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Laminates Supported on Cubes

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We study the relationship between rank-one convexity and quasiconvexity in the space of 2×2 matrices. We show that a certain procedure for constructing homogeneous gradient Young measures from periodic deformations, that arises from V. Šverák's celebrated counterexample in higher dimensions, always yields laminates in the 2×2 case.

Keywords: Gradients, quasiconvexity, rank-one convexity, laminates.

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1. Introduction

A continuous function $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ is said to be *quasiconvex* if

$$\int_{\mathbb{T}^n} f\big(A+Du(x)\big)\,dx \geq f(A) \tag{1}$$

for any matrix $A\in\mathbb{R}^{m\times n}$ and any Lipschitz function $u:\mathbb{R}^n\to\mathbb{R}^m$ which is periodic with respect to the lattice $\mathbb{Z}^n$. Here $\mathbb{T}^n$ is the unit cube of $\mathbb{R}^n$. Equivalently, quasiconvexity may be defined by using smooth periodic or smooth compactly supported test functions $u\in C_c^\infty(\Omega;\mathbb{R}^m)$ for any bounded domain with Lipschitz boundary (see [8, 15, 9]). It is well known [8] that quasiconvexity of f is equivalent to the weak star lower-semicontinuity of the functional $u\mapsto \int_\Omega f\big(Du(x)\big)\,dx$ in $W^{1,\infty}(\Omega;\mathbb{R}^m)$.

Because of its fundamental importance in the calculus of variations, it is of interest to find necessary and sufficient conditions for quasiconvexity. The most well-known necessary condition is the rank-one convexity of f, namely that for any $A,B\in\mathbb{R}^{m\times n}$ with $\operatorname{rank}(B)=1$

$$t\mapsto f(A+tB) \quad \text{is convex.}$$

This arises by using test functions of the form

$$u(x)=\mathbf{a}h(x\cdot\mathbf{n}), \tag{2}$$

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where $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{n} \in \mathbb{Z}^n$ and $h : \mathbb{R} \to \mathbb{R}$ is the 1-periodic extension of the saw-tooth function

$$h(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1/2, \\ 1-t & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Indeed, by direct calculation

$$Du(x) = h'(x \cdot \mathbf{n})\, \mathbf{a} \otimes \mathbf{n} \quad \text{ for a.e. } x$$

and

$$\int_{\mathbb{T}^n} f\big(A + Du(x)\big)\, dx = \frac{1}{2} f(A + \mathbf{a} \otimes \mathbf{n}) + \frac{1}{2} f(A - \mathbf{a} \otimes \mathbf{n}).$$

Thus, (1) implies that

$$t \mapsto f(A + t\mathbf{a} \otimes \mathbf{n}) \text{ is convex} \tag{3}$$

for any $\mathbf{a} \in R^m$ and $\mathbf{n} \in \mathbb{Z}^n$. Since for any $\mathbf{n} \in \mathbb{Q}^n$ there exists $\lambda \neq 0$ such that $\lambda\mathbf{n} \in \mathbb{Z}^n$, by using that $\mathbf{a} \otimes \mathbf{n} = (\frac{1}{\lambda}\mathbf{a}) \otimes (\lambda\mathbf{n})$, one can easily extend (3) to all $\mathbf{n} \in \mathbb{Q}^n$, and then, using the continuity of f, to all $\mathbf{n} \in \mathbb{R}^n$. Thus f is rank-one convex.

The question whether the converse implication holds, i.e. whether rank-one convexity is also sufficient for quasiconvexity, has attracted a lot of attention since Morrey's seminal paper [8], not only because of the relevance to the calculus of variations, but also because of surprising and deep connections to other areas [1]. In the case where $m \geq 3$, V. Šverák constructed in [16] an ingenious example showing that rank-one convexity is not sufficient for quasiconvexity. The case $m = n = 2$, however, remains wide open. Indeed, there is evidence that for this case rank-one convexity might be sufficient after all [1, 2, 4, 11, 14].

