq B(a, \varepsilon) \quad \forall\, n \geq n_\varepsilon \tag{3}$$

$$\text{if } \ w_n \in (a_n - E \cup \{0\}) \cap A_n, \ \text{ then } \quad w_n \to a \quad \text{strongly} \tag{4}$$

Proof. We assume, by contradiction, that $y_k \in \big((a_{n_k} - E) \cap A_{n_k}\big) \setminus B(a, \varepsilon_o)$ for some $\varepsilon_o > 0$ and some subsequence $(n_k)_{k\in\mathbb{N}}$. Since it results $\|y_k - a_{n_k}\| \geq \varepsilon_o/2$ for large k, thanks to assumption (iii), we may suppose $(y_k)_{k\in\mathbb{N}}$ to be bounded. Up to a subsequence we have $y_k \rightharpoonup y_o \in Y$ and, thanks to our assumptions, $y_o \in a - (E \cup \{0\}) \cap (A + C)$. Moreover, by the assumption on E and (1), there exists a weakly closed set $K_{\varepsilon_o} \subset Y \setminus \{0\}$ such that $(E - a_{n_k}) \setminus B(-a_{n_k}, \varepsilon_o/2) \subset K_{\varepsilon_o} - a_{n_k}$. Since $-y_k \in (E - a_{n_k}) \setminus B(-a, \varepsilon_o) \subset (E - a_{n_k}) \setminus B(-a_{n_k}, \varepsilon_o/2)$ for large k, by passing to the limit, we get $-y_o \in K_{\varepsilon_o} - a$. This implies that $y_o \neq a$, because $0 \notin K_{\varepsilon_o}$. Then, for some $e \in E$, $a' \in A$, $c \in C$, it results $y_o = a - e = a' + c$, but $A \ni a' = a - (e + c)$ is impossible, because $e + c \in E$, while, from the definition of $O^E(A)$, is $a - E \cap A = \emptyset$.

The proof of (4) easily follows from (3), because from any subsequence of $(w_n)_{n\in\mathbb{N}}$, which we still denote by $(w_n)_{n\in\mathbb{N}}$, we can obtain a further subsequence $(w_{n_k})_{k\in\mathbb{N}}$ such that $\|w_{n_k} - a\| < 1/k$. $\square$

A reinforcement of assumption (ii') in the previous lemma, permits a slightly weaker assumption in place of (iii), which will be useful in the applications. We state this in the following lemma, whose proof is analogous to the previous one.

Lemma 4.2. *Let Y be a reflexive Banach space, $C \subset Y$ a given cone, $E \subset Y$ a $C-$i.s. and $A_n, A \subseteq Y$ satisfying the following hypotheses*

(i) $E \cup \{0\}$ *is weakly closed and E is w-closely supported;*

(ii) $A \subseteq s - lim\, inf\, A_n$;

(a) $w - lim\, sup(A_n + C) \subseteq A + C$;

(b) *if $\big((a_{n_k} - E) \cap (A_{n_k} + C)\big) \setminus B(a_{n_k}, \varepsilon) \neq \emptyset$ for some $\varepsilon > 0$ and a subsequence $(n_k)_{k\in\mathbb{N}}$, where $a_{n_k} \in A_{n_k}$ and $(a_{n_k})_{k\in\mathbb{N}}$ is bounded, then there exists $y_k \in \big((a_{n_k} - E) \cap (A_{n_k} + C)\big) \setminus B(a_{n_k}, \varepsilon)$ for every $k \in \mathbb{N}$, such that $(y_k)_{k\in\mathbb{N}}$ is bounded.*

Then, if $a \in O^E(A)$, $a_n \in A_n$ and $a_n \xrightarrow{s} a$, the following properties hold

$$\forall\, \varepsilon > 0 \quad \exists\, n_\varepsilon \in \mathbb{N} \text{ such that } (a_n - E) \cap (A_n + C) \subseteq B(a, \varepsilon) \quad \forall\, n \geq n_\varepsilon \tag{5}$$

$$\text{if } \; w_n \in (a_n - E \cup \{0\}) \cap (A_n + C), \quad \text{then} \quad w_n \to a \quad \text{strongly.} \tag{6}$$

Now we give sufficient conditions which imply the assumption (iii) of Lemma 4.1 or (b) of Lemma 4.2.

Definition 4.3. Let Y be a normed space and $(S_n)_{n\in\mathbb{N}}$ a sequence of subsets.

(I) We say that $(S_n)_{n\in\mathbb{N}}$ satisfies the *asymptotic condition* iff there exist cones C_1, C_2 in Y and $w \in Y$ such that:

$$\begin{aligned} &-E \subseteq C_1,\ S_n \subseteq C_2 + w \ \text{ for every } \ n \in \mathbb{N}, \\ &\lim_{r\to\infty} d(C_1 \setminus B(0,r), C_2 \setminus B(0,r)) = +\infty, \end{aligned} \tag{7}$$

where $d(A, B)$ denotes the usual distance between the sets A and B defined as $d(A, B) := \inf\{\|a - b\|,\ a \in A,\ b \in B\}$.

(II) We say that $(S_n)_{n\in\mathbb{N}}$ satisfies the *path connection condition* iff the set $y - (E \cup \{0\}) \cap S_n$ is path connected for every $n \in \mathbb{N}$ and $y \in S_n$.

(III) If $C \subset Y$ is a given cone, we say that $(S_n)_{n\in\mathbb{N}}$ satisfies the *C-path connection condition* iff $y - (E \cup \{0\}) \cap (S_n + C)$ is path connected for every $n \in \mathbb{N}$, $y \in S_n$.

Remark 4.4.

(a) If Y is a normed space and $(A_n)_{n\in\mathbb{N}}$ a sequence of subsets of Y which satisfies the asymptotic condition, then $(A_n)_{n\in\mathbb{N}}$ satisfies the assumption (iii) of the Lemma 4.1.

In fact we prove that the sequence $(a_n - E) \cap A_n$, $n \in \mathbb{N}$ is bounded, if $a_n \in A_n$ and $(a_n)_{n\in\mathbb{N}}$ is bounded. On the contrary, if $y_k \in (a_{n_k} - E) \cap A_{n_k}$, $\|y_{n_k}\| \to \infty$

for a subsequence $(n_k)_{k\in\mathbb{N}}$, by assumption, there exist $c_k \in C_1$, $c'_k \in C_2$ such that

$$y_{n_k} = a_{n_k} + c_k = w + c'_k \tag{8}$$

Therefore, from (8), $\|c_k\| \to \infty$, $\|c'_k\| \to \infty$, so that (7) implies that $\|c_k - c'_k\| \to \infty$, which is impossible, since $c_k - c'_k = w - a_{n_k}$.

(b) If Y is a normed space and $(A_n)_{n\in\mathbb{N}}$ a sequence of subsets of Y which satisfies the path connection condition, which is true for example when $E \cup \{0\}$ and A_n are convex, then $(A_n)_{n\in\mathbb{N}}$ satisfies the assumption (iii) of Lemma 4.1.

Let $y_k \in \big((a_{n_k} - E) \cap A_{n_k}\big) \setminus B(a_{n_k}, \varepsilon)$ for every $k \in \mathbb{N}$. We can assume $(y_k)_{k\in\mathbb{N}}$ to be unbounded, so, up to a subsequence, we get $\|y_k - a_{n_k}\| \geq 3\varepsilon$ for large k. By the path connection there exists a regular map $\varphi : [0,1] \to Y$ such that $\varphi([0,1]) \subseteq a_{n_k} - (E \cup \{0\}) \cap A_{n_k}$, $\varphi(0) = a_{n_k}$, $\varphi(1) = y_k$, and then $\lim\limits_{t\to 0} \varphi(t) = a_{n_k}$. Consequently, if we let $\bar{t}_k = \inf\{t \in [0,1] : \varphi(t) \notin B(a_{n_k}, 2\varepsilon)\}$ it results $\|\varphi(\bar{t}_k) - a_{n_k}\| = 2\varepsilon$, so that the sequence $\big(\varphi(\bar{t}_k)\big)_{k\in\mathbb{N}}$ is bounded and satisfies, in place of $(y_k)_{k\in\mathbb{N}}$ the condition in (iii) of Lemma 4.1.

(c) If Y is a normed space, $C \subset Y$ is a given cone, and $(A_n)_{n\in\mathbb{N}}$ a sequence of subsets of Y which satisfies the C-path connection condition, which is true for example when $E \cup \{0\}$ and $A_n + C$ are convex, then $(A_n)_{n\in\mathbb{N}}$ satisfies the assumption (b) of Lemma 4.2.

The proof is analogous to the case (b), only with obvious changes.

If Y is a reflexive Banach space, the following two propositions are a generalization of Theorems 31 and 32 in [14], where the local assumption iii) implies the boundedness of the sequence $\big((a_n - E) \cap A_n\big)_{n\in\mathbb{N}}$.

Proposition 4.5. *Let Y be a reflexive Banach space, $C \subset Y$ a given cone, $E \subset Y$ a C–i.s. and $A_n, A \subseteq Y$ satisfying* (ii), (ii') *of Lemma* 4.1, *and*

(i) *E is weakly closed;*

(x) $(A_n)_{n\in\mathbb{N}}$ *satisfies the asymptotic condition of Definition* 4.3 (I) *or the path connection condition of Definition* 4.3 (II).

Then $O^E(A) \subseteq s - \liminf O^E(A_n)$.

