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Existence of Solutions for a Nonlocal Variational Problem in $\mathbb{R}^2$ with Exponential Critical Growth

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We study the existence of nontrivial solutions for the following class of nonlocal problem,

$$-\Delta u + V(x)u = \Big(I_\mu * F(x,u)\Big)f(x,u) \quad \text{in } \mathbb{R}^2,$$

where V is a positive periodic potential, $I_\mu = |x|^{-\mu}$, $0 < \mu < 2$ and $F(x,s)$ is the primitive function of $f(x,s)$ in the variable s. By assuming that the nonlinearity $f(x,s)$ has an exponential critical growth at infinity, we prove the existence of solutions by variational methods.

Keywords: Nonlocal nonlinearities, exponential critical growth, ground state solution.

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1. Introduction and main results

During recent years much attention has been given to the problem

$$\begin{cases} -\Delta u + V(x)u = \Big(I_\mu * F(x,u)\Big)f(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \\ u(x) > 0 \ \text{ for all } \ x \in \mathbb{R}^N, \end{cases} \tag{P}$$

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where $0 < \mu < N$, $I_\mu = |x|^{-\mu}$, $F(x,s)$ is the primitive function of $f(x,s)$ in the variable s and V, f are continuous verifying some conditions. Here $I_\mu * F(x,u)$ denotes the convolution between I_μ and $F(.,u(.))$.

This problem comes from looking for standing waves of the nonlinear nonlocal Schrödinger equation which is known to influence the propagation of electromagnetic waves in plasmas [8] and also plays an important role in the theory of Bose-Einstein condensation [13]. It is used in the description of the quantum theory of a polaron at rest by S. Pekar in 1954 [23] and the modeling of an electron trapped in its own hole in 1976 in the work of P. Choquard, in a certain approximation to Hartree-Fock theory of one-component plasma [17].

If $F(x,s) = |s|^q$, then we arrive at the Choquard-Pekar equation,

$$-\Delta u + V(x)u = \Big(\frac{1}{|x|^\mu} * |u|^q\Big)|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N. \tag{1}$$

For the case $N \geq 3$, $V(x) = 1$, Lieb [17] proved the existence and uniqueness, up to translations, of the ground state to equation (1). Later, in [19], Lions showed that the solution of (1) is radially symmetric. Involving the properties of the ground state solutions, Ma and Zhao [20] proved that every positive solution of equation (1) is radially symmetric and monotone decreasing about some point, under the assumption that a certain set of real numbers, defined in terms of N, μ and q, is nonempty. Under the same assumption, Cingolani, Clapp and Secchi [12] proved the existence and multiplicity results in the electromagnetic case, and established the regularity and decay behavior at infinity of the ground states of (1). Moroz and Van Schaftingen [21] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and derived decay property at infinity as well. When V is a continuous periodic function with $\inf_{\mathbb{R}^N} V(x) > 0$, noticing that the nonlocal term is invariant under translation, one can obtain the existence results easily by applying the Mountain Pass Theorem, see [1] for example. For periodic potential V that changes sign and 0 lies in the gap of the spectrum of the Schrödinger operator $-\Delta + V$, the problem is strongly indefinite and has been considered in [9]. In that paper, the existence of nontrivial solutions with $\mu = 1$ and $F(u) = u^2$ was obtained by using the reduction methods. For a general class of response function Q and nonlinearity f, Ackermann [1] proposed an approach to prove the existence of infinitely many geometrically distinct weak solutions.

In the study of the above papers, the following Hardy-Littlewood-Sobolev inequality is crucial.

Proposition 1.1 ([18], Hardy-Littlewood-Sobolev inequality). *Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$. If $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(t,\mu,r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} \leq C(t,\mu,r)|f|_t|h|_r.$$

This inequality permits to use variational methods to show study the existence of nontrivial solutions for problem (P) when the nonlinearity f has a subcritical growth. However, we can observe that the Hardy-Littlewood-Sobolev inequality still holds for $N = 2$, motivated by this fact, it seems to be interesting, at least from a mathematical point of view, to ask if the existence of nontrivial solution still holds for nonlinearities f with "*exponential subcritical growth*" or "*exponential critical growth*" in $\mathbb{R}^2$. Indeed, if the nonlinearity f has an exponential growth, a lot of estimates made for the case $N \geq 3$ cannot be repeated easily for the case $N = 2$ and the well known Trudinger-Moser inequality will be a crucial tool to work with this type of nonlinearities. Here, we focus our attention to this much more difficult case, that is, f has an "*exponential critical growth*" in $\mathbb{R}^2$. However, we would like to point out that we cannot say that problem (P) in $\mathbb{R}^2$ is a *nonlinear Choquard equation*, because the kernel associated with a Choquard equation in dimension 2, namely the term I_μ, must involve a logarithmic convolution potential.

Since we intend to work with a nonlinearity of "*exponential critical growth*" in $\mathbb{R}^2$, we recall that the function $f(x, s)$ has an *exponential critical growth* when it behaves like $e^{\alpha s^2}$ as $|s| \to +\infty$. More exactly, there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^2}} = 0, \quad \forall \alpha > \alpha_0, \quad \text{and} \quad \lim_{|s|\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^2}} = +\infty, \quad \forall \alpha < \alpha_0. \tag{2}$$

The above notion of criticality was introduced by Adimurthi and Yadava [4], see also de Figueiredo, Miyagaki and Ruf [10].

To work with exponential critical growth problems, one of the most important tools is the Trudinger-Moser inequality, which says that if Ω is a bounded domain in $\mathbb{R}^2$, then for all $\alpha > 0$ and $u \in H_0^1(\Omega)$, $e^{\alpha u^2} \in L^1(\Omega)$. Moreover, there exists a positive constant C such that

$$\sup_{u \in H_0^1(\Omega)\,:\,\|\nabla u\|_2 \leq 1} \int_\Omega e^{\beta u^2} \leq C|\Omega| \quad \text{if } \alpha \leq 4\pi,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. This inequality is optimal in the sense that for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the corresponding supremum is infinite. In the present paper, we are working in the whole space $\mathbb{R}^2$, in this way, it is much more convenient to use the following Trudinger-Moser type inequality in $H^1(\mathbb{R}^2)$ due to Cao [11].