Returning to necessary conditions for quasiconvexity, consider now test functions of the form

$$u(x) = \sum_{i=1}^{N} \mathbf{a}_i h(x \cdot \mathbf{n}_i + c_i), \tag{4}$$

where $\mathbf{a}_i \in \mathbb{R}^m$, $\mathbf{n}_i \in \mathbb{Z}^2$ and $c_i \in \mathbb{R}$. As above, for a.e. $x \in \mathbb{T}^n$

$$Du(x) = \sum_{i=1}^{N} h'(x \cdot \mathbf{n}_i + c_i)\, \mathbf{a}_i \otimes \mathbf{n}_i = \sum_{i=1}^{N} \epsilon_i(x)\, \mathbf{a}_i \otimes \mathbf{n}_i,$$

where $\epsilon_i(x) \in \{-1, +1\}$ for each i. The set of possible values of $Du(x)$ are precisely the 2^N vertices of a *rank-one hypercube*, i.e. an N-dimensional cube immersed in $\mathbb{R}^{m \times n}$, whose sides are given by the rank-one matrices $C_i := \mathbf{a}_i \otimes \mathbf{n}_i$, $i = 1, \ldots, N$. Let us denote the vertices by

$$X_\epsilon = \sum_{i=1}^{N} \epsilon_i C_i, \quad \epsilon \in \{-1, +1\}^N.$$

The integral in (1) defines a probability measure ν supported on the vertices of this hypercube - in fact ν is a homogeneous gradient Young measure, see [5, 15, 9], with barycenter 0. More precisely,

$$\int_{\mathbb{T}^n} f\big(Du(x)\big)\,dx = \sum_{\epsilon\in\{-1,+1\}^N} \nu_\epsilon f(X_\epsilon), \tag{5}$$

so that a consequence of the quasiconvexity of f would be the inequality

$$\sum_{\epsilon\in\{-1,+1\}^N} \nu_\epsilon f(X_\epsilon) \geq f(0). \tag{6}$$

Our aim in this paper is to analyse in more detail this inequality in the case $n = m = 2$. As a first observation, note that the weights ν_ϵ are determined by the volume fractions in $\mathbb{T}^n$ where the functions $h'(x\cdot\mathbf{n}_i + c_i)$ each take the value ±1 respectively. In particular it does not depend on the choices of the vectors $\mathbf{a}_i$.

It turns out that with $N = 2$ nothing more is gained from (4) with respect to (2) - see Lemma 2.1 below. Indeed, with $N = 2$ the inequality (6) becomes

$$\frac{1}{4}\Big(f(X_{++}) + f(X_{+-}) + f(X_{-+}) + f(X_{--})\Big) \geq f(0),$$

which is clearly satisfied by all rank-one convex functions. The situation, however, becomes much more interesting if $N \geq 3$. Indeed, the example of Šverák can be understood, following R. James' modification (see Section 4.7 in [9]), precisely in this way. To this end let $N = 3$, and set $\mathbf{n}_1 = (1,0)$, $\mathbf{n}_2 = (0,1)$, $\mathbf{n}_3 = (1,1)$ and the phases are $c_1 = c_2 = 0$ and $c_3 = 1/4$. A direct calculation (see e.g. [9]) easily shows that the measure ν in (5) is given in this case by

$$\begin{aligned} &\nu_{+++} = \nu_{+--} = \nu_{-+-} = \nu_{--+} = 1/16,\\ &\nu_{---} = \nu_{++-} = \nu_{+-+} = \nu_{-++} = 3/16. \end{aligned} \tag{7}$$

Whether any rank-one convex function satisfies the corresponding inequality (6) now requires specific knowledge of the vectors $\mathbf{a}_i$. Indeed, for the 3×2 case, where $\mathbf{a}_1 = (1,0,0)$, $\mathbf{a}_2 = (0,1,0)$ and $\mathbf{a}_3 = (0,0,1)$, there exist rank-one convex functions which do not satisfy (6) ([16]).

On the other hand, the same example does not work in the 2×2 case: if $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (0,1)$ and $\mathbf{a}_3 = (1,1)$, it was shown by P. Pedregal in [11] (see also [13, 14]) that every rank-one convex function satisfies (6) - in other words the measure ν in (7) is a laminate (see [10] and below for definitions). It was recently suggested in [12] that for better choices of $\mathbf{a}_i \in \mathbb{R}^2$ the measure ν might not be a laminate. Our main result in this paper is to show that this is not the case:

Theorem 1.1. *Given any C_1, C_2, C_3 rank-1 matrices in $\mathbb{R}^{2\times 2}$, for any rank-one convex function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ we have*

$$\begin{aligned} f(0) \leq &\frac{1}{16}\big(f(X_{+++}) + f(X_{+--}) + f(X_{-+-}) + f(X_{--+})\big) + \\ &\frac{3}{16}\big(f(X_{---}) + f(X_{++-}) + f(X_{+-+}) + f(X_{-++})\big), \end{aligned} \tag{8}$$

where $X_{\pm\pm\pm}$ denotes the matrix $\pm C_1 \pm C_2 \pm C_3$.