Proof. Since $0 \notin E$ and E is weakly closed, clearly it is w-closely supported. If $a \in O^E(A)$, $a_n \in A_n$ and $a_n \xrightarrow{s} a$, then, by Lemma 4.1 and Remark 4.4, it follows that (3) and (4) hold. We prove that $a_n \in O^E(A_n)$, eventually. In fact, on the contrary, if for a subsequence $(n_j)_{j\in\mathbb{N}}$ there exist $w_j \in (a_{n_j} - E) \cap A_{n_j}$, then $w_j = a_{n_j} - e_j$ for some $e_j \in E$. From (4) it results $e_j = a_{n_j} - w_j \to 0$. This is a contradiction, because the closedness of E implies that $d(E, 0) > 0$. □

Proposition 4.6. *Let Y be a reflexive Banach space, $C \subset Y$ a given cone, $E \subset Y$ a C–i.s. and $A_n, A \subseteq Y$ satisfying* (ii), (ii') *of Lemma* 4.1 *such that*

(i) $d(0, E) = 0$, $E \cup \{0\}$ *is weakly closed and* $E + E \subseteq E \cup \{0\}$*;*

(iv) *E is w-closely supported;*

(v) *for every $n \in \mathbb{N}$, the set A_n has the E-domination property;*

(x) *$(A_n)_{n\in\mathbb{N}}$ satisfies the asymptotic condition of Definition* 4.3 (I) *or the path connection condition of Definition* 4.3 (II)*.*

Then $\quad O^E(A) \subseteq s - \liminf O^E(A_n)$.

Proof. Let $a \in O^E(A)$ and $A_n \ni a_n \to a$. From the E-domination property it results $(a_n - (E \cup \{0\})) \cap O^E(A_n) \neq \emptyset$. It is evident that $a_n - (E \cup \{0\})) \cap O^E(A_n) \subseteq O^E(A_n \cap a_n - (E \cup \{0\}))$, so, for each $n \in \mathbb{N}$, there exists

$$w_n \in O^E(A_n \cap a_n - (E \cup \{0\})). \tag{9}$$

We claim that $w_n \in O^E(A_n)$ for every $n \in \mathbb{N}$. On the contrary, it would be $w_n - e_n \in A_n$ for some $e_n \in E$, but, at the same time, from (9), $w_n = a_n - e'_n$ with some $e'_n \in E \cup \{0\}$. This is a contradiction to (9), because it implies that $w_n - e_n = a_n - (e'_n + e_n)$, which belongs to $(w_n - E) \cap A_n \cap (a_n - (E \cup \{0\}))$. Now, thanks to Lemma 4.1, the property (4) implies that $a = \lim_{n\to\infty} w_n$, and this means that $a \in s-\liminf O^E(A_n)$. □

Remark 4.7. Thanks to Lemma 4.2 and Remark 4.4 (c), Propositions 4.5 and 4.6 still hold if we replace assumptions (ii′) (the same of Lemma 4.1) and (x) with, respectively,

(a) $w-lim\,sup(A_n + C) \subseteq A + C$;

(x') $(A_n)_{n\in\mathbb{N}}$ satisfies the asymptotic condition of Definition 4.3 (I) or the C-path connection condition of Definition 4.3 (III).

Proposition 4.8. *If Y is a reflexive Banach space, then the assumption* (v) *of Proposition* 4.6 *can be replaced by*

$$A_n \textit{ is s.}\ell.{-}E \textit{ closed} \qquad \textit{or} \qquad A_n + C \textit{ is s.}\ell.{-}E \textit{ closed.}$$

Proof. In fact, since E is w-closely supported, from Remark 17 (b) of [14], it follows that Y has the $(w-s)$ property with respect to E. Therefore, since in the proof of Proposition 4.6, from (3) the sets $(a_n - E) \cap A_n$ are bounded, we can apply Corollary 3.5 in the first case, or Corollary 3.6 in the second one, in order to get the E-domination property for A_n. □

About the upper convergence of the optimal points we recall, for the reader's convenience, some definitions and theorems given in [14].

Proposition 4.9. (see [14], Theorem 37) *Let Y be a reflexive Banach space, $C \subset Y$ a given convex cone with $int\, C \neq \emptyset$. Let E, $E_n \subset Y$ be C–i.s. and $A_n, A \subseteq Y$ satisfying*

(i) *$E \subseteq s-\liminf E_n$,*

(ii) $\quad A \subseteq s-\liminf A_n$ *and* $s-\limsup A_n \subseteq A + C$.

Then $\quad s-\limsup O^{int\, E_n}(A_n) \subseteq O^{int\, E}(A+C) \subseteq O^{int\, E}(A) + C$.

Definition 4.10. Let Y be a normed space, A, C, $E \subset Y$ such that C is a cone, E is a $C-i.s.$ and $int\, A \neq \emptyset$. We define

$$\widetilde{O}^E(int\, A) := \{a \in A : \ (a - E) \cap int\, A = \emptyset\}.$$

The following definition is the lower part of the Attouch-Wets convergence (see [11]) .

Definition 4.11. Let Y be a normed space, $A_n \subseteq Y$, $n \in \mathbb{N}$. We say that A satisfies the *Attouch-Wets lower limit* of $(A_n)_{n\in\mathbb{N}}$, and we will denote it as $AW_{-}\lim(A, A_n) = 0$, iff

$$\lim_{n\to\infty} \sup_{x\in A\cap B(0,\rho)} d(x, A_n) = 0 \quad \forall \rho > 0\,.$$

Proposition 4.12. *Let Y be a normed space, $C \subset Y$ a closed convex cone, $E \subset Y$ a $C-i.s.$ and A, $A_n \subseteq Y$, $n \in \mathbb{N}$ such that A_n are closed convex sets, $int\, A \neq \emptyset$ and*

$$AW_{-}\lim(A, A_n) = 0\,, \ s-\limsup A_n \subseteq A + C\,.$$

Then $\quad s-\limsup \widetilde{O}^E(int\, A_n) \subseteq \widetilde{O}^E(int\, A) + C \quad$ *and*

$$(y - E) \cap (int\, A + C) = \emptyset \quad \forall\, y \in s-\limsup \widetilde{O}^E(int\, A_n).$$

Proof. The proof is the same as that of Theorem 42 in [14] after previously remarking that Lemma 41 of the same paper still holds if A is not assumed to be closed and convex. □

5. Convergence of functions and their E-optimal points

Definition 5.1. (see [7],[12]) Let X be a normed space and $A, A_n \subset X$ $(n \in \mathbb{N})$.

(a) We say that A is a *recessively w-compact set* iff for every unbounded sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A, there exist a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, a sequence $(t_k)_{k\in\mathbb{N}}$ of positive numbers, $t_k \to 0$ and $\bar{x} \in X \setminus \{0\}$ such that $t_k x_{n_k} \rightharpoonup \bar{x}$.

(b) We say that the sequence $(A_n)_{n\in\mathbb{N}}$ is *recessively w-compact* iff for every unbounded sequence $(x_k)_{k\in\mathbb{N}}$, $x_k \in A_{n_k}$, where $(n_k)_{k\in\mathbb{N}}$ is any subsequence of $\mathbb{N}$, there exist a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ and a sequence $(t_j)_{j\in\mathbb{N}}$ of positive numbers, $t_j \to 0$ and $\bar{x} \in X \setminus \{0\}$ such that $t_j x_{k_j} \rightharpoonup \bar{x}$.

Remark 5.2. Clearly a sufficient condition for the recessive w-compactness of a sequence $(A_n)_{n\in\mathbb{N}}$ is the recessive w-compactness of $\underset{n\in\mathbb{N}}{\cup} A_n$, and, in the finite dimensional case, any closed set which does not contain the origin is recessively compact. Other sufficient conditions for a set A to be recessively w-compact are given in [7], where, among others, the following sufficient property is introduced.

(I) A set A in X satisfies the (CB) condition, iff there exists a bounded set $V \subseteq X$ such that the closure of the cone generated by $A \setminus V$ is equal to the cone generated by some compact set H in X not containing the origin.

In the case of a reflexive Banach space endowed with the weak topology, the following condition ensures that $A \subset X$ is recessively w-compact.

(II) There exist a weakly closed set $K \subset X \setminus \{0\}$ and two positive numbers α, β such that $\alpha \dfrac{a}{\|a\|} \in K$ for every $a \in A$ such that $\|a\| \geq \beta$. In fact, if $(x_k)_{k\in\mathbb{N}}$ is a sequence in A such that $\|x_k\| \to \infty$, then $\|x_k\| \geq \beta$ for large k. Hence $\alpha \dfrac{x_k}{\|x_k\|} \in K$, so that, up to a subsequence, $\alpha \dfrac{x_k}{\|x_k\|} \rightharpoonup x \in K$, thanks to the reflexivity of the space X.

Moreover, we point out that this property is not stronger than the (CB) condition. In fact, if we assume (CB) to be true, and H be the compact set given by condition (I), there exist $\delta, \mu > 0\,(\delta < \mu)$ such that $H \subset B(0,\mu) \setminus B(0,\delta)$. Moreover, if V is the bounded set given by the (CB) property, then $V \subset B(0,\beta)$ for some $\beta > 0$. We prove that condition (II) is true with $\alpha = \mu$, the same value of β, and $K = \{\lambda y : y \in H,\ \lambda \geq 1\}$. It is easy to verify that K is weakly closed and $0 \notin K$. Moreover, if $a \in A$ and $\|a\| \geq \beta$, we have $\frac{\alpha}{\|a\|} a \in \mathit{cone}(A \setminus V)$, so, by the (CB) condition, $\frac{\alpha}{\|a\|} a \in \mathit{cone}(H)$, namely $\frac{\alpha}{\|a\|} a = \tau y$ with some $\tau > 0$, $y \in H$. But $\tau = \frac{\|\tau y\|}{\|y\|} = \frac{\alpha}{\|y\|} \geq \frac{\alpha}{\mu} = 1$, so that $\frac{\alpha}{\|a\|} a \in K$.