Lemma 1.2. *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} \left[e^{\alpha|u|^2} - 1\right] < \infty. \tag{3}$$

Moreover, if $\|\nabla u\|^2_{L^2(\mathbb{R}^2)} \leq 1$, $\|u\|_{L^2(\mathbb{R}^2)} \leq M < \infty$ and $\alpha < \alpha_0 = 4\pi$, then there exists a constant C, which depends only on M and α, such that

$$\int_{\mathbb{R}^2} \left[e^{\alpha|u|^2} - 1\right] \leq C(M, \alpha). \tag{4}$$

We assume that $V : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies: There are $c_0 > 0$ and a continuous 1-periodic function $V_0 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$(V_1) \qquad 0 < c_0 \leq V(x) \leq V_0(x) \ \ \forall x \in \mathbb{R}^2$$

and

$$(V_2) \qquad |V(x) - V_0(x)| \to 0 \ \ |x| \to +\infty.$$

In the sequel, E denotes the Sobolev space $H^1(\mathbb{R}^2)$ equipped with the norm

$$\|u\| := \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) \right)^{1/2}$$

and $L^s(\mathbb{R}^2)$, for $1 \leq s \leq \infty$, denotes the Lebesgue space endowed with the usual norm $| \ \ |_s$. Since the imbedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ is continuous for any $p \in (2, +\infty)$, from the Hardy-Littlewood-Sobolev inequality, there is a best constant S_p verifying

$$S_p = \inf_{u \in E, u \neq 0} \frac{\left(\int_{\mathbb{R}^2} (|\nabla u|^2 + |V|_\infty |u|^2) \right)^{1/2}}{\left(\int_{\mathbb{R}^2} \left(I_\mu * |u|^p \right) |u|^p \right)^{\frac{1}{2p}}}.$$

Moreover, a standard minimizing argument shows that S_p is achieved by a positive radial function $u_p \in E$.

With respect to the function f, we consider the following assumptions:

(f_1) There is a 1-periodic continuous function $\tilde{f}(x, s)$ such that

$$0 \leq \tilde{f}(x, s) < f(x, s) \leq \ \ Ce^{4\pi s^2} \ \ \forall s \geq 0.$$

(f_2) $\displaystyle\lim_{s \to 0^+} \frac{f(x, s)}{s^{\frac{2-\mu}{2}}} = 0, \quad \lim_{s \to 0^+} \frac{\tilde{f}(x, s)}{s^{\frac{2-\mu}{2}}} = 0.$

(f_3) There exist $\theta \geq \tilde{\theta} > 2$ such that

$$0 < \theta F(x, s) \leq 2f(x, s)s, \quad 0 < \tilde{\theta}\tilde{F}(x, s) \leq 2\tilde{f}(x, s)s \qquad \forall s > 0,$$

where $F(x, t) = \int_0^t f(x, s)ds$ and $\tilde{F}(x, s) = \int_0^t \tilde{f}(x, s)ds$.

(f_4) There exists $p > \frac{4-\mu}{2}$ such that $F(x, s) \geq C_p s^p$ for all $s \geq 0$ where

$$C_p > \frac{\left[\frac{4\theta(p-1)}{(2-\mu)(\theta-2)}\right]^{\frac{p-1}{2}} S_p^p}{p^{\frac{p}{2}}}.$$

(f_5) For any fixed $x \in \mathbb{R}^2$, the functions $s \to f(x,s)$, $\tilde{f}(x,s)$ are increasing.

(f_6) $\tilde{F}(x,s) < F(x,s)$ for any $s \neq 0$ and there exists $A \in L^\infty(\mathbb{R}^2)$ such that $A(x) \to 0$ as $|x| \to \infty$, and

$$|\tilde{f}(x,s) - f(x,s)| \leq A(x)\Big(|s|^{\frac{2-\mu}{2}} + e^{4\pi s^2}\Big) \;\; \forall x \in \mathbb{R}^2 \text{ and } \forall s \in \mathbb{R}.$$

Here, we would like to point out that the above assumptions related to f hold for

$$f((x,y),s) = \left(\sin(x+y) + 4 - \frac{1}{x^2+y^2+1}\right) 2\alpha C_p |s|^{p-2} s e^{\alpha s^2}$$

with

$$0 < \alpha < 4\pi \quad \text{and} \quad p > \frac{2-\mu}{2}.$$

The first result of this paper is about the periodic case which says the following

Theorem 1.3. *Assume $N = 2$, $0 < \mu < 2$, (V_1) with $V = V_0$ and $(f_1) - (f_5)$ with $\tilde{f} = f$. Then, (P) has a ground state solution in $H^1(\mathbb{R}^2)$.*

Concerning the problem where the nonlinearity is asymptotically periodic, our main result is the following

Theorem 1.4. *Assume $N = 2, 0 < \mu < 2$, (V) and $(f_1) - (f_6)$. Then, (P) has a ground state solution in $H^1(\mathbb{R}^2)$.*

In this paper, C, C_i will denote positive constants and B_R will denote the open ball centered at the origin with radius $R > 0$. If E is a real Hilbert space and $I : E \to \mathbb{R}$ is a functional of class $C^1(E, \mathbb{R})$, we say that $(u_n) \subset E$ is a $(PS)_c$ sequence for I, when (u_n) satisfies

$$I(u_n) \to c \;\text{ and }\; I'(u_n) \to 0 \;\text{ as }\; n \to \infty.$$

Moreover, we say that I satisfies the $(PS)_c$ condition, if any $(PS)_c$ sequence possesses a convergent subsequence.

To conclude this introduction, we would like to cite some recent works involving exponential critical growth for the elliptic problem of the form

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^2.$$

See for example, Adimurthi and K. Sandeep [3], Adimurthi and Yang [4], Albuquerque, Alves and Medeiros [5], do Ó, Medeiros and Severo [14], do Ó and de Souza [15], Li and Ruf [16] and their references.