A different generalization of [14] was analysed in [3] – here the rank-one cube is the same, but the barycenter of the measure is varied.

2. Interaction of frequencies

In this section we take a closer look at those probability measures ν that arise from the construction in (5) with u given by (4) and $n = m = 2$. Note that for 2×2 matrices the determinant is a quadratic function with the identity

$$\det(X + Y) = \det X + \langle \operatorname{cof} X, Y\rangle + \det Y, \tag{9}$$

where $\operatorname{cof} X$ denotes the cofactor matrix and $\langle X, Y\rangle = \sum_{1\leq i,j\leq 2} X_{ij}Y_{ij}$ is the standard Hilbert-Schmidt scalar product on matrices.

First of all, since $u : \mathbb{T}^2 \to \mathbb{R}^2$ in (4) is a periodic Lipschitz function, integration by parts shows that

$$\int_{\mathbb{T}^2} Du(x)\,dx = 0, \int_{\mathbb{T}^2} \det Du(x)\,dx = 0.$$

Consequently

$$\sum_{\epsilon\in\{-1,+1\}^N} \nu_\epsilon X_\epsilon = 0, \quad \sum_{\epsilon\in\{-1,+1\}^N} \nu_\epsilon \det(X_\epsilon) = 0. \tag{10}$$

From these equations we can deduce the following simple lemma:

Lemma 2.1. *If $N = 2$ and $\mathbf{n}_1, \mathbf{n}_2$ are linearly independent vectors, then the measure ν satisfies*

$$\nu_{++} = \nu_{+-} = \nu_{-+} = \nu_{--} = 1/4.$$

Proof. Since ν is independent of $\mathbf{a}_i$, let us set without loss of generality $\mathbf{a}_i = \mathbf{n}_i$, $i = 1, 2$, and $C_i = \mathbf{n}_i \otimes \mathbf{n}_i$. Hence C_1, C_2 are linearly independent (in $\mathbb{R}^{2\times 2}$) and, working in coordinates $(x, y) \sim xC_1 + yC_2$, the first equation in (10) leads to

$$\begin{aligned} \nu_{++} + \nu_{+-} - \nu_{-+} - \nu_{--} &= 0, \\ \nu_{++} - \nu_{+-} + \nu_{-+} - \nu_{--} &= 0. \end{aligned}$$

Next, a quick calculation based on (9) shows that

$$\det(C_1 \pm C_2) = \pm\langle \operatorname{cof} C_1, C_2\rangle = \pm(\mathbf{n}_1 \cdot \mathbf{n}_2^\perp)^2 \neq 0.$$

Therefore the second equation in (10) leads to

$$\nu_{++} - \nu_{+-} - \nu_{-+} + \nu_{--} = 0.$$

Finally, since ν is a probability measure, we also have

$$\nu_{++} + \nu_{+-} + \nu_{-+} + \nu_{--} = 1.$$

It is easy to check that the four equations we obtained for $\nu_{\pm\pm}$ has the unique solution

$$\nu_{++} = \nu_{+-} = \nu_{-+} = \nu_{--} = 1/4. \qquad \square$$

Next, let us consider the situation where $N = 3$ in (4). Using the periodicity of f, we may assume without loss of generality that $c_1 = c_2 = 0$, $c_3 = c$. Moreover, we will assume that no two of the vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are collinear. The measure ν defined in (5) is now supported on the 8 vertices of a rank-one cube. Using Lemma 2.1 we see that the sum of the 2 weights on any edge of the cube is equal to 1/4, see Figure 2.1.

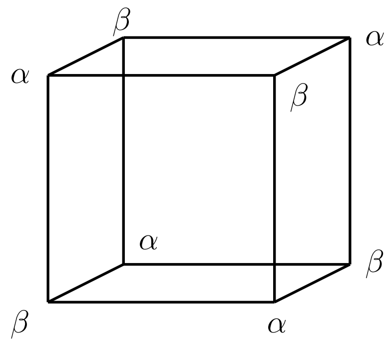


Figure 2.1: Symmetric measures

This motivates the following

Definition 2.2 (Symmetric measures). A probability measure ν supported on the vertices of a cube is said to be *symmetric* if the weights satisfy

$$\begin{aligned} \nu_{+++} = \nu_{+--} = \nu_{-+-} = \nu_{--+} &= \alpha, \\ \nu_{---} = \nu_{++-} = \nu_{+-+} = \nu_{-++} &= \beta, \end{aligned}$$

with $\alpha + \beta = 1/4$.