Definition 5.3. (see [3], [18]) Let X, Y be normed spaces, and Y partially ordered by a cone $C \subset Y$. Let $A \subset X$ be a convex set and $f : X \to Y$ a given function. We say that f is *properly quasi C-convex* on A iff for every $x, y \in A$ and every $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \in (f(x) - C) \cup (f(y) - C).$$

When $\mathit{int}\, C \neq \emptyset$, we say that f is *strictly properly quasi C-convex* on A, iff for every $x, y \in A$, $x \neq y$ and every $\lambda \in]0,1[$ we have

$$f(\lambda x + (1-\lambda)y) \in (f(x) - \mathit{int}\, C) \cup (f(y) - \mathit{int}\, C).$$

We say that f is *C-convex* on A, iff for every $x, y \in A$ and every $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \preceq_Y \lambda f(x) + (1-\lambda) f(y).$$

When $int\,C \neq \emptyset$, we say that f is *strictly C-convex* on A, iff for every $x, y \in A$, $x \neq y$ and every $\lambda \in]0,1[$ we have

$$\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y) \in int\,C.$$

Remark 5.4. If $Y = \mathbb{R}$, it is straightforward to see that C-convexity ($C = \mathbb{R}_+$) implies the properly quasi C-convexity, but the converse is not true. If the dimension of Y is greater than 1 the definitions are each other independent (see [3], Proposition 4.2).

The following kind of convergence, which is the well known Γ-convergence (or epiconvergence) in the case of real functions, has been introduced in [19] in order to study the convergence of efficient solutions for vector valued functions in a finite dimensional framework. Here, in infinite dimensional spaces, we need to deal with the Mosco convergence of the epigraphs.

Definition 5.5. Let X, Y be normed spaces, where Y is partially ordered by a pointed convex cone $C \subset Y$. If A, $A_n \subset X$ $(n \in \mathbb{N})$ and f, $f_n : X \to Y$ $(n \in \mathbb{N})$ are given, we say that $f_{n|A_n}$ *M-epiconverges* to $f_{|A}$, iff

$$epi\,f_{n|A_n} \xrightarrow{M} epi\,f_{|A}$$

where, for a given $g : X \to Y$, $epi\,g_{|A} := \{(x,y) \in A \times Y : \ g(x) \preceq_Y y\}$, and on $X \times Y$ the product topologies $w \times w$ and $s \times s$ are considered.

Lemma 5.6. *Let X, Y be normed spaces with Y partially ordered by a closed convex and pointed cone $C \subset Y$. Let A, $A_n \subset X$ and f, $f_n : X \to Y$ $(n \in \mathbb{N})$ be given. Assume that*

(a) $(f_n(A_n))_{n\in\mathbb{N}}$ *is recessively w-compact in Y,*

(b) $epi\ f_{n|A_n} \xrightarrow{M} epi\ f_{|A}$.

Let $x_n \in A_n$ be such that $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to $x \in A$ and $(y_n)_{n\in\mathbb{N}}$ is weakly convergent to $\bar{y} \in Y$. If $f_n(x_n) \preceq_Y y_n$ for every $n \in \mathbb{N}$, then $(f_n(x_n))_{n\in\mathbb{N}}$ is bounded in Y.

Proof. We assume, by contradiction and by passing to a subsequence, if necessary, that $\|f_n(x_n)\| \to \infty$. Thanks to the recessive w-compactness, there exist a positive sequence $t_k \to 0$, a subsequence $(n_k)_{k\in\mathbb{N}}$ and $y_o \in Y \setminus \{0\}$, such that $t_k f_{n_k}(x_{n_k}) \rightharpoonup y_o$. From $t_k f_{n_k}(x_{n_k}) \preceq_Y t_k y_{n_k}$, by passing to the limit, we obtain $y_o \preceq_Y 0$. Let now $\lambda > 0$. Since $f_{n_k}(x_{n_k}) - y_{n_k} \preceq_Y 0$, and, for great k, $\lambda t_k < 1$, it results $f_{n_k}(x_{n_k}) \preceq_Y \lambda t_k (f_{n_k}(x_{n_k}) - y_{n_k}) + y_{n_k}$, where the last sequence weakly converges to $\lambda y_o + \bar{y}$. From Definition 5.5, therefore, $(x, \lambda y_o + \bar{y}) \in epi f_{|A}$, namely $f(x) \preceq_Y \lambda y_o + \bar{y}$. Since λ is arbitrary, remembering Proposition 7(iii) of [14], this is impossible, because $y_o \in -C \setminus \{0\}$. □

Proposition 5.7. *Let X be a normed space, Y a reflexive Banach space, partially ordered by a closed convex and pointed cone $C \subset Y$. Let f, $f_n : X \to Y$,*

A, $A_n \subset X$ be such that epi $f_{n|A_n} \xrightarrow{M}$ epi $f_{|A}$ and $(f_n(A_n))_{n\in\mathbb{N}}$ is recessively w-compact in Y, then

$$\forall\, x \in A \text{ there exist } x_n \in A_n, \text{ such that } x_n \xrightarrow{s} x \text{ and } f_n(x_n) \rightharpoonup f(x).$$

Moreover, if C is w-closely supported, then the following property holds:

$$\forall\, x \in A \text{ there exist } x_n \in A_n\,, \text{ such that } x_n \xrightarrow{s} x \text{ and } f_n(x_n) \xrightarrow{s} f(x).$$

Proof. If $x \in A$, by the Mosco epiconvergence of $(f_{n|A_n})_{n\in\mathbb{N}}$ to $f_{|A}$, we can take $(x_n, y_n) \in epi f_{n|A_n}$, $(x_n, y_n) \to (x, f(x))$ strongly. Since $f_n(x_n) \preceq_Y y_n$ for every $n \in \mathbb{N}$, Lemma 5.6 implies that $(f_n(x_n))_{n\in\mathbb{N}}$ is a bounded sequence. So, by reflexivity of Y, up to a subsequence, it results $f_n(x_n) \rightharpoonup y' \in Y$. But $x_n \xrightarrow{s} x$, so, by the M-epiconvergence, $f(x) \preceq_Y y'$.

On the other hand, if $p \in C^* \setminus \{0\}$ and $\varepsilon > 0$ are given, when n is great enough, it results $\langle p, y'\rangle - \varepsilon \le \langle p, f_n(x_n)\rangle \le \langle p, y_n\rangle \to \langle p, f(x)\rangle$. Thanks to the arbitrariness of p and ε, this implies $y' \preceq_Y f(x)$, so $y' = f(x)$, because the cone is pointed. Since every subsequence of $(f_n(x_n))_{n\in\mathbb{N}}$ has a further subsequence weakly convergent to $f(x)$, we conclude that $f_n(x_n) \rightharpoonup f(x)$.

If, in addition, C is w-closely supported, we prove that $f_n(x_n) \xrightarrow{s} f(x)$. On the contrary, if $f_n(x_n) \notin B(f(x), 2\varepsilon)$ for a subsequence and some $\varepsilon > 0$, we consider K_ε given by condition (1) of Definition 2.3, applied to $-C$. Then $K_\varepsilon + y_n \supset -(C \setminus B(0,\varepsilon)) + y_n = (y_n - C) \setminus B(y_n, \varepsilon) \supset (y_n - C) \setminus B(f(x), 2\varepsilon)$, when n is sufficiently large. Since $f_n(x_n) \in (y_n - C) \setminus B(f(x), 2\varepsilon)$, it follows that $f_n(x_n) \in K_\varepsilon + y_n$ and, by passing to the limit, $f(x) \in K_\varepsilon + f(x)$, as K_ε is weakly closed. This is impossible because $0 \notin K_\varepsilon$. $\square$

Definition 5.8. Let X be a normed space, $A \subset X$ and Y a partially ordered topological vector space with the partial order denoted by $\preceq_Y$. If $f : X \to Y$ is a given function, we define the recession cone of A as

$$A_\infty := \{d \in X : a + td \in A \;\forall\, a \in A,\, t \ge 0\}.$$

Moreover, if $z \in Y$, we define the sublevel set of $f_{|A}$ as

$$f^z_{|A} = \{x \in A : f(x) \preceq_Y z\},$$

and

$$O^+(f^z_{|A}) = \{d \in A_\infty : f(x + \lambda d) \preceq_Y z \;\; \forall\, x \in f^z_{|A},\, \lambda \ge 0\}.$$

Remark 5.9. In [10, 19, 6], the condition $O^+(f^z_{|A}) = \{0\}$ is assumed in order to prove various results about the convergence of optimal points. We also need to introduce this assumption in most of the following results.

Theorem 5.10. *Let X, Y be Banach spaces, with X reflexive, and Y partially ordered by a closed convex and pointed cone $C \subset Y$. Let f, $f_n : X \to Y$ and A, $A_n \subset X$ be given. Moreover,*

(i) *A_n are convex;*

(ii) *$epi\, f_{n|A_n} \xrightarrow{M} epi\, f_{|A}$;*

(iii) *$(A_n)_{n\in\mathbb{N}}$ is recessively w-compact in X;*

(iv) *each $f_{n|A_n}$ is properly quasi C-convex or C-convex;*

(v) *$O^+(f^z_{|A}) = \{0\}$ if $z \in Y$ is such that $f^z_{|A} \neq \emptyset$.*

Then $w - \limsup(f_n(A_n) + C) \subset f(A) + C$.