2. Mountain Pass Geometry

Since we are going to study the existence of positive solutions by variational methods, we will assume that

$$f(x,s)=0 \quad \forall s\leq 0 \text{ and } \forall x\in\mathbb{R}^2.$$

From $(f_1)-(f_3)$, for any $\varepsilon>0$, $p\geq 1$ and $\beta>1$, there exists $C_\varepsilon>0$ such that

$$|f(x,s)|\leq \varepsilon|s|^{\frac{2-\mu}{2}}+C(\varepsilon,p,\beta)|s|^{p-1}\big[e^{\beta 4\pi s^2}-1\big] \quad \forall s\in\mathbb{R}, \tag{5}$$

and

$$|F(x,s)|\leq \varepsilon|s|^{\frac{4-\mu}{2}}+C(\varepsilon,p,\beta)|s|^{p}\big[e^{\beta 4\pi s^2}-1\big] \quad \forall s\in\mathbb{R}. \tag{6}$$

From Lemma 1.2 and the Hölder inequality, we know that $F(x,u)\in L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$ for any $u\in H^1(\mathbb{R}^2)$. Then, by applying the Hardy-Littlewood-Sobolev inequality with $t=r=\frac{4}{4-\mu}$, we see that

$$\Big(I_\mu * F(x,u)\Big)F(x,u)\in L^1(\mathbb{R}^2),$$

and so, the energy functional $I:E\to\mathbb{R}$ associated with problem (P) given by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2+V(x)|u|^2)-\frac{1}{2}\int_{\mathbb{R}^2}\Big(I_\mu * F(x,u)\Big)F(x,u)$$

is well defined on E. Furthermore, $I\in C^1(E,\mathbb{R})$ and

$$I'(u)\varphi=\int_{\mathbb{R}^2}(\nabla u\nabla\varphi+V(x)u\varphi)-\int_{\mathbb{R}^2}\Big(I_\mu * F(x,u)\Big)f(x,u)\varphi \ \ \forall u,\varphi\in E.$$

Next, we will show that I verifies the Mountain Pass Geometry.

Lemma 2.1. *Assume* $0<\mu<2$, $(f_1)-(f_3)$ *and* (V_1). *Then,*

(1) *There exist* $\rho,\delta_0>0$ *such that* $I|_{S_\rho}\geq\delta_0>0$ $\forall u\in S_\rho=\{u\in E:\|u\|=\rho\}$.

(2) *There is* $e\in E$ *with* $\|e\|>\rho$ *such that* $I(e)<0$.

Proof. (1) For any $\varepsilon>0$, $p>1$ and $\beta>1$, there exists $C_\varepsilon>0$ such that

$$|F(x,s)|\leq \varepsilon|s|^{\frac{4-\mu}{2}}+C(\varepsilon,p,\beta)|s|^{p}\big[e^{\beta 4\pi s^2}-1\big] \ \forall s\in\mathbb{R},$$

from where it follows

$$|F(x,u)|_{\frac{4}{4-\mu}}\leq \varepsilon C|u|_2^{\frac{4-\mu}{2}}+C(\varepsilon,p,\beta)\big|u^p\big[e^{\beta 4\pi u^2}-1\big]\big|_{\frac{4}{4-\mu}}. \tag{7}$$

Since the imbedding $E \hookrightarrow L^p(\mathbb{R}^2)$ is continuous, for each $p \in (2, +\infty)$, there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^2} |u|^{\frac{4p}{4-\mu}} \left[e^{\beta 4\pi u^2} - 1\right]^{\frac{4}{4-\mu}} \leq (\int_{\mathbb{R}^2} |u|^{\frac{8p}{4-\mu}})^{\frac{1}{2}} (\int_{\mathbb{R}^2} \left[e^{\beta 4\pi u^2} - 1\right]^{\frac{4}{4-\mu}})^{\frac{1}{2}}$$
$$\leq C_1 \|u\|^{\frac{4p}{4-\mu}} \Big(\int_{\mathbb{R}^2} \left[e^{(\frac{4\beta}{4-\mu} 4\pi u^2)} - 1\right]\Big)^{\frac{1}{2}}.$$

Observing that

$$\int_{\mathbb{R}^2} \left[e^{(\frac{4\beta}{4-\mu} 4\pi u^2)} - 1\right] = \int_{\mathbb{R}^2} \left[e^{(\frac{4\beta}{4-\mu} \|u\|^2 4\pi \frac{u^2}{\|u\|^2})} - 1\right],$$

by fixing $\xi \in (0,1)$ and $\frac{4\beta}{4-\mu}\|u\|^2 = \xi < 1$,we conclude from Lemma 1.2 that

$$\int_{\mathbb{R}^2} \left[e^{\xi 4\pi \frac{u^2}{\|u\|^2}} - 1\right] \leq C_2 \quad \text{for} \quad \|u\| = \left(\frac{\xi(4-\mu)}{4\beta}\right)^{\frac{1}{2}},$$

for some positive constant C_2. Gathering the last estimate and (7), there exists $C_3 > 0$ such that

$$|F(x,u)|_{\frac{4}{4-\mu}} \leq \varepsilon \|u\|^{\frac{4-\mu}{2}} + C_3 \|u\|^p \quad \text{for} \quad \|u\| = \left(\frac{\xi(4-\mu)}{4\beta}\right)^{\frac{1}{2}}.$$

Hence, by the Hardy-Littlewood-Sobolev inequality, we get

$$\int_{\mathbb{R}^2} \Big(I_\mu * F(x,u)\Big) F(x,u) \leq \varepsilon^2 C \|u\|^{4-\mu} + 2C_3 \|u\|^{2p} \quad \text{for} \quad \|u\| = \left(\frac{\xi(4-\mu)}{4\beta}\right)^{\frac{1}{2}},$$

and so,

$$I(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon^2 C \|u\|^{4-\mu} - C_3 \|u\|^{2p}.$$

Since $0 < \mu < 2$ and $p > 1$, the conclusion (1) follows by choosing $\rho = \left(\frac{\xi(4-\mu)}{4\beta}\right)^{\frac{1}{2}}$ with $\xi \approx 0^+$.