Returning to the formula (5) we see that the weights α, β can be obtained from calculating

$$\int_{\mathbb{T}^2} \chi(x)\, dx = 4(\alpha - \beta),$$

where $\chi(x) = \prod_{i=1}^{3} h'(\mathbf{n}_i \cdot x + c_i)$. Using the periodicity and an affine change of variables, we may then assume that $\mathbf{n}_1 = (1,0)$, $\mathbf{n}_2 = (0,1)$ and $\mathbf{n}_3 \in \mathbb{Z}^2$. As a consequence of Lemma (2.1), it then suffices to calculate

$$\int_Q h'(\mathbf{n}_3 \cdot x + c)\, dx = \alpha - \beta,$$

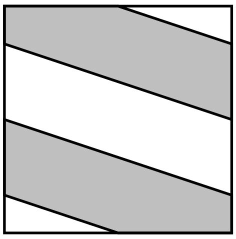
where $Q = (0,1/2) \times (0,1/2)$. In the following lemma we perform this calculation, however for simplicity we rescale Q to be the unit square.

Lemma 2.3. *Let $\mathbf{n} = (k,l)$ with $k,l \in \mathbb{N}$, $Q = (0,1)^2$, and let f be the 2-periodic function*

$$f(t) = \begin{cases} +1 & \text{if } t \in [0,1) \\ -1 & \text{if } t \in [1,2) \end{cases}.$$

If either k or l is even, then $\int_Q f(x \cdot \mathbf{n} + c)\, dx = 0$. Otherwise

$$\max_c \int_Q f(x \cdot \mathbf{n} + c)\, dx = \frac{1}{2kl}.$$



$(k,l) = (1,3)$

$\int_Q f(x \cdot n - \frac{1}{2})\, dx = \frac{1}{6}$

Figure 2.2: Calculation of volume fractions

Proof. Note that $f(t) = f(t+2)$ for all t, and $\int_I f(t)\, \mathrm{d}t = 0$ for any interval I of length 2. For any c we have

$$\int_0^1 f(kx_1 + c)\, \mathrm{d}x_1 = \frac{1}{k} \int_c^{k+c} f(t)\, \mathrm{d}t,$$

and so if k is even, the above integral is zero. Then Fubini gives $\int_Q f(x \cdot \mathbf{n})\, \mathrm{d}x = 0$ whenever k or l is even. Moreover, if k is odd, then

$$\frac{1}{k} \int_c^{k+c} f(t)\, \mathrm{d}t = \frac{1}{k} \Big(\int_c^{c+1} + \int_{c+1}^{c+k} \Big) f(t)\, \mathrm{d}t = \frac{1}{k} \int_c^{c+1} f(t)\, \mathrm{d}t =: g(c),$$

where $g(c) = g(c+2)$ and $g(c) + g(c+1) = 0$. Furthermore, it is easy to verify that

$$g(c) = \begin{cases} \frac{1}{k}(1-2c) \text{ if } c \in [0,1) \\ \frac{1}{k}(2c-3) \text{ if } c \in [1,2) \end{cases}.$$

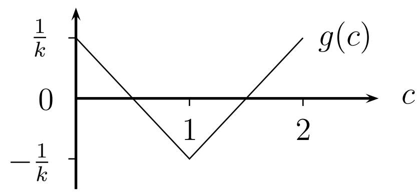


Figure 2.3: $\int_0^1 f(kx_1 + c)\,\mathrm{d}x_1$ is a 2-periodic function of c.

Note that $\int_I g(s)\,\mathrm{d}s = 0$ for any interval of length 2. Now using Fubini

$$I(c) = \int_Q f(x\cdot\mathbf{n} + c)\,\mathrm{d}x = \int_0^1 g(lx_2 + c)\,\mathrm{d}x_2 = \frac{1}{l}\int_c^{c+1} g(s)\,\mathrm{d}s$$

by the same argument as before. So I is 2-periodic and $I(c) + I(c+1) = 0$. Now let $c \in (0,1)$. Then

$$I(c) = \frac{1}{kl}\left(\int_c^1 (1-2s)\,\mathrm{d}s + \int_1^{1+c} (2s-3)\,\mathrm{d}s\right) = \frac{2c(c-1)}{kl}. \qquad \square$$

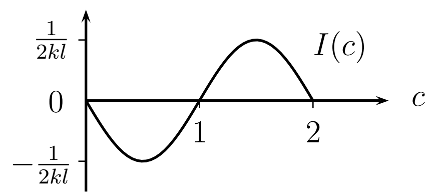


Figure 2.4: $I(c) = \int_Q f(x\cdot\mathbf{n} + c)\,\mathrm{d}x$ is a 2-periodic function of c.