Proof. Let us consider $y_o = w - \lim f_{n_k}(x_k) + c_k \in w - \limsup(f_n(A_n) + C)$, where $c_k \in C$ and $x_k \in A_{n_k}$ for some subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$. We prove that $(x_k)_{k\in\mathbb{N}}$ is bounded. If we assume the contrary, then, up to a subsequence, we get $\|x_k\| \to \infty$. The assumption about the recessive w-compactness of $(A_n)_{n\in\mathbb{N}}$ gives the existence, possibly for a subsequence, of $\bar{x} \in X \setminus \{0\}$ such that $t_k x_k \rightharpoonup \bar{x}$, where $t_k > 0$ and $t_k \to 0$. First we prove that

$$\bar{x} \in A_\infty. \tag{10}$$

If $x \in A$ is given, thanks to M-epiconvergence, there exists a sequence $(x'_n, y_n) \in epif_{n|A_n}$ strongly convergent to $(x, f(x))$. We can assume one or both the following assumptions to be true:

- $f_{n_k|A_{n_k}}$ are properly quasi C-convex for infinitely many $k \in \mathbb{N}$,
- $f_{n_k|A_{n_k}}$ are C-convex for infinitely many $k \in \mathbb{N}$.

If $\lambda > 0$ is given, let $v_{\lambda,k} = (1 - \lambda t_k)x'_{n_k} + \lambda t_k x_k$. In the case where $f_{n_k|A_{n_k}}$ is properly quasi C-convex we get

$$f_{n_k}(v_{\lambda,k}) \preceq_Y f_{n_k}(x'_{n_k}) \preceq_Y y_{n_k} \quad \text{or} \quad f_{n_k}(v_{\lambda,k}) \preceq_Y f_{n_k}(x_k) + c_k.$$

Therefore, by taking M-epiconvergence into account and by passing to a subsequence if necessary, $(v_{\lambda,k}, y_{n_k})$ weakly converges to $(x + \lambda\bar{x}, f(x)) \in epif_{|A}$ for every $\lambda \in [0, \infty[$, or, otherwise, $(v_{\lambda,k}, f_{n_k}(x_k) + c_k)$ weakly converges to $(x + \lambda\bar{x}, y_o) \in epif_{|A}$ for every $\lambda \in [0, \infty[$.

In the case where $f_{n_k|A_{n_k}}$ is C-convex we get

$$f_{n_k}(v_{\lambda,k}) \preceq_Y (1-\lambda t_k) f_{n_k}(x'_{n_k}) + \lambda t_k f_{n_k}(x_k) \preceq_Y (1-\lambda t_k) y_{n_k} + \lambda t_k (f_{n_k}(x_k) + c_k).$$

The last side converges to $f(x)$, so again, thanks to M-epiconvergence, we get clearly $(x + \lambda\bar{x}, f(x)) \in epif_{|A}$ for every $\lambda \in [0, \infty[$. Hence assertion (10) is proved. Moreover, such condition implies that $\bar{x} \in O^+(f^z_{|A})$, if $z \in Y$ is such that $f^z_{|A} \neq \emptyset$, in contradiction with the assumption (v). Also in the case where $(x + \lambda\bar{x}, y_o) \in epif_{|A}$ for every $\lambda \in [0, \infty[$, we conclude that, for any $x \in f^{y_o}_{|A}$, it results $\bar{x} \in O^+(f^{y_o}_{|A})$, and this contradicts the assumption (v) again. Then, since $(x_k)_{k\in\mathbb{N}}$ is bounded and X is reflexive, up to a subsequence, it weakly converges to some $x_o \in X$. In virtue of M-epiconvergence, we obtain that $(x_o, y_o) \in epif_{|A}$, and this concludes the proof. □

The following propositions are a direct consequence of the above results about the convergence of the images $f_n(A_n)$ and the theorems about the stability of optimal points of Section 4.

Proposition 5.11. *Let X, Y be reflexive Banach spaces, with Y partially ordered by a closed convex and pointed cone $C \subset Y$. Let $E \subset Y$ be a C–i.s. and A, $A_n \subset X$, f, $f_n : X \to Y$ satisfying the following assumptions*

(i) *C is w-closely supported and E is weakly closed;*

(ii) *A_n are convex;*

(iii) *$(A_n)_{n\in\mathbb{N}}$ is recessively w-compact in X and $(f_n(A_n))_{n\in\mathbb{N}}$ is recessively w-compact in Y;*

(iv) *epi $f_{n|A_n} \xrightarrow{M}$ epi $f_{|A}$;*

(v) *each $f_{n|A_n}$ is properly quasi C-convex or C-convex;*

(vi) *$O^+(f^z_{|A}) = \{0\}$ if $z \in Y$ is such that $f^z_{|A} \neq \emptyset$;*

(vii) *condition* (I) *or* (III) *of Definition* 4.3 *is satisfied by $\big(f_n(A_n)\big)_{n\in\mathbb{N}}$.*

Then $\quad O^E(f(A)) \subset s-\liminf O^E(f_n(A_n))$.

Proof. By Proposition 5.7, it results $f(A) \subset s-\liminf f_n(A_n)$, while Theorem 5.10 gives $w - \limsup(f_n(A_n) + C) \subset f(A) + C$. Then the conclusion follows from Proposition 4.5 and Remark 4.7. □

Proposition 5.12. *Let X, Y be reflexive Banach spaces, with Y partially ordered by a closed convex and pointed cone $C \subset Y$. Let $E \subset Y$ be a C–i.s. and A, $A_n \subset X$, f, $f_n : X \to Y$ satisfying the following assumptions*

(i) *$d(0, E) = 0$, $E \cup \{0\}$ is weakly closed and $E + E \subseteq E \cup \{0\}$;*

(i') *E is w-closely supported;*

(ii) *A_n are convex;*

(iii) *$(A_n)_{n\in\mathbb{N}}$ is recessively w-compact in X and $(f_n(A_n))_{n\in\mathbb{N}}$ is recessively w-compact in Y;*

(iv) *epi $f_{n|A_n} \xrightarrow{M}$ epi $f_{|A}$;*

(v) *each $f_{n|A_n}$ is properly quasi C-convex or C-convex;*

(vi) *if $z \in Y$ is such that $f^z_{|A} \neq \emptyset$, then $O^+(f^z_{|A}) = \{0\}$;*

(vii) *condition* (I) *or* (III) *of Definition* 4.3 *is satisfied by $\big(f_n(A_n)\big)_{n\in\mathbb{N}}$;*

(viii) *$f_n(A_n)$ satisfies the E-domination property for any $n \in \mathbb{N}$.*

Then $\quad O^E(f(A)) \subset s-\liminf O^E(f_n(A_n))$.

Proof. We notice that, since $d(0, E) = 0$ and $E \cup \{0\}$ is weakly closed, then $E \cup \{0\} \supset C$, so C itself is w-closely supported. Hence, by Proposition 5.7, it results $f(A) \subset s-\liminf f_n(A_n)$, while Theorem 5.10 gives $w-\limsup(f_n(A_n)+C) \subset f(A)+C$. Then the conclusion follows from Proposition 4.6 and Remark 4.7. □

Remark 5.13. If X, Y, C and E are as in the previous proposition, $A_n \subset X$, $f_n : X \to Y$, the following assumption (I) is sufficient to ensure that the hypothesis (viii) of the aforesaid proposition is true.

(I) Y has the $(w{-}s)$ property with respect to E and, for every $n \in \mathbb{N}$, the set $f_n(A_n)$ or, alternatively, $f_n(A_n)+C$ is $s.\ell.-E$ closed and $(y-E)\cap f_n(A_n)$ bounded for any $y \in f_n(A_n)$.

In fact it is sufficient to apply Corollary 3.5 or Corollary 3.6 with $f_n(A_n)$ in place of A, and $f_n(a)$, $a \in A_n$, in place of a.

The next proposition gives sufficient conditions to ensure that $f_n(A_n) + C$ are closed.

Proposition 5.14. *Let X be a reflexive Banach space and Y a topological vector space ordered by a closed convex cone $C \subset Y$. Let $f : X \to Y$ be given and A a convex closed and recessively w-compact subset of X. Moreover,*

(i) *$epif_{|A}$ is closed in the product topology $w_X \times s_Y$,*

(ii) *f is C-convex or properly quasi C-convex,*

(iii) *$O^+(f^z_{|A}) = \{0\}$ if $z \in Y$ is such that $f^z_{|A} \neq \emptyset$.*

Then $f(A) + C$ is strongly closed.

Proof. Let $y_n \in f(A) + C$, $n \in \mathbb{N}$, such that $y_n \xrightarrow{s} y$. We prove that $y \in f(A) + C$. We claim that, if $x_n \in A$, $f(x_n) \preceq_Y y_n$, $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence. The proof is similar to the case of Theorem 5.10.

If, by contradiction, for a subsequence, it results $\|x_n\| \to \infty$, in virtue of recessive w-compactness there exist a positive sequence $t_k \searrow 0$ and $\bar{x} \in X \setminus \{0\}$ such that $t_k x_{n_k} \rightharpoonup \bar{x}$, for a suitable subsequence $(n_k)_{k\in\mathbb{N}}$.

If $x \in A$, since A is convex and closed, then $x + t\bar{x} = w - \lim(1 - t_k t)x + t_k t x_{n_k} \in A$. Moreover, thanks to assumption (ii), $f((1 - t_k t)x + t_k t x_{n_k}) \preceq_Y (1 - t_k t)f(x) + t_k t f(x_{n_k}) \preceq_Y (1 - t_k t)f(x) + t_k t y_{n_k}$ for large k, when f is C-convex.