(2) Fixing $u_0 \in E$ with $u_0^+(x) = \max\{u_0(x), 0\} \neq 0$, we set

$$\mathcal{A}(t) = \Psi(\frac{tu_0}{\|u_0\|}) > 0 \quad \text{for} \quad t > 0,$$

where

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^2} \Big(I_\mu * F(x,u)\Big) F(x,u).$$

A straightforward computation yields

$$\frac{\mathcal{A}'(t)}{\mathcal{A}(t)} \geq \frac{\theta}{t} \text{ for all } t > 0.$$

Then, integrating this over $[1, s\|u_0\|]$ with $s > \frac{1}{\|u_0\|}$, we find

$$\Psi(su_0) \geq \Psi(\frac{u_0}{\|u_0\|})\|u_0\|^\theta s^\theta.$$

Therefore

$$\Phi(su_0) \leq C_1 s^2 - C_2 s^\theta \text{ for } s > \frac{1}{\|u_0\|},$$

and the conclusion (2) holds for $e = su_0$ with s large enough. □

As a consequence of the Mountain Pass Theorem without (PS) condition in [27], there is a $(PS)_{c_V}$ sequence $(u_n) \subset E$, that is,

$$I(u_n) \to c_V \text{ and } I'(u_n) \to 0, \tag{8}$$

where c_V is the mountain pass level characterized by

$$0 < c_V := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \tag{9}$$

with

$$\Gamma := \{\gamma \in \mathcal{C}^1([0,1], E) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

The next lemma is crucial in our arguments, because it establishes an important estimate involving the level c_V.

Lemma 2.2. *The mountain pass level c_V satisfies $c_V \in [\rho, \frac{(2-\mu)(\theta-2)}{8\theta})$, the $(PS)_{c_V}$ sequence (u_n) is bounded and its weak limit u satisfies $I'(u) = 0$.*

Proof. From (f_3),

$$c_V = \lim \Big(I(u_n) - \frac{1}{\theta} I'(u_n)u_n\Big) \geq (\frac{1}{2} - \frac{1}{\theta}) \limsup \|u_n\|^2$$

which means

$$\limsup \|u_n\|^2 \leq \frac{2\theta}{\theta - 2} c_V. \tag{10}$$

Let $u_p \in E$ be a positive radial function verifying

$$S_p = \inf_{u\in E, u\neq 0} \frac{\Big(\int_{\mathbb{R}^2} (|\nabla u_p|^2 + |V|_\infty |u_p|^2)\Big)^{1/2}}{\Big(\int_{\mathbb{R}^2} \Big(I_\mu * |u_p|^p\Big)|u_p|^p\Big)^{\frac{1}{2p}}}.$$

By (f_4), it is easy to see that

$$\begin{aligned} c_V &= \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) \le \inf_{u\in E\setminus\{0\}}\max_{t\ge 0} I(tu) \le \max_{t\ge 0} I(tu_p) \\ &\le \max_{t\ge 0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^2}(|\nabla u_p|^2 + |V|_\infty |u_p|^2) - \frac{t^{2p}C_p^2}{2}\int_{\mathbb{R}^2}\Big(I_\mu * |u_p|^p\Big)|u_p|^p\right\} \\ &= \frac{(p-1)S_p^{\frac{2p}{p-1}}}{2p^{\frac{p}{p-1}}C_p^{\frac{2}{p-1}}} < \frac{(2-\mu)(\theta-2)}{8\theta} \ . \end{aligned}$$

Consequently, from (10),

$$\limsup \|u_n\|^2 < \frac{(2-\mu)}{4}.$$

In the following, we may assume that there are $n_0 \in \mathbb{N}$ and $m \in (0, \frac{(2-\mu)}{4})$ such that

$$\|u_n\|^2 \le m \ \ \forall n \ge n_0. \tag{11}$$

Without loss of generality, in what follows we suppose that $n_0 = 1$.

Claim 2.3. *There exists $C > 0$ such that*

$$|I_\mu * F(x, u_n)|_\infty < C \quad \forall n \in \mathbb{N}.$$

Proof. For each $\beta > 1$, there exists $C_0 > 0$ such that

$$F(x, s) \le C_0\Big(|s|^{\frac{4-\mu}{2}} + |s|\big[e^{\beta 4\pi s^2} - 1\big]\Big) \ \ \forall s \in \mathbb{R}.$$

Hence,

$$\begin{aligned} |I_\mu * F(x, u_n)(x)| &\le \Big|\int_{\mathbb{R}^2}\frac{F(x,u_n)}{|x-y|^\mu}\Big| = \Big|\int_{|x-y|\le 1}\frac{F(x,u_n)}{|x-y|^\mu}\Big| + \Big|\int_{|x-y|\ge 1}\frac{F(x,u_n)}{|x-y|^\mu}\Big| \\ &\le C_0\int_{|x-y|\le 1}\frac{|u_n|^{\frac{4-\mu}{2}} + |u_n|\big[e^{\beta 4\pi |u_n|^2} - 1\big]}{|x-y|^\mu} + \\ &\qquad + C_0\int_{|x-y|\ge 1}\Big(|u_n|^{\frac{4-\mu}{2}} + |u_n|\big[e^{\beta 4\pi|u_n|^2} - 1\big]\Big). \end{aligned}$$

Since

$$\frac{1}{|y|^\mu} \in L^{\frac{2+\delta}{\mu}}(B_1^c(0)) \ \ \forall \ \ \delta > 0\,,$$

we take $\delta \approx 0^+$ such that

$$q_{1,\delta} = \frac{(4-\mu)}{2}\frac{(2+\delta)}{(2+\delta)-\mu} > 2\ .$$

Using the Hölder inequality, we derive that

$$\int_{|x-y|\geq 1} \frac{|u_n|^{\frac{4-\mu}{2}}}{|x-y|^\mu} \leq C_0 \left(\int_{|x-y|\geq 1} |u_n|^{q_{1,\delta}} \right)^{\frac{(2+\delta)-\mu}{2+\delta}} \leq C_1 \ \ \forall n \in \mathbb{N}. \tag{12}$$

By (11), we can fix $\beta > 1$ close to 1, such that $2\beta m \in (0,1)$. Then, using the Trudinger-Moser inequality, there exists $C_2 > 0$ such that

$$\int_{|x-y|\geq 1} |u_n| \left[e^{\beta 4\pi u_n^2} - 1 \right] \leq |u_n|_2 \int_{\mathbb{R}^2} \left(\left[e^{2\beta m 4\pi \frac{u_n^2}{\|u_n\|^2}} - 1 \right] \right)^{\frac{1}{2}} \leq C_2 \ \ \forall n \in \mathbb{N}. \tag{13}$$