Summarizing the results of this section we obtain:

Corollary 2.4. *If* $u : \mathbb{T}^2 \to \mathbb{R}^2$ *is given by*

$$u(x) = \sum_{i=1}^{3} h'(\mathbf{n}_i \cdot x + c_i)$$

for some $\mathbf{n}_i, c_i$ *and* ν *is the probability measure obtained from the formula* (5), *then* ν *is a symmetric probability measure supported on* $\{\sum_{i=1}^{3} \pm\mathbf{a}_i \otimes \mathbf{n}_i\}$ *with*

$$\begin{aligned} \nu_{+++} = \nu_{+--} = \nu_{-+-} = \nu_{--+} &= \alpha, \\ \nu_{---} = \nu_{++-} = \nu_{+-+} = \nu_{-++} &= \beta, \end{aligned}$$

such that $\alpha + \beta = 1/4$ *and* $-1/8 \le \alpha - \beta \le 1/8$, *where* $\gamma \in [-\frac{1}{16kl}, \frac{1}{16kl}]$ *if* k *and* l *are both odd, or* $\gamma = 0$ *if* k *or* l *is even.*

In particular if $\mathbf{n}_1 = (1,0)$, $\mathbf{n}_2 = (0,1)$ and $\mathbf{n}_3 = (k,l)$ $(k,l \in \mathbb{Z})$, then $\alpha - \beta \in [-\frac{1}{8kl}, \frac{1}{8kl}]$ if k and l are both odd, and $\alpha = \beta$ if k or l is even. The extremal cases where $|\alpha - \beta| = 1/8$ arise from $\mathbf{n}_1 = (1,0)$, $\mathbf{n}_2 = (0,1)$ and $\mathbf{n}_3 = (1,1)$.

There is one parameter giving symmetric measures, either $\alpha - \beta$ as above, or, equivalently,

$$\frac{\nu(X_0)}{\nu(X_1)} = \frac{\alpha}{\beta} \in [1/3, 3].$$

In the rest of this paper we show that for any ratio $\alpha/\beta \in [1/3, 3]$ the corresponding symmetric measures laminates. As a consequence no counterexample to the equivalence of rank-one convexity and quasiconvexity can arise from such configurations.

3. Laminates and semiconvex hulls

Let $\mathcal{P}(\mathbb{R}^{2\times 2})$ denote the set of all compactly supported probability measures on $\mathbb{R}^{2\times 2}$. For $\nu \in \mathcal{P}$ we denote by $\overline{\nu} = \int_{\mathbb{R}^{2\times 2}} X d\nu(X)$ the center of mass or *barycenter* of ν.

Definition 3.1. A measure $\nu \in \mathcal{P}$ is called a *laminate*, denoted $\nu \in \mathcal{L}$, if

$$f(\overline{\nu}) \le \int_{\mathbb{R}^{2\times 2}} f d\nu \tag{11}$$

for all rank-one convex functions f. The set of laminates with barycenter 0 and supported in a compact set $K \subset \mathbb{R}^{2\times 2}$ is denoted by $\mathcal{L}_0(K)$.

With this definition, the question of whether an inequality of the type (6) holds for all rank-one convex functions amounts to the question of whether ν is a laminate. We note in passing that $\mathcal{L}_0(K)$ is a convex set.

Definition 3.2. We call $\mathcal{PL}(\mathbb{R}^{2\times 2})$ the set of *prelaminates*. This is the smallest class of probability measures on $\mathbb{R}^{2\times 2}$ which

- contains all measures of the form $\lambda\delta_A + (1-\lambda)\delta_B$ with $\lambda \in [0, 1]$ and $\operatorname{rank}(A - B) = 1$;
- is closed under splitting in the following sense: if $\lambda\delta_A + (1-\lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(\mathbb{R}^{2\times 2})$ for some $\tilde{\nu} \in \mathcal{P}(\mathbb{R}^{2\times 2})$ and μ also belongs to $\mathcal{PL}(\mathbb{R}^{2\times 2})$ with $\overline{\mu} = A$, then also $\lambda\mu + (1-\lambda)\tilde{\nu}$ belongs to $\mathcal{PL}(\mathbb{R}^{2\times 2})$.