In the case where f is properly quasi C-convex it results either $f((1 - t_k t)x + t_k t x_{n_k}) \preceq_Y f(x)$ or $f((1 - t_k t)x + t_k t x_{n_k}) \preceq_Y f(x_{n_k}) \preceq_Y y_{n_k}$. In any case, by passing to a subsequence, if necessary, there exists a sequence $(z_k)_{k\in\mathbb{N}}$ in Y, strongly converging either to $f(x)$, or to y, such that $f((1-t_k t)x+t_k t x_{n_k}) \preceq_Y z_k$. In virtue of assumption (i), this implies that $f(x + t\bar{x}) \preceq_Y s - \lim z_k$ for every $t \geq 0$. Hence, in both cases, this is in contradiction with the assumption (iii).

Thus, the boundedness of $(x_n)_{n\in\mathbb{N}}$ being proved, up to a subsequence, it results $x_n \rightharpoonup x \in A$, so, by the closedness of $epif_{|A}$, we obtain $(x, y) \in epif_{|A}$. This proves that $y \in f(A) + C$. □

Definition 5.15. (see [16], Proposition 1.7, Ch.II) Let Y be a normed space ordered by a cone C. Such a cone is *normal* for the norm topology, iff there exists a constant $\gamma > 0$ such that $\gamma\|x\| \le \|y\|$, whenever $0 \preceq_Y x \preceq_Y y$. Equivalently (see [16], Proposition 1.3, Ch.II), for any two nets $(x_\beta)_{\beta\in I}$, $(y_\beta)_{\beta\in I}$, such that $0 \preceq_Y x_\beta \preceq_Y y_\beta$ for all $\beta \in I$, and $(y_\beta)_{\beta\in I}$ converges to 0, then $(x_\beta)_{\beta\in I}$ also converges to 0.

Remark 5.16. A sufficient condition for the closedness of $epi f_{|A}$, assumed in the previous proposition, is the weak lower semicontinuity of $f_{|A} : A \to Y$. More precisely, following [15], we say that $\varphi : A \to Y$ is $w_X \times s_Y$-lower semicontinuous at $x \in A$ iff *for any neighbourhood V of $\varphi(x)$ in (Y, s_Y), there exists a neighbourhood U of $x \in A$, for the weak topology on A induced by X, such that $\varphi(U) \subset V + C$.* We say that φ is $w_X \times s_Y$-lower semicontinuous in A, iff it is $w_X \times s_Y$-lower semicontinuous at x for every $x \in A$. When the cone C is assumed to be closed and convex, Proposition 1.3 of [15] can be applied, so that the $w_X \times s_Y$ lower semicontinuity of $f_{|A}$ implies the closedness of $epi f_{|A}$ with respect to the $w_X \times s_Y$ topology of $X \times Y$.

Remark 5.17. When $A = X$, or A is a closed convex subset of G, where G is a convex open set in X, if the cone C is assumed to be closed and normal with respect to the norm topology of Y, and f is C-convex and locally upper bounded on G, then the $w_X \times s_Y$ closedness of $epi f_{|A}$ follows as consequence. So this assumption can be omitted in Proposition 5.14.

In fact, by [12], Theorem 3.5, f is continuous on A with respect to the norm topologies of X and Y. Therefore, if $(x_n, y_n) \in epi f_{|A}$ and $x_n \rightharpoonup x$, $y_n \to y$, by Mazur's Lemma, we may consider a sequence $(u_k)_{k\in\mathbb{N}}$ of X, such that $u_k \to x$ and $u_k = \sum_{n=k}^{i(k)} \alpha_{(k,n)} x_n$, where $\alpha_{(k,n)}$ are nonnegative real numbers such that $\sum_{n=k}^{i(k)} \alpha_{(k,n)} = 1$. Hence, by convexity of A and C-convexity of f, we get $u_k \in A$ and $f(u_k) \preceq_Y \sum_{n=k}^{i(k)} \alpha_{(k,n)} f(x_n) \preceq_Y \sum_{n=k}^{i(k)} \alpha_{(k,n)} y_n$. Since $f(u_k)$ strongly converges to $f(x)$ and the last term in such inequality converges to y, by the closedness of C, it follows that $f(x) \preceq_Y y$. This concludes the proof.

Let us now deal with the upper stability of the optimal points of $f_n(A_n)$.

Proposition 5.18. *Let X be a normed space, Y a reflexive Banach space, partially ordered by a closed, convex and pointed cone $C \subset Y$ with $int\, C \neq \emptyset$. Let E, $E_n \subset Y$ be $C-$i.s. and assume that $A_n, A \subseteq Y$ satisfy*

(o) *C is w-closely supported;*

(i) *$E \subseteq s - \liminf E_n$;*

(ii) *A_n are convex, $(A_n)_{n\in\mathbb{N}}$ is recessively w-compact in X.*

Let f, $f_n : X \to Y$ be such that

(iii) *$epi\, f_{n|A_n} \xrightarrow{M} epi\, f_{|A}$;*

(iv) $(f_n(A_n))_{n\in\mathbb{N}}$ *is recessively* w*-compact in* Y*;*

(v) *each* $f_{n|A_n}$ *is properly quasi* C*-convex or* C*-convex;*

(vi) $O^+(f^z_{|A}) = \{0\}$ *if* $z \in Y$ *is such that* $f^z_{|A} \neq \emptyset$*.*

Then $\quad s{-}\limsup O^{int\,E_n}(f_n(A_n)) \subseteq O^{int\,E}(f(A)+C) \subseteq O^{int\,E}(f(A))+C.$

Proof. Proposition 5.7 and Theorem 5.10 imply

$$f(A) \subseteq s{-}\liminf f_n(A_n) \subseteq w{-}\limsup f_n(A_n) \subseteq f(A)+C\,.$$

Then the conclusion follows from Proposition 4.9 applied to $f_n(A_n)$ and $f(A)$ in place of A_n and A, respectively. $\square$

With the further assumption of strict properly quasi C-convexity or strict C-convexity on f, we can join the lower and the upper stability thanks to the following result.

Proposition 5.19. *Let* X, Y *be reflexive Banach spaces,* Y *partially ordered by a closed, convex and pointed cone* $C \subset Y$ *such that* $int\,C \neq \emptyset$*. Let* $E \subset Y$ *be a* $C{-}i.s.$ *and let* $A_n, A \subseteq Y$ *satisfy*

(o) E *is* w*-closely supported,* $d(0,E)=0$*,* $E\cup\{0\}$ *is weakly closed and* $E+E \subseteq E\cup\{0\}$*;*

(i) A, A_n *are convex, and* $(A_n)_{n\in\mathbb{N}}$ *is recessively* w*-compact in* X*.*

Let $f,\ f_n : X \to Y$ *be such that*

(ii) $epi\, f_{n|A_n} \xrightarrow{M} epi\, f_{|A}$*;*

(iii) $(f_n(A_n))_{n\in\mathbb{N}}$ *is recessively* w*-compact in* Y*;*

(iv) *each* $f_{n|A_n}$ *is properly quasi* C*-convex or* C*-convex;*

(v) *if* $z \in Y$ *is such that* $f^z_{|A} \neq \emptyset$*, then* $O^+(f^z_{|A}) = \{0\}$*;*

(vi) *condition* (I) *or* (III) *of Definition* 4.3 *is satisfied by* $\big(f_n(A_n)\big)_{n\in\mathbb{N}}$*;*

(vii) $f_n(A_n)$ *satisfies the* E*-domination property for any* $n \in \mathbb{N}$*;*

(viii) f *is strictly properly quasi* C*-convex on* A *or, alternatively,* $\lambda E \subseteq E$ *for any* $\lambda \in]0,1]$ *and* f *is strictly* C*-convex.*

Then
$$f(A) \cap s-K\lim O^E(f_n(A_n)) = O^E(f(A)) \tag{11}$$

Proof. From Proposition 5.12 we conclude $O^E(f(A)) \subset s{-}\liminf O^E(f_n(A_n))$. Moreover, thanks to the strict C-convexity or the strict proper quasi C-convexity of f, we get $O^E(f(A)) \supseteq O^{int\,E}(f(A))$. In fact, if $y = f(x) \in O^{int\,E}(f(A))$ and, by contradiction, $y - e = f(a)$ for some $e \in E$, $a \in A$, then $a \neq x$, since $e \neq 0$. Therefore, for example when f is strictly C-convex and $\lambda E \subseteq E$ for any $\lambda \in]0,1]$,

$$\frac{f(x)+f(a)}{2} - f(\frac{x+a}{2}) \in int\,C.$$

But, $\dfrac{f(x)-f(a)}{2} = \dfrac{e}{2} \in E$, thanks to the further property of E and, as follows from [17], Proposition 2.2, it results $E + int\, C \subset int\, E$. So,

$$f(x) - f(\frac{x+a}{2}) = \frac{f(x)+f(a)}{2} - f(\frac{x+a}{2}) + \frac{f(x)-f(a)}{2} \in int\, E,$$

which contradicts $f(x) \in O^{int\, E}(f(A))$. If f is strictly properly quasi C-convex and

$$f(x) - f(\frac{x+a}{2}) \in int\, C \quad \text{or} \quad f(a) - f(\frac{x+a}{2}) \in int\, C,$$

we have again a contradiction, because $int\, C \subset int\, E$, and, also in the last case,

$$f(x) - f(\frac{x+a}{2}) = f(a) - f(\frac{x+a}{2}) + f(x) - f(a) \in int\, E.$$

This proves that $O^{int\, E}(f(A)) \subseteq O^{E}(f(A))$. Then the conclusion follows if we prove that $f(A) \cap s - \limsup O^{int\, E}(f_n(A_n)) \subset O^{int\, E}(f(A))$.