Choosing $t \in \left(\frac{2}{2-\mu}, +\infty\right)$, we see that $\frac{(4-\mu)t}{2} > 2$ and $1 - \frac{t\mu}{t-1} > -1$. Thus, by Hölder's inequality,

$$\begin{aligned} \int_{|x-y|\leq 1} \frac{|u_n|^{\frac{4-\mu}{2}}}{|x-y|^\mu} &\leq \Big(\int_{|x-y|\leq 1} |u_n|^{\frac{(4-\mu)t}{2}} \Big)^{\frac{1}{t}} \Big(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}} \Big)^{\frac{t-1}{t}} \\ &\leq C_2 \Big(\int_{|r|\leq 1} |r|^{1-\frac{t\mu}{t-1}} dr \Big)^{\frac{t-1}{t}} \leq C_3 \ \ \forall n \in \mathbb{N}, \end{aligned} \tag{14}$$

for some $C_3 > 0$. Now, for $t > \frac{2}{2-\mu}$ and close to $\frac{2}{2-\mu}$, we can assume that $2\beta t m \in (0,1)$. Thus, combining the Trudinger-Moser inequality and the boundedness of (u_n) in E we get

$$\begin{aligned} \int\limits_{|x-y|\leq 1} \frac{|u_n| \left[e^{\beta 4\pi u_n^2} - 1 \right]}{|x-y|^\mu} &\leq \Big(\int\limits_{|x-y|\leq 1} |u_n \left[e^{\beta 4\pi u_n^2} - 1 \right]|^t \Big)^{\frac{1}{t}} \Big(\int\limits_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}} \Big)^{\frac{t-1}{t}} \\ &\leq \Big(\int\limits_{|x-y|\leq 1} |u_n|^{2t} \Big)^{\frac{1}{2t}} \Big(\int\limits_{|x-y|\leq 1} \left[e^{2\beta t m 4\pi \frac{u_n^2}{\|u_n\|^2}} - 1 \right] \Big)^{\frac{1}{2t}} \Big(\int\limits_{|r|\leq 1} |r|^{1-\frac{t\mu}{t-1}} dr \Big)^{\frac{t-1}{t}}, \end{aligned}$$

implying that there is $C_4 > 0$ such that

$$\int_{|x-y|\leq 1} \frac{|u_n| \left[e^{\beta 4\pi u_n^2} - 1 \right]}{|x-y|^\mu} \leq C_4 \ \ \forall n \in \mathbb{N}. \tag{15}$$

Now, the claim follows from (12)–(15). □

Claim 2.4. *Let (u_n) be the $(PS)_{c_V}$ sequence with weak limit u. Then, u satisfies $I'(u) = 0$.*

Indeed, since (u_n) is bounded in E, going to a subsequence still denoted by (u_n), there is $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^q_{loc}(\mathbb{R}^2)$ for all $q \in [1, +\infty)$, and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^2$.

For each $\varphi \in C_0^\infty(\mathbb{R}^2)$, the Hardy-Littlewood-Sobolev inequality together with the Hölder inequality leads to

$$\Big|\int_{\mathbb{R}^2}\Big(I_\mu * F(x,u_n)\Big)f(x,u_n)\varphi\Big| \le |F(x,u_n)|_{\frac{4}{4-\mu}}|f(x,u_n)|_{\frac{4r}{4-\mu}}|\varphi|_{\frac{4r'}{4-\mu}} \tag{16}$$

where $r=\frac{4-\mu}{2-\mu}$ and $r'>1$ satisfies $\frac{1}{r}+\frac{1}{r'}=1$. From $(f_1)-(f_3)$, for each $\varepsilon>0$ and $\beta>1$, there is $C(\varepsilon,\beta)>0$ such that

$$|f(x,s)| \le \varepsilon|s|^{\frac{2-\mu}{2}} + C(\varepsilon,\beta)\big[e^{\beta 4\pi s^2}-1\big] \quad \forall s \in \mathbb{R},$$

and

$$|F(x,s)| \le \varepsilon|s|^{\frac{4-\mu}{2}} + C(\varepsilon,\beta)|s|\big[e^{\beta 4\pi s^2}-1\big] \quad \forall s \in \mathbb{R}.$$

Then

$$\begin{aligned}|f(x,u_n)|_{\frac{4r}{4-\mu}} &\le \varepsilon|u_n|_2^{\frac{2-\mu}{2}} + C(\varepsilon,\beta)|e^{\beta 4\pi u_n^2}-1|_{\frac{4r}{4-\mu}}\\ &\le C_1\|u_n\|^{\frac{2-\mu}{2}} + C_1\Big(\int_{\mathbb{R}^2}[e^{(\frac{4\beta r}{4-\mu}\|u_n\|^2 4\pi\frac{u_n^2}{\|u_n\|^2})}-1]\Big)^{\frac{4-\mu}{4r}}.\end{aligned}$$

Now, choosing $\beta>1$ sufficiently close to 1, the estimate (11) and the Trudinger-Moser imply that the sequence $(f(x,u_n))$ is bounded in $L^{\frac{4}{2-\mu}}(\mathbb{R}^2)$. Moreover, by a similar argument, we also know that the sequence $(F(x,u_n))$ is bounded in $L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$.

In the sequel, we will prove that for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ the limit below holds:

$$\int_{\mathbb{R}^2}\Big(I_\mu * F(x,u_n)\Big)f(x,u_n)\varphi \to \int_{\mathbb{R}^2}\Big(I_\mu * F(x,u)\Big)f(x,u)\varphi.$$

In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\Big(\big(I_\mu * F(x,u_n)\big)f(x,u_n) - \big(I_\mu * F(x,u)\big)f(x,u)\Big)\varphi\Big| \le\\ &\quad\le \Big|\int_{\mathbb{R}^2}\big(I_\mu * F(x,u_n)\big)\Big(f(x,u_n)-f(x,u)\Big)\varphi\Big| + \\ &\qquad + \Big|\int_{\mathbb{R}^2}\Big(I_\mu * \big(F(x,u_n)-F(x,u)\big)\Big)f(x,u)\varphi\Big|.\end{aligned} \tag{17}$$

For the above first term, we recall that $(I_\mu * F(x,u_n))$ is bounded in $L^\infty(\mathbb{R}^2)$. Then,

$$\Big|\int_{\mathbb{R}^2}\big(I_\mu * F(x,u_n)\big)\Big(f(x,u_n)-f(x,u)\Big)\varphi\Big| \le C\Big|\int_{\mathbb{R}^2}\big(f(x,u_n)-f(x,u)\big)\varphi\Big|.$$