The order of a prelaminate denotes the number of splittings required to obtain the measure from a Dirac measure. It is clear from the definition that $\mathcal{PL}(\mathbb{R}^{2\times 2})$ consists of atomic measures. Also, from a repeated application of Jensen's inequality it follows that $\mathcal{PL} \subset \mathcal{L}$. Furthermore, $\nu \in \mathcal{L}$ if and only if there exists a sequence $\nu_k \in \mathcal{PL}$ with uniformly bounded support, such that $\nu_k \overset{*}{\rightharpoonup} \nu$ (see [10]).

We remind the reader that prelaminates, as defined above, are precisely those probability measures $\nu = \sum_{i=1}^N \lambda_i \delta_{X_i}$ for which the sequence $\{(\lambda_i, X_i)\}_{1\le i\le N}$ satisfies the (H_N)-condition (see [10]). It is also worth noting that in [11, 14] the inequality (8) is verified for rank-one convex functions in the case

$$\mathbf{n}_1 = \mathbf{a}_1 = (1,0),\ \mathbf{n}_2 = \mathbf{a}_2 = (0,1),\ \mathbf{n}_3 = \mathbf{a}_3 = (1,1)$$

by showing that the associated measure ν is a convex combination of 3 prelaminates of order 6.

Next, we introduce the various semiconvex hulls. Let $K \subset \mathbb{R}^{2\times 2}$ be a compact set. The lamination-convex hull is defined as follows. We first set $K^{lc,0} := K$, and for any $i \geq 0$

$$K^{lc,i+1} := \left\{\lambda X + (1-\lambda)Y :\ X, Y \in K^{lc,i},\ \mathrm{rank}\,(X-Y) = 1,\ \lambda \in [0,1]\right\}.$$

Then, the lamination-convex hull is defined as $K^{lc} = \bigcup_{i\geq 0} K^{lc,i}$. The rank-one convex hull is defined in terms of separation with rank-one convex functions:

$$K^{rc} := \left\{X :\ f(X) \leq \sup_K f \quad \text{for all rank-one convex } f\right\}.$$

It is not difficult to see (e.g. [9, 7, 6]) that $K^{rc} \supset K^{lc}$, but equality does not necessarily hold. Moreover, another characterization of K^{rc} follows from duality:

$$K^{rc} = \{\bar{\nu} :\ \nu \text{ is a laminate with } \mathrm{supp}\,\nu \subset K\}.$$

Finally, the polyconvex hull of K is defined as

$$K^{pc} = \left\{\bar{\nu} :\ \nu \in \mathcal{P}(\mathbb{R}^{2\times 2}) \text{ with } \mathrm{supp}\,\nu \subset K \text{ and } \int \det(X)d\nu(X) = \det(\bar{\nu})\right\}$$

Since the functions $X \mapsto \pm\det X$ are rank-one convex, we see that $K^{rc} \subset K^{pc}$.

For calculating lamination hulls, the following will be useful:

Lemma 3.3. *Suppose $K = \{X_1, X_2, X_3, X_4\} \subset \mathbb{R}^{2\times 2}$ is a rank-one square, in other words suppose*

$$\det(X_1 - X_2) = 0,\ \det(X_2 - X_3) = 0,\ \det(X_3 - X_4) = 0,\ \det(X_4 - X_1) = 0.$$

Then $K^{lc} = K^{pc}$. For the hull there are three cases depending on the determinant of the 'diagonals':

1. *If $\det(X_1 - X_3) = 0$ and $\det(X_2 - X_4) \neq 0$, then X_1, X_2, X_3 and X_1, X_3, X_4 both lie on a rank-one plane, and*

$$K^{lc} = \{X_1, X_2, X_3\}^{co} \cup \{X_1, X_3, X_4\}^{co}.$$

Furthermore if in addition $\det(X_2 - X_4) = 0$, then $K^{lc} = K^{co}$.

2. *If $\det(X_1 - X_3)$ and $\det(X_2 - X_4)$ have the same sign, then*

$$K^{rc} = K^{lc,1} = [X_1, X_2] \cup [X_2, X_3] \cup [X_3, X_4] \cup [X_4, X_1].$$

3. *If* $\det(X_1 - X_3)$ *and* $\det(X_2 - X_4)$ *have opposite sign, then*

$$K^{pc} = K^{lc,2},$$

and there exists a continuous increasing function $f\colon [0,1] \to [0,1]$ *with* $f(0) = 0$ *and* $f(1) = 1$ *such that for any* $t \in [0,1]$ *we have*

$$\det\left((tX_1 + (1-t)X_2) - (sX_4 + (1-s)X_3)\right) = 0,$$

where $s = f(t)$.