Let $f(A) \ni y = s - \lim y_n$, $y_n \in O^{int\, E}(f_n(A_n))$, we prove that $y \in O^{int\, E}(f(A))$. In fact, if, on the contrary, $y - e = f(x)$ for some $e \in int\, E$, $x \in A$, thanks to assumption (ii), there exists a sequence $(x_n, v_n) \in epi\, f_{n|A_n}$ such that $(x_n, v_n) \to (x, f(x))$ strongly. Since $e \in int\, E$ and $y_n - v_n \xrightarrow{s} y - f(x) = e$, then $y_n - v_n \in int\, E$ for large n. On the other hand, $v_n - f_n(x_n) \in C$, so, $y_n - f_n(x_n) = y_n - v_n + v_n - f_n(x_n) \in int\, E + C \subset int\,(E + C) = int\, E$, where the last inclusion is justified, as before, by Proposition 2.2 of [17]. This contradicts the fact that $y_n \in O^{int\, E}(f_n(A_n))$ and our proof is achieved. □

When E is weakly closed we can drop the assumption (vii) of the previous proposition, in the case where f is strictly properly quasi C-convex.

Proposition 5.20. *Let X, Y be reflexive Banach spaces, Y partially ordered by a closed, convex and pointed cone $C \subset Y$, such that $int\, C \neq \emptyset$. Let $E \subset Y$ be a $C-$i.s. and $A_n, A \subseteq Y$, f, $f_n : X \to Y$ satisfying the assumptions from* (i) *to* (vi) *of the previous proposition. Moreover, in place of* (o) *we assume*

(o') *C is w-closely supported, E is weakly closed*

and in place of (viii)

(viii') *f is strictly properly quasi C-convex on A.*

Then the same conclusion as in Proposition 5.19 *holds.*

Proof. The inclusion $O^{E}(f(A)) \subset s - \liminf O^{E}(f_n(A_n))$ follows now from Proposition 5.11, then the remaining part of the proof is the same of the previous proposition. □

Remark 5.21.

(a) The following example shows that, under the assumption of Proposition

5.19, it may happen that $s-\limsup O^{int\,E_n}(f_n(A_n)) \not\subseteq O^{int\,E}(f(A))$, also in finite dimension.

Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $C=E=[0,\infty[\times[0,\infty[$, $A=A_n=[0,1]$ and $f_n(x)=(x^2,\dfrac{1}{(x+1)^{n^2}}+g(x))$, where $g(x)=x(x-2)$. It is easy to verify that $epi\, f_{n|A} \xrightarrow{M} epi\, f_{|A}$, where $f(x)=(x^2,g(x))$, $x\in[0,1]$, and f is strictly C-convex. In fact, if $x>0$ and $(x,\alpha,\beta)\in epi\, f_{|A}$, $\alpha\geq x^2$, $\beta\geq g(x)$, it results $(x,\alpha,\beta)=\lim_{n\to\infty}(x,\alpha,\dfrac{1}{(x+1)^{n^2}}+g(x)+\beta-g(x))\in\liminf(epi\, f_{n|A})$. If $x=0$ and $\beta\geq 0$, then $(0,\alpha,\beta)=\lim_{n\to\infty}(\dfrac{1}{n},\dfrac{1}{n^2}+\alpha,\dfrac{1}{(\frac{1}{n}+1)^{n^2}}+g(\dfrac{1}{n})+\beta)=\lim_{n\to\infty}(\dfrac{1}{n},f_n(\dfrac{1}{n})+(\alpha,\beta))\in\liminf(epi\, f_{n|A})$. Hence, $epi\, f_{|A}\subseteq\liminf(epi\, f_{n|A})$. Moreover, if $(x,y,z)\in\limsup(epi\, f_{n|A})$, we have $(x,y,z)=\lim_{k\to\infty}(x_k,y_k,z_k)$, $y_k\geq x_k^2$, $z_k\geq\dfrac{1}{(x_k+1)^{n_k^2}}+g(x_k)$ and, since $\dfrac{1}{(x_k+1)^{n_k{}^2}}+g(x_k)\geq g(x_k)\to g(x)$, we conclude that $y\geq x^2$ and $z\geq g(x)$, so, $(x,y,z)\in epi\, f_{|A}$. Thus is proved that $epi\, f_{n|A}\xrightarrow{M} epi\, f_{|A}$. We now show that

$$f_n(A)\xrightarrow{s-K}\Gamma:=\{0\}\times[0,1]\cup\{(x^2,g(x)):\,x\in]0,1]\}. \tag{12}$$

We begin with the inclusion $\Gamma\subseteq\liminf f_n(A)$. When $(0,y)\in\{0\}\times]0,1]$ we consider $x_n=\frac{1}{n^2}(-\ln y)$, while, when $y=0$, $x_n=\frac{1}{n}$, so, anyway, $f_n(x_n)\to(0,y)$ and $(0,y)\in\liminf f_n(A)$. Besides, $(x^2,g(x))\in\liminf f_n(A)$, because $f_n(x)\to f(x)$ for any $x\in]0,1]$.
For the opposite inclusion, we consider a sequence $(x_n)_{n\in\mathbb{N}}$ in A, such that $(x_n^2,\dfrac{1}{(x_n+1)^{n^2}}+g(x_n))\to(t,y)$. Obviously, $t\geq 0$ and $x_n\to\sqrt{t}$. If $t=0$ then $g(x_n)\to 0$ and $y=\lim_{n\to\infty}\dfrac{1}{(x_n+1)^{n^2}}$, but $\lim_{n\to\infty}n^2\ln(1+x_n)\geq 0$, so $y\in[0,1]$. If $t\in]0,1]$, clearly $(x_n^2,\dfrac{1}{(x_n+1)^{n^2}}+g(x_n))\to(x^2,g(x))$, where $x=\sqrt{t}$. This concludes the proof of the opposite inclusion. It is simple to see that $f_n(A_n)$ and $f(A)$ are graphics of strictly decreasing functions on $[0,1]$, so it is clear that $O^{int\,E}(f_n(A_n))=f_n(A)$ and $O^{int\,E}(f(A))=\{(x^2,g(x)),\,x\in[0,1]\}=f(A)$, hence, by taking (12) into account, our assertion is proved.

(b) By another example, we point out that, when the improvement set E is closed, so that $d(0,E)>0$, it may happen that $O^{int\,E}(f(A))\not\subseteq O^E(f(A))$ and $s-\limsup O^{int\,E_n}(f_n(A_n))\not\subseteq O^E(f(A))$, also if f is strictly C-convex or strictly properly quasi C-convex on A.

Let $A=A_n=\mathbb{R}^2$, f, $f_n:\mathbb{R}^2\to\mathbb{R}^2$ defined as $f_n(x,y)=f(x,y)=(x^2+y^2,\exp(-x)+y^2)$ and $C=\{(x,y)\in\mathbb{R}^2:\,x\geq 0,y\geq 0\}$, $E=(1,1)+C$.

Since $x \in \mathbb{R} \mapsto x^2 \in \mathbb{R}$ and $x \in \mathbb{R} \mapsto \exp(-x) \in \mathbb{R}$ are strictly C-convex and properly strictly quasi C-convex, the same holds for f. Clearly $f(x,1) \in O^{int\,E}(f(A)) \setminus O^E(f(A))$ for every $x \in \mathbb{R}$.

In the next result about the upper inclusion of sequences of optimal points we avoid the assumption of nonempty interior on the cone, thanks to the use of the Attouch-Wets lower limit.

Proposition 5.22. *Let X, Y be normed spaces, with Y partially ordered by a closed convex and pointed cone $C \subset Y$, and $E \subset Y$ a C-i.s. Let f, $f_n : X \to Y$ and A, $A_n \subset X$, we assume*

(i) *f_n are C-convex, $f_n(A_n) + C$ are strongly closed, and $int\, f(A) \neq \emptyset$*

(ii) *$AW_- \lim(f(A), f_n(A_n) + C) = 0$.*

Then
$$f(A) \cap s - \limsup O^E(f_n(A_n)) \subset \widetilde{O}^E(int\, f(A)) \qquad (13)$$

Proof. Let $y = s - \lim f_{n_k}(a_k)$, $a_k \in A_{n_k}$, $y \in f(A)$, and $f_{n_k}(a_k) - E \cap f_{n_k}(A_{n_k}) = \emptyset$, $k \in \mathbb{N}$. We assume, by contradiction, that $y \notin \widetilde{O}^E(int\, f(A))$, so, we have $y - e = f(a)$ for some $e \in E$ and $f(a) \in int\, f(A)$. It is by an obvious argument that $f_n(A_n) + C$ results to be convex. We notice that Lemma 41 of [14] still holds if the limit set is not assumed to be nor closed neither convex. Since by assumption $f_n(A_n) + C$ are also closed, if $B(f(a), \delta) \subset B(f(a), \delta') \subset f(A)$, $0 < \delta < \delta'$, from such lemma it follows that there exists $n_\delta \in \mathbb{N}$ such that $B(f(a), \delta) \subset f_n(A_n) + C$ for each $n \geq n_\delta$. On the other hand, as $f_{n_k}(a_k) - e \to f(a)$, there exists $\nu_\delta \in \mathbb{N}$ such that $f_{n_k}(a_k) - e \in B(f(a), \delta)$ for every $k \geq \nu_\delta$. As we can assume $n_k \geq n_\delta$ when $k \geq \nu_\delta$, it follows that $f_{n_k}(a_k) - e = f_{n_k}(a'_k) + c_k \in int(f_{n_k}(A_{n_k}) + C)$. This means that $f_{n_k}(a_k) - e - c_k = f_{n_k}(a'_k) \in f_{n_k}(A_{n_k})$, which is in contradiction with $f_{n_k}(a_k) \in O^E(f_{n_k}(A_{n_k}))$. □

Remark 5.23.
(a) Let X, Y, C, E be like in the previous proposition, with the further assumption that $\frac{1}{2}\, e \in E$ for any $e \in E$; if we assume that $A \subset X$ and $f : A \to Y$ are given and satisfy

(i) A is closed and convex, $int\, f(A) \neq \emptyset$ and $f(\frac{a+b}{2}) \in int\, f(A)$ $\forall\, a, b \in A$;

(ii) f is C-convex;

then $O^E(f(A)) = \widetilde{O}^E(int\, f(A))$.