Since $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^2$, the continuity of f implies $f(x, u_n(x)) \to f(x, u(x))$ a.e. in $\mathbb{R}^2$. This fact combined with the boundedness of $(f(x, u_n))$ in $L^{\frac{4}{2-\mu}}(\mathbb{R}^2)$ leads to

$$f(x, u_n) \rightharpoonup f(x, u) \quad \text{in} \quad L^{\frac{4}{2-\mu}}(\mathbb{R}^2),$$

from where it follows that

$$\int_{\mathbb{R}^2} \big(f(x, u_n) - f(x, u)\big)\varphi \to 0\,.$$

Consequently,

$$\Big| \int_{\mathbb{R}^2} \big(I_\mu * F(x, u_n)\big)\Big(f(x, u_n) - f(x, u)\Big)\varphi\Big| \to 0, \tag{18}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2)$. For the second term, one notices that

$$\begin{aligned}\Big| \int_{\mathbb{R}^2} \Big(I_\mu * \big(F(x, u_n) - F(x, u)\big)\Big) f(x, u)\varphi\Big| \\ = \Big| \int_{\mathbb{R}^2} \big(F(x, u_n) - F(x, u)\big) I_\mu * (f(x, u)\varphi)\Big|.\end{aligned}$$

Since $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^2$, the continuity of F implies $F(x, u_n(x)) \to F(x, u(x))$ a.e. in $\mathbb{R}^2$. Using the boundedness of $(F(x, u_n))$ in $L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$, we conclude that

$$F(x, u_n) \rightharpoonup F(x, u) \quad \text{in} \quad L^{\frac{4}{4-\mu}}(\mathbb{R}^2).$$

As $I_\mu * (f(x, u)\varphi) \in L^{\frac{4}{\mu}}(\mathbb{R}^2)$, we must have

$$\Big| \int_{\mathbb{R}^2} \Big(I_\mu * \big(F(x, u_n) - F(x, u)\big)\Big) f(x, u)\varphi\Big| \to 0\,, \tag{19}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2)$. Now, the result follows by using the density of $C_0^\infty(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$. □

3. Proof of the main results

In this section, we will prove Theorems 1.3 and 1.4.

3.1. Proof of Theorem 1.3.

Let (u_n) be the $(PS)_{c_V}$ sequence. Since (u_n) is bounded and $\limsup \|u_n\|^2 < \frac{(2-\mu)}{4}$, we have either (u_n) is vanishing, i.e., there exists $r > 0$ such that

$$\limsup_{y \in \mathbb{R}^2} \int_{B_r(y)} |u_n|^2 = 0\,,$$

or non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $(y_n) \subset \mathbb{Z}^2$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \geq \delta\,.$$

If (u_n) is vanishing, then Lions' result ensures that

$$u_n \to 0 \text{ in } L^s(\mathbb{R}^2), \quad 2 < s < +\infty.$$

Using the Hardy-Littlewood-Sobolev inequality and condition (f_3), we derive

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big) f(x,u_n)u_n\Big| &\leq C|F(x,u_n)|_{\frac{4}{4-\mu}} |f(x,u_n)u_n|_{\frac{4}{4-\mu}} \\ &\leq C|f(x,u_n)u_n|^2_{\frac{4}{4-\mu}}.\end{aligned}$$

For any $\varepsilon > 0$, $p > 1$ and $\beta > 1$, there exists $C(\varepsilon, p, \beta) > 0$ such that

$$|f(x,s)| \leq \varepsilon |s|^{\frac{2-\mu}{2}} + C(\varepsilon,p,\beta)|s|^{p-1}\big[e^{\beta 4\pi s^2} - 1\big] \quad \forall s \in \mathbb{R}.$$

Then,

$$|f(x,u_n)u_n|_{\frac{4}{4-\mu}} \leq \varepsilon |u_n|_2^{\frac{4-\mu}{2}} + C(\varepsilon,p,\beta)|u_n|^{\frac{4-\mu}{4t'}}_{\frac{4pt'}{4-\mu}} \Big(\int_{\mathbb{R}^2} [e^{(\frac{4\beta t}{4-\mu}\|u_n\|^2 4\pi \frac{u_n^2}{\|u_n\|^2})} - 1]\Big)^{\frac{4-\mu}{4t}}$$

where $t, t' > 1$ satisfying $\frac{1}{t} + \frac{1}{t'} = 1$. Now, using (11) and the Trudinger-Moser inequality, if $\beta, t > 1$ are fixed close to 1, we deduce that

$$\Big(\int_{\mathbb{R}^2} [e^{(\frac{4\beta t}{4-\mu}\|u_n\|^2 4\pi \frac{u_n^2}{\|u_n\|^2})} - 1]\Big)^{\frac{4-\mu}{4t}} \leq \Big(\int_{\mathbb{R}^2} [e^{(\frac{4\beta m t}{4-\mu} 4\pi \frac{u_n^2}{\|u_n\|^2})} - 1]\Big)^{\frac{4-\mu}{4t}} \leq C_1 \quad \forall n \in \mathbb{N},$$

for some $C_1 > 0$. Then,

$$\Big|\int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big) f(x,u_n)u_n\Big| \leq \varepsilon |u_n|_2^{\frac{4-\mu}{2}} + C_2 |u_n|^{\frac{4-\mu}{4t'}}_{\frac{4pt'}{4-\mu}}.$$

Since $t > 1$ is close to 1, we have that $\frac{4pt'}{4-\mu} > 2$. Consequently,

$$\int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big) f(x,u_n)u_n \to 0\,,$$

which implies that $u_n \to 0$ in E as $n \to \infty$. Recalling that I is a continuous functional, we must have $I(u_n) \to 0$, implying $c_V = 0$, a contradiction. Hence, the vanishing case does not hold.

From now on, we set $v_n = u_n(\cdot - y_n)$. Therefore, $\|v_n\| = \|u_n\|$ and

$$\int_{B_r(0)} |v_n|^2 \geq \delta\,.$$

Using the definition of I, we see that I and I' are both invariant with respect to a $\mathbb{Z}^2$-translation. Then, $I(v_n) \to c_V$ and $I'(v_n) \to 0$.