Remark 3.4. In fact, it is not difficult to check that if $X_i \in \mathbb{R}^{2\times 2}$, $i = 1\dots 4$, are not coplanar, then there exists $R \in \operatorname{span}\{X_1, \dots, X_4\}$ in the affine span and $\alpha \in \mathbb{R}$ such that for all i

$$\det(X_i - R) = \alpha. \tag{12}$$

To see this, subtract equation i from equation 1 in (12) and use (9) to obtain

$$\langle \operatorname{cof} X_i - \operatorname{cof} X_1, R\rangle = \det X_i - \det X_1, \quad i = 2,3,4.$$

Since we assumed that the matrices $X_1, \dots, X_4$ are not coplanar, the above linear system uniquely determines $R \in \operatorname{span}\{X_1, \dots, X_4\}$. The scalar α is then obtained by back-substitution.

Applying this observation to Case 3. in the Lemma above shows that $X_1, \dots, X_4$ in this case lie on a one-sheeted hyperboloid (a doubly ruled surface) given by the equation $\det(X - R) = \alpha$.

Proof of Lemma 3.3. Let $P = \sum_{i=1}^4 \lambda_i X_i$, where $\sum_{i=1}^4 \lambda_i = 1$. By direct calculation

$$\begin{aligned}\det P &= \sum_{i=1}^4 \lambda_i \det X_i - \frac{1}{2}\sum_{i,j=1}^4 \lambda_i\lambda_j \det(X_i - X_j)\\ &= \sum_i \lambda_i \det X_i - \frac{1}{2}(\lambda_1\lambda_3 d_{13} + \lambda_2\lambda_4 d_{24}),\end{aligned}$$

where $d_{ij} = \det(X_i - X_j)$. In particular $P \in K^{pc}$ if and only if

$$\lambda_1\lambda_3 d_{13} + \lambda_2\lambda_4 d_{24} = 0. \tag{13}$$

So if d_{13} and d_{24} have the same sign, then

$$K^{pc} = K^{lc,1} = [X_1, X_2] \cup [X_2, X_3] \cup [X_3, X_4] \cup [X_4, X_1].$$

Suppose that $d_{13} > 0$ and $d_{24} < 0$, and let $P \in K^{pc}$. Consider the points

$$P_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} X_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} X_2, \quad P_2 = \frac{\lambda_3}{\lambda_3 + \lambda_4} X_3 + \frac{\lambda_4}{\lambda_3 + \lambda_4} X_4.$$

Clearly $P \in [P_1, P_2]$. Furthermore, using (13),

$$
\begin{aligned}
\det(P_1 - P_2) &= \det(P_1) + \det(P_2) - \langle \operatorname{cof} P_1, P_2 \rangle \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}\det X_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\det X_2 + \frac{\lambda_3}{\lambda_3+\lambda_4}\det X_3 + \frac{\lambda_4}{\lambda_3+\lambda_4}\det X_4 - \\
&\quad - \frac{1}{(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)}\Big(\lambda_1\lambda_3(\det X_1 + \det X_3 - \det(X_1 - X_3)) + \\
&\quad + \lambda_2\lambda_4(\det X_2 + \det X_4 - \det(X_2 - X_4)) + \\
&\quad + \lambda_1\lambda_4(\det X_1 + \det X_4) + \lambda_2\lambda_3(\det X_2 + \det X_3)\Big) \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}\det X_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\det X_2 + \frac{\lambda_3}{\lambda_3+\lambda_4}\det X_3 + \frac{\lambda_4}{\lambda_3+\lambda_4}\det X_4 - \\
&\quad - \frac{1}{(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)}\Big((\lambda_1+\lambda_2)(\lambda_3\det X_3 + \lambda_4 \det X_4) + \\
&\quad + (\lambda_3+\lambda_4)(\lambda_1 \det X_1 + \lambda_2 \det X_2)\Big) \\
&= 0.
\end{aligned}
$$

Thus if $P \in K^{pc}$, then P lies on the rank-one segment $[P_1, P_2]$, hence in $K^{lc,2}$. For completeness we find now f. Let $t \in (0,1)$ and $s = f(t)$. From above we have

$$
t = \frac{\lambda_1}{\lambda_1+\lambda_2} \text{ and } s = \frac{\lambda_4}{\lambda_3+\lambda_4}.
$$