So, if we join such assumptions with those of the Propositions 5.11 and 5.22, we obtain
$$f(A) \cap s - K \lim O^E(f_n(A_n)) = O^E(f(A))\,.$$

Proof. Obviously, the only inclusion to prove is $\widetilde{O}^E(int\, f(A)) \subset O^E(f(A))$. Let $f(x) \in f(A)$ be such that $(f(x) - E) \cap int\, f(A) = \emptyset$ and, on the contrary,

$f(x) - e = f(a)$ for some $a \in A$, $e \in E$. From assumption (ii), we have $\frac{f(x)+f(a)}{2} - f(\frac{x+a}{2}) \in C$, while, thanks to the additional property of E, $\frac{f(x)-f(a)}{2} \in E$. This leads to $f(x) - f(\frac{x+a}{2}) \in E$, which is a contradiction, since $f(\frac{x+a}{2}) \in int\, f(A)$ by virtue of assumption (i). $\square$

(b) The condition on the strong closure of $f_n(A_n) + C$ of assumption (i) in the previous proposition is a consequence, whereas the assumptions of Proposition 5.14 are satisfied by f_n and A_n in place of f and A.

In order to avoid the intersection with $f(A)$ in the left hand of (11) and (13), we need a stronger convergence which is known, in the finite dimensional case, as 'continuous convergence'.

Definition 5.24. Let X, Y be normed spaces, f, $f_n : X \to Y$, A, $A_n \subset X$. We say that $(f_{n|A_n})_{n\in\mathbb{N}}$ converges to $f_{|A}$ in the sense of weak-weak [respectively weak-strong] continuous convergence, and we denote $f_{n|A_n} \xrightarrow{c.ww} f_{|A}$ [respectively $f_{n|A_n} \xrightarrow{c.ws} f_{|A}$], iff for every $A_{n_k} \ni x_k \rightharpoonup x \in A$, where $(n_k)_{k\in\mathbb{N}}$ is a subsequence of $\mathbb{N}$, it results $f_{n_k}(x_k) \rightharpoonup f(x)$ [respectively $f_{n_k}(x_k) \xrightarrow{s} f(x)$].

Remark 5.25. Assuming that Y is partially ordered by a pointed convex and closed cone $C \subset Y$, then

(i) if $w - \limsup A_n \subseteq A$, the weak-weak continuous convergence implies the upper part of the M-epiconvergence $w - \limsup epi\, f_{n|A_n} \subset epi\, f_{|A}$. In fact, if $epi\, f_{n_k|A_{n_k}} \ni (x_k, y_k) \rightharpoonup (x, y) \in X \times Y$, thanks to the assumption $w - \limsup A_n \subseteq A$, it results $x \in A$ and $f_{n_k}(x_k) \rightharpoonup f(x)$. So, since $f_{n_k}(x_k) \preceq_Y y_k$ and the cone is weakly closed, we obtain that $y - f(x) \in C$ and $(x, y) \in epi\, f_{|A}$.

(ii) If $A_n \xrightarrow{M} A$ the weak-strong continuous convergence is stronger than the M-epiconvergence. In fact, let us assume that $f_{n|A_n} \xrightarrow{c.ws} f_{|A}$. Thanks to part (i) of this Remark, we only need to prove the inclusion $epi\, f_{|A} \subset s - \liminf epi\, f_{n|A_n}$. Let $(x, y) \in epi\, f_{|A}$ and $x_n \in A_n$, $x_n \to x$, then, by definition of weak-strong continuous convergence, $f_n(x_n) \to f(x)$, thus $epi\, f_{n|A_n} \ni (x_n, y - f(x) + f_n(x_n)) \to (x, y)$ and $(x, y) \in s - \liminf epi\, f_{n|A_n}$.
An example of M-epiconvergent sequence which is not weak-strongly continuously convergent is the one given in Remark 5.21 (a).

Proposition 5.26. *Let X, Y be normed spaces, such that X is a reflexive Banach space, and Y is partially ordered by a closed convex and pointed cone $C \subset Y$. Let f, $f_n : X \to Y$ and A, $A_n \subset X$ such that*

(i) *A_n are convex, and $A_n \xrightarrow{w-K} A$;*

(ii) *$(A_n)_{n\in\mathbb{N}}$ is recessively w-compact in X;*

(iii) *each $f_{n|A_n}$ is properly quasi C-convex or C-convex;*

(iv) $f_{n|A_n} \xrightarrow{c.ww} f_{|A}$;

(v) $O^+(f^z_{|A}) = \{0\}$ *if* $z \in Y$ *is such that* $f^z_{|A} \neq \emptyset$.

Then $w{-}\limsup f_n(A_n) \subset f(A)$.

Proof. Let $y = w - \lim_{k\to\infty} f_{n_k}(x_k)$, where $x_k \in A_{n_k}$, we prove that $(x_k)_{k\in\mathbb{N}}$ is bounded in X. If, by contradiction, up to a subsequence, $\|x_k\| \to \infty$, by assumption (ii), we get $t_k x_k \rightharpoonup \bar{x} \in X \setminus \{0\}$ for some positive sequence $t_k \to 0$. Let $\lambda > 0$ and $x \in A$ be given. In virtue of assumption (i), we may consider $A_{n_k} \ni x'_k \rightharpoonup x$. Since $\lambda t_k \in [0,1]$, when k is large, and A_n are convex, it results $(1-\lambda t_k)x'_k + \lambda t_k x_k \in A_{n_k}$, while the same sequence weakly converges to $x + \lambda\bar{x}$. So, $x + \lambda\bar{x} \in A$ and $\bar{x} \in A_\infty$. Thanks to assumption (iii) and to the weak-weak continuous convergence we easily obtain that $f(x+\lambda\bar{x}) \preceq_Y f(x)$ or $f(x+\lambda\bar{x}) \preceq_Y y$ for any $\lambda \in [0,\infty[$, and this contradicts the assumption (v). Hence $(x_k)_{k\in\mathbb{N}}$ is bounded, and, by reflexivity, $x_k \rightharpoonup a$ for some subsequence, and $a \in A$. The conclusion follows, because, by weak-weak continuous convergence, $f_{n_k}(x_k) \rightharpoonup f(a)$ and this says that $y = f(a) \in f(A)$. □

Lemma 5.27. *Let X be a normed space, Y a reflexive Banach space partially ordered by a closed convex and pointed cone $C \subset Y$, and f, $f_n : X \to Y$, A, $A_n \subset X$. We assume the conditions (ii), (iii), (v) of Proposition* 5.26 *plus the following ones*

(i') A_n are convex, and $w{-}\limsup A_n \subseteq A$;

(vi) the sequence $(f_n(A_n))_{n\in\mathbb{N}}$ is recessively w-compact in Y;

(vii) epi $f_{n|A_n} \xrightarrow{M}$ epi $f_{|A}$.

Then, if $(f_{n_k}(x_k))_{k\in\mathbb{N}}$, where $x_k \in A_{n_k}$, is strongly convergent in Y, the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in X

Proof. The proof is analogous to the one in the previous proposition, with the difference that the sequence $(f_{n_k}(x'_k))_{k\in\mathbb{N}}$ weakly convergent to $f(x)$ is given by Proposition 5.7. □

Definition 5.28. Let X, Y be normed spaces, with Y partially ordered by a closed convex and pointed cone $C \subset Y$, and $E \subset Y$ be a C-i.s. Let $f : X \to Y$ and $A \subset X$, we say that $x \in A$ is an *E-efficient point*, iff $f(x) \in O^E(f(A))$ and we denote the set of such efficient points by $\mathrm{Eff}(f, A, E)$.

Theorem 5.29. *Let X, Y be reflexive Banach spaces, with Y partially ordered by a closed convex and pointed cone $C \subset Y$, and $E \subset Y$ a C-i.s. such that $d(0, E) = 0$. Let f, $f_n : X \to Y$ and A, $A_n \subset X$. We assume that all the conditions from* (i) *to* (viii) *of Proposition* 5.12*, or, alternatively, from* (i) *to* (vii) *of Proposition* 5.11 *are true. In addition*

(a) $w - \limsup A_n \subseteq A$, *and*

(b) $\lambda f(a) + (1-\lambda)f(b) - f(\lambda a + (1-\lambda)b) \in C\setminus\{0\}$ *if* a, $b \in A$, $a \neq b$, $\lambda \in]0,1[$;

then $\mathrm{Eff}(f,A,E) \subseteq w-\liminf \mathrm{Eff}(f_n,A_n,E)$.