Since (v_n) is also bounded, we may assume $v_n \rightharpoonup v$ in E and $v_n \to v$ in $L^2_{loc}(\mathbb{R}^2)$. From the last inequality we know $v \neq 0$, and following the same arguments as in Lemma 2.2 we get $I'(v) = 0$.

Let $\mathcal{N}$ be the Nehari manifold defined by

$$\mathcal{N} = \{u \in E : u \neq 0, I'(u)u = 0\}.$$

From (f_5), it is standard to check that the mountain pass level can be characterized by

$$c_V = \inf_{u \in E\{0\}} \max_{t \geq 0} I(tu) = \inf_{u \in \mathcal{N}} I(u).$$

The above characterization together with (f_3) gives $I(v) = c_V$, showing that v is a ground state solution.

3.2. Proof of Theorem 1.4.

In what follows, we will denote by $\tilde{I} : H^1(\mathbb{R}^2) \to \mathbb{R}$ the energy functional associated with the problem

$$\begin{cases} -\Delta u + V_0(x)u = \Big(I_\mu * \tilde{F}(x,u)\Big)\tilde{f}(x,u) \quad \text{in} \quad \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \\ u(x) > 0 \ \ \text{for all} \ \ x \in \mathbb{R}^2, \end{cases}$$

where $0 < \mu < 2$, $\tilde{F}(x,s)$ is the primitive function of $\tilde{f}(x,s)$ in the variable s and $V_0, \tilde{f}$ are continuous and 1-periodic. Thus,

$$\tilde{I}(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + V_0(x)|u|^2) - \frac{1}{2}\int_{\mathbb{R}^2} \Big(I_\mu * \tilde{F}(x,u)\Big)\tilde{F}(x,u) \ \ \forall u \in H^1(\mathbb{R}^2).$$

By Theorem 1.3, we know that there is $u_0 \in H^1(\mathbb{R}^2)$ such that

$$\tilde{I}'(u_0) = 0 \ \text{ and } \ \tilde{I}(u_0) = \tilde{c}_V,$$

where $\tilde{c}_V$ is the mountain pass level associated with $\tilde{I}$.

Using the same arguments as employed in the proof of Lemma 2.1, we can show that I verifies the Mountain Pass Geometry. Consequently, there is a (PS) sequence $(u_n) \subset E$ such that

$$I(u_n) \to c_V \ \text{ and } \ I'(u_n) \to 0, \tag{20}$$

where c_V is the mountain pass level characterized by

$$0 < c_V := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) = \inf_{u \in E\{0\}} \max_{t \geq 0} I(tu) = \inf_{u \in \mathcal{N}} I(u) \tag{21}$$

with

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0 \ \text{ and } \ I(\gamma(1)) < 0\}$$

and

$$\mathcal{N} = \{u \in E : u \neq 0, I'(u)u = 0\}.$$

Therefore, using the above notation,

$$\begin{aligned} c_V &= \inf_{u \in E\{0\}} \max_{t \geq 0} I(tu) \leq \max_{t \geq 0} I(tu_0) = I(t_0 u_0) \\ &< \tilde{I}(t_0 u_0) = \max_{t \geq 0} \tilde{I}(tu_0) = \tilde{I}(u_0) = \tilde{c}_V, \end{aligned}$$

that is,

$$c_V < \tilde{c}_V. \tag{22}$$

From Lemma 2.2 we get $\tilde{c}_V \in \big(0, \frac{(2-\mu)(\theta-2)}{8\theta}\big)$, and hence, $c_V \in \big(0, \frac{(2-\mu)(\theta-2)}{8\theta}\big)$. Recalling that $\theta \geq \tilde{\theta}$, it follows from (f_3),

$$\tilde{\theta} F(x,s) \leq f(x,s)s \ \ \forall x \in \mathbb{R}^2 \ \text{ and } \ s \in \mathbb{R}.$$

Arguing as in the proof Lemma 2.2, it follows that (u_n) is bounded in E with

$$\limsup \|u_n\|^2 < \frac{(2-\mu)}{4}\,.$$

Moreover, there is $u \in E$ such that for a subsequence,

$$u_n \rightharpoonup u \ \text{ in } \ E \ \text{ and } \ \tilde{I}'(u) = 0.$$

We claim that $u \neq 0$. To see why, we will argue by contradiction, supposing that $u = 0$. Notice that, for any $\varphi \in E$,

$$I(u_n) = \tilde{I}(u_n) + \frac{1}{2}\int_{\mathbb{R}^2} \Big(I_\mu * \tilde{F}(x,u_n)\Big)\tilde{F}(x,u_n) - \frac{1}{2}\int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big)F(x,u_n)$$

and

$$I'(u_n)\varphi = \tilde{I}'(u_n)\varphi + \int_{\mathbb{R}^2} \Big(I_\mu * \tilde{F}(x,u_n)\Big)\tilde{f}(x,u_n)\varphi - \int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big)f(x,u_n)\varphi.$$

By (f_6), it is possible to prove that

$$\int_{\mathbb{R}^2} \Big(I_\mu * \tilde{F}(x,u_n)\Big)\tilde{F}(x,u_n) - \int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big)F(x,u_n) \to 0 \tag{23}$$

and

$$\int_{\mathbb{R}^2} \Big(I_\mu * \tilde{F}(x,u_n)\Big)\tilde{f}(x,u_n)\varphi - \int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big)f(x,u_n)\varphi \to 0 \tag{24}$$

uniformly for $\|\varphi\| \leq 1, \varphi \in E$. Next, we will show only (24), because (23) follows with the same type of argument. For any $\varphi \in E$, considering $r = \frac{4-\mu}{2-\mu}$ and $r' > 1$ such that $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}^2} \Big(I_\mu * \tilde{F}(x,u_n)\Big)\tilde{f}(x,u_n)\varphi - \int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big) f(x,u_n)\varphi\Big| \leq \\
&\quad \leq \Big| \int_{\mathbb{R}^2} \Big(I_\mu * (\tilde{F}(x,u_n) - F(x,u_n))\Big)\tilde{f}(x,u_n)\varphi\Big| + \\
&\qquad\qquad + \Big| \int_{\mathbb{R}^2} \Big(I_\mu * F(x,u_n)\Big)(\tilde{f}(x,u_n) - f(x,u_n))\varphi\Big| \leq \\
&\quad \leq |\tilde{F}(x,u_n) - F(x,u_n)|_{\frac{4}{4-\mu}} |f(x,u_n)|_{\frac{4r}{4-\mu}} |\varphi|_2 \\
&\qquad\qquad + |F(x,u_n)|_{\frac{4}{4-\mu}} |\tilde{f}(x,u_n) - f(x,u_n)|_{\frac{4r}{4-\mu}} |\varphi|_2 \leq \\
&\quad \leq C|\tilde{F}(x,u_n) - F(x,u_n)|_{\frac{4}{4-\mu}} |f(x,u_n)|_{\frac{4r}{4-\mu}} + \\
&\qquad\qquad + C|F(x,u_n)|_{\frac{4}{4-\mu}} |\tilde{f}(x,u_n) - f(x,u_n)|_{\frac{4r}{4-\mu}}.
\end{aligned}
$$

Since $\limsup \|u_n\|^2 < \frac{2-\mu}{4}$, the proof used in the previous section shows that $(\tilde{F}(x,u_n))$ and $(\tilde{f}(x,u_n))$ are bounded sequences in $L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$ and $L^{\frac{4r}{4-\mu}}(\mathbb{R}^2)$ respectively.

On the other hand, from (f_6), there exists $C > 0$ such that

$$
|\tilde{f}(x,s) - f(x,s)| \leq CA(x)\Big(|s|^{\frac{2-\mu}{2}} + s\big[e^{\beta 4\pi s^2} - 1\big]\Big) \quad \forall s \in \mathbb{R} \text{ and } x \in \mathbb{R}^2.
$$

We then have

$$
\begin{aligned}
&|\tilde{f}(x,u_n) - f(x,u_n)|_{\frac{4}{2-\mu}} \leq \\
&\leq C(\int_{\mathbb{R}^2} |A(x)|^{\frac{4}{2-\mu}} |u_n|^2)^{\frac{2-\mu}{4}} + C(\int_{\mathbb{R}^2} |A(x)u_n|^{\frac{4}{2-\mu}} [e^{\frac{4\beta}{2-\mu}4\pi u_n^2} - 1])^{\frac{2-\mu}{4}} \\
&\leq C(\int_{\mathbb{R}^2} |A(x)|^{\frac{4}{2-\mu}} |u_n|^2)^{\frac{2-\mu}{4}} + C(\int_{\mathbb{R}^2} |A(x)u_n|^{\frac{4t'}{2-\mu}})^{\frac{2-\mu}{4t'}} (\int_{\mathbb{R}^2} [e^{(\frac{4\beta m t}{4-\mu}4\pi \frac{u_n^2}{\|u_n\|^2})} - 1])^{\frac{2-\mu}{4t}}.
\end{aligned}
$$

Fixing $t > 1$ sufficiently close to 1 and by Lemma 1.2 again, we get

$$
(\int_{\mathbb{R}^2} [e^{(\frac{4\beta m t}{4-\mu}4\pi \frac{u_n^2}{\|u_n\|^2})} - 1])^{\frac{2-\mu}{4t}} < C \ \ \forall n \in \mathbb{N},
$$

for some $C > 0$. Since $A \in L^\infty(\mathbb{R}^2)$ and $A(x) \to 0$ as $|x| \to \infty$, it easy to obtain

$$
(\int_{\mathbb{R}^2} |A(x)|^{\frac{4}{2-\mu}} |u_n|^2)^{\frac{2-\mu}{4}} \to 0 \quad \text{and} \quad (\int_{\mathbb{R}^2} |A(x)u_n|^{\frac{4t'}{2-\mu}})^{\frac{2-\mu}{4t'}} \to 0.
$$

Therefore

$$|\tilde{f}(x,u_n) - f(x,u_n)|_{\frac{4}{2-\mu}} \to 0,$$

and by a similar argument,

$$|\tilde{F}(x,u_n) - F(x,u_n)|_{\frac{4}{4-\mu}} \to 0,$$

showing (24). Consequently, the sequence $(u_n) \subset E$ satisfies

$$\tilde{I}'(u_n) \to 0 \text{ and } \tilde{I}(u_n) \to .c_V, \tag{25}$$

Repeating the arguments explored in the proof of Theorem 1.3, we will find a nontrivial critical point $\tilde{u}$ of $\tilde{I}$ verifying the estimate $\tilde{I}(\tilde{u}) \leq c_V$. However, this is a contradiction, because by definition of $\tilde{c}_V$, we must have

$$\tilde{c}_V \leq \tilde{I}(\tilde{u})$$

implying that $\tilde{c}_V \leq c_V$, which is absurd in view of (22). Thereby, the weak limit u of the $(PS)_{c_V}$ is nontrivial, finishing the proof of Theorem 1.4.

4. Final comments

The positive solution u obtained in Theorem 1.3 belongs to $L^\infty(\mathbb{R}^2)$ and decays to zero as $|x| \to \infty$. First of all, we would like to point out that the arguments used in the proof of Claim 2.3 imply that

$$I_\mu * F(x,v) \in L^\infty(\mathbb{R}^2) \;\; \forall v \in H^1(\mathbb{R}^2).$$

Thus, if u is a solution of

$$-\Delta u + V(x)u = \Big(I_\mu * F(x,u)\Big) f(x,u) \text{ in } \mathbb{R}^2,$$

since $V, I_\mu * F(x,u) \in L^\infty(\mathbb{R}^2)$ and $f(x,u) \in L^q(\mathbb{R}^2)$ for q large enough, we deduce by bootstrap arguments that $u \in L^\infty(\mathbb{R}^2)$ and

$$|u(x)| \to 0 \text{ as } |x| \to +\infty.$$

We would like to point out that it is possible to prove with a few modifications the existence of a solution for (P) for other classes of potentials V, such as:

1. $V(x) = V(|x|)$ and $f(x,s) = f(|x|,s)$ for all $x \in \mathbb{R}^2$ and $s \in \mathbb{R}$.

2. For all $M > 0$, we assume that

$$|\{x \in \mathbb{R}^2 \,:\, V(x) < M\}| < +\infty,$$

where $|\,.\,|$ stands for the Lebesgue measure in $\mathbb{R}^2$.

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