Moreover, with the further assumptions

(c) $f_{n|A_n} \xrightarrow{c.ws} f_{|A}$,

(d) $int\, C \neq \emptyset$,

(e) f *is strictly* C*-convex,*

(f) $\lambda E \subseteq E$ *for any* $\lambda \in]0,1]$,

it follows that $\mathrm{Eff}(f_n,A_n,E) \xrightarrow{w-K} \mathrm{Eff}(f,A,E)$.

Proof. Let $x \in A$ be such that $f(x) \in O^E(f(A))$. We prove that $x \in w-\liminf \mathrm{Eff}(f_n,A_n,E)$. Thanks to our assumptions, we can apply the Proposition 5.12 or 5.11 to obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \mathrm{Eff}(f_n,A_n,E)$ and $f(x) = s-\lim f_n(x_n)$. Since among our assumptions, there are those of Lemma 5.27, such sequence is bounded, so, up to a subsequence, $x_n \rightharpoonup \tilde{x}$ and $\tilde{x} \in A$. By taking the M-epiconvergence into account, it results $f(\tilde{x}) \preceq_Y f(x)$, namely $f(x) - c = f(\tilde{x})$ for some $c \in C$. If $c \neq 0$, then $c \in E$, because $d(0,E) = 0$ implies that $C \setminus \{0\} \subseteq E$. This is impossible, because $f(x) \in O^E(f(A))$, so $f(x) = f(\tilde{x})$.

Now, assumption (b) implies that $x = \tilde{x}$, in fact, on the contrary,

$$C \setminus \{0\} \ni \frac{f(x)+f(\tilde{x})}{2} - f(\frac{x+\tilde{x}}{2}) + \frac{f(x)-f(\tilde{x})}{2} = f(x) - f(\frac{x+\tilde{x}}{2}),$$

which again contradicts $f(x) \in O^E(f(A))$. Hence $x = \tilde{x} = w-\lim x_n$, and the inclusion $\mathrm{Eff}(f,A,E) \subseteq w-\liminf \mathrm{Eff}(f_n,A_n,E)$ is proved.

We now consider the upper inclusion $w-\limsup \mathrm{Eff}(f_n,A_n,E) \subseteq \mathrm{Eff}(f,A,E)$, under the further assumptions from (c) to (f). If $x_k \in \mathrm{Eff}(f_{n_k},A_{n_k},E), k \in \mathbb{N}$ and $x_k \rightharpoonup x \in X$, then assumption (a) gives $x \in A$. Hence, by assumption (c), $f_{n_k}(x_k) \xrightarrow{s} f(x)$ and, by Proposition 5.19, we have that $f(x) \in O^E(f(A))$, which means $x \in \mathrm{Eff}(f,A,E)$. This concludes the proof. $\square$

Theorem 5.30. *Let* X, Y *be reflexive Banach spaces, with* Y *partially ordered by a closed convex and pointed cone* $C \subset Y$*, and* $E \subset Y$ *a* C*-i.s. such that* $d(0,E) = 0$*. Let* $f, f_n : X \to Y$ *and* $A, A_n \subset X$*. We assume that all the conditions from* (i) *to* (viii) *of Proposition* 5.12 *are true. In addition*

(a) $w-\limsup A_n \subseteq A$;

(b) *for any* $n \in \mathbb{N}$*, any* $\lambda \in]0,1[$*, and any* $a, b \in A$*,* $a \neq b$*, we have*
$f(\lambda a + (1-\lambda)b) \in (f(a) - C \setminus \{0\}) \cup (f(b) - C \setminus \{0\})$*, and*
$f_n(\lambda a + (1-\lambda)b) \in (f_n(a) - C \setminus \{0\}) \cup (f_n(b) - C \setminus \{0\})$.

Then $\mathrm{Eff}(f_n,A_n,E) \xrightarrow{w-K} \mathrm{Eff}(f,A,E)$.

Proof. The proof of the inclusion $\text{Eff}(f, A, E) \subseteq w-\liminf \text{Eff}(f_n, A_n, E)$ is analogous to the one of the previous proposition.

For the opposite inclusion, we consider $x_k \in \text{Eff}(f_{n_k}, A_{n_k}, E), k \in \mathbb{N}$, $x_k \rightharpoonup x \in X$, then assumption (a) gives $x \in A$. We prove that $f(x) \in O^E(f(A))$. If, on the contrary

$$f(x) - e = f(a) \tag{14}$$

for some $e \in E$, $a \in A$, thanks to the epiconvergence, there exists a sequence $(a_n, y_n)_{n\in\mathbb{N}}$ such that $epi\, f_{n|A_n} \ni (a_n, y_n) \xrightarrow{s} (a, f(a))$. Clearly $a_{n_k} \neq x_k$ for large k, otherwise $a = x$ against (14).

Let now $s_k = \frac{1}{k} a_{n_k} + (1 - \frac{1}{k}) x_k$, which, by convexity, belongs to A_{n_k}, and weakly converges to x. In virtue of assumption (b), we have $f_{n_k}(x_k) - f_{n_k}(s_k) \in C \setminus \{0\}$ or $f(a_{n_k}) - f_{n_k}(s_k) \in C \setminus \{0\}$. The first case is impossible, because $s_k \neq x_k$ and $f(x_k) \in O^E(f_{n_k}(A_{n_k}))$. Therefore, $f_{n_k}(s_k) \preceq_Y f_{n_k}(a_k) \preceq_Y y_{n_k} \xrightarrow{s} f(a)$, so, by epiconvergence, $f(x) \preceq_Y f(a) = f(x) - e$. This implies that $-e \in C$, which is impossible, because it would entail $0 = e - e \in E$, while E is a C-i.s. This concludes the proof. □

Remark 5.31.
(a) Assumption (d) in Theorem 5.29 is essential as can be shown by the following example.

Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $E = C \setminus \{(0, 0)\} = \{(0, \lambda) : \lambda > 0\}$, $A_n = A = [0, \infty[$ and $f_n \colon A_n \to Y$, $f \colon A \to Y$, defined by $f_n(t) = (\dfrac{t}{n}, t^2)$, $f(t) = (0, t^2)$, $t \in [0, \infty[$, $n \in \mathbb{N}$, $n \geq 1$. It is clear that $f_{n|A_n} \xrightarrow{c.ws} f_{|A}$ and this, thanks to Remark 5.25 (ii), implies that $epi\, f_{n|A_n} \xrightarrow{M} epi\, f_{|A}$. Also the other assumptions in Theorem 5.29 are easily verified, in particular the E-domination property of each $f_n(A_n)$, because any its point is in $O^E(f_n(A_n))$. However, $\text{Eff}(f_n, A_n, E) = [0, \infty[$ for every $n \in \mathbb{N}$, while $\text{Eff}(f, A, E) = \{0\}$.

(b) Also assumption (c) in Theorem 5.29 is essential as can be shown by the following example.

Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $E = C \setminus \{(0, 0)\} = [0, \infty[\times [0, \infty[\setminus \{(0, 0)\}$, $A_n = A = [0, \infty[$, $f_n \colon A_n \to Y$, $f \colon A \to Y$, defined by $f_n(t) = (t, -\dfrac{1}{n}\sqrt{t})$, $f(t) = (t^2, t^4)$, $t \in [0, \infty[$, $n \in \mathbb{N}$, $n \geq 1$. Obviously $epi\, f_{n|A_n} = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0,\ y_2 \geq -\dfrac{1}{n}\sqrt{y_1}\}$, $epi\, f_{|A} = C$ and $epi\, f_{n|A_n} \xrightarrow{M} epi\, f_{|A}$. Moreover f, f_n are strictly C-convex and all the other assumptions of Theorem 5.29 are verified, except (c). The conclusion of Theorem 5.29 fails, because $\text{Eff}(f_n, A_n, E) = [0, \infty[$ for every $n \in \mathbb{N}$, while $\text{Eff}(f, A, E) = \{0\}$.

Errata Corrige. We recall Theorem 3.21 of [12].

Let X be a normed space, Y an ordered topological vector space with positive cone C such that $int\, C \neq \emptyset$. Let $f,\ f_n : X \to Y$ and $A,\ A_n \subset X$ be such that

- the sequence $(A_n)_{n\in\mathbb{N}}$ is recessively w-compact in X and $(A_n)_{n\in\mathbb{N}}$ converges to A in the sense of Mosco;
- $f,\ f_n$ are C-convex and locally upper bounded (i.e. for any $x \in X$ there exist a neighborhood U of x and $y \in Y$ such that $y - f(x') \in C\ \forall\, x' \in U$);
- $\forall\, x \in A$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in A_n$, $x_n \xrightarrow{s} x$, such that $f_n(x_n) \xrightarrow{s} f(x)$;
- $\forall\, x \in A, x_n \in A_n$, $x_n \rightharpoonup x$ and every $\varepsilon \in int\, C$ there exists $k_{\varepsilon,x} \in \mathbb{N}$ such that $f_n(x_n) - f(x) + \varepsilon \in C\ \ \forall n \geq k_{\varepsilon,x}$;
- there not exists $\bar{x} \in X \setminus \{0\}$, $y \in Y$ such that $x + \lambda\bar{x} \in A, y - f(x + \lambda\bar{x}) \in C,\ \ \forall x \in A, \lambda \geq 0$

Then

$$s\!-\!\limsup O^{intC}(f_n(A_n)) \subseteq O^{intC}(f(A)). \tag{15}$$

This conclusion is incorrect and must be replaced by

$$f(A) \cap s\!-\!\limsup O^{intC}(f_n(A_n)) \subseteq O^{intC}(f(A)). \tag{16}$$

In fact, it is easy to check that the example in Remark 5.21 satisfies all the above assumptions, however, as already noticed about Propositions 5.19 and 5.20, the inclusion (15) fails, while the same proof shows that (16) holds.

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