Thus $\lambda_2 = \frac{1-t}{t}\lambda_1$ and $\lambda_3 = \frac{1-s}{s}\lambda_4$. Substituting this into (13) gives

$$
(1-s)td_{13} + (1-t)sd_{24} = 0
$$

which after rearranging gives

$$
f(t) = \frac{td_{13}}{td_{13} - (1-t)d_{24}}.
$$

If $\det(X_1 - X_3) = 0$, then we have the following degenerate cases. Either K lies in a plane, which then is a rank-one plane (hence $K^{pc} = K^{lc,1} = K^c$), or we have $\det(X_2 - X_4) \neq 0$ and then condition (13) reduces to $\lambda_2\lambda_4 = 0$, so that $K^{pc} = K^{lc,1}$ consists of the convex hulls of the two triangles X_1, X_2, X_3 and X_1, X_3, X_4. $\square$

4. Symmetric laminates on cubes

In this section we prove our main theorem, Theorem 1.1. Let us recall the setting. Let $C_1, C_2, C_3 \in \mathbb{R}^{2\times 2}$ be rank-one matrices, and for $\epsilon \in \{-1,+1\}^3$ let

$$X_\epsilon = \sum_{i=1}^{3} \epsilon_i C_i$$

and $K = \{X_\epsilon : \epsilon \in \{-1,+1\}^3\}$, see Figure 4.1.

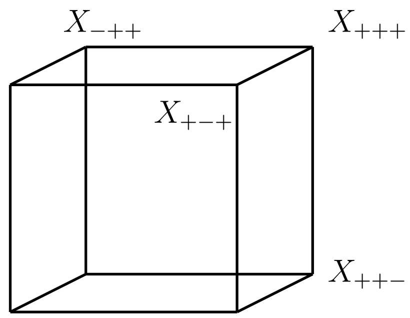


Figure 4.1: The rank-one cube.

From now on we work in the coordinates given by $\{C_1, C_2, C_3\}$, i.e. (x, y, z) corresponds to $xC_1 + yC_2 + zC_3$. The determinant in these coordinates is

$$\det(x, y, z) = axy + bxz + cyz,$$

being a quadratic form vanishing on the axes, and we have

$$\begin{aligned}
\det(1, 1, 1) &= a + b + c, \\
\det(-1, 1, 1) &= -a - b + c, \\
\det(1, -1, 1) &= -a + b - c, \\
\det(1, 1, -1) &= a - b - c.
\end{aligned}$$

Lemma 4.1. *If $abc = 0$, then any symmetric measure in the sense of Definition 2.2 is a laminate.*

Proof. We may assume without loss of generality that $a = 0$. Then the $\{z = 0\}$ plane is a rank-one plane, and symmetric measures on K may be obtained by splitting (c.f. Definition 3.1) as follows:

$$\begin{aligned}
\delta_{(0,0,0)} &\mapsto \frac{1}{2}\delta_{(0,0,1)} + \frac{1}{2}\delta_{(0,0,-1)} \\
&\mapsto \frac{1}{2}\left(\frac{1}{2}\delta_{(1,1,1)} + \frac{1}{2}\delta_{(-1,-1,1)}\right) + \frac{1}{2}\left(\frac{1}{2}\delta_{(1,1,-1)} + \frac{1}{2}\delta_{(-1,-1,-1)}\right).
\end{aligned}$$

This is a symmetric laminate in the sense of Definition 2.2 with $\alpha = 1/4$ and $\beta = 0$. In a similar way we can obtain a symmetric laminate with $\alpha = 0$ and $\beta = 1/4$. Since symmetric laminates form a convex set and every symmetric measure can be written as a convex combination of these two laminates, we are done. $\square$

In light of the preceding lemma, in the following we will assume $a, b, c \neq 0$. Furthermore, by swapping the signs $\pm x, y, z$ and multiplying X_i by J if necessary, we can assume without loss of generality that

$$a, b, c > 0. \tag{14}$$

Remark 4.2. We note in passing that the assumption (14) in particular implies that $\{C_1, C_2, C_3\}$ is linearly independent. Indeed, if $xC_1 + yC_2 = C_3$, then $0 = \det C_3 = axy$ and consequently $x = 0$ or $y = 0$. If (without loss of generality) $y = 0$, then $0 = \det(xC_1 - C_3) = -bx$, hence $x = 0$.

Our aim in this section is to construct symmetric laminates supported on the vertices of the cube K. The following lemma, which gives a simpler criterion for laminates to be symmetric, will be useful.

Lemma 4.3. *If ν is a laminate supported on K with barycenter $\bar{\nu} = 0$, and if*

$$\nu_{+--} = \nu_{-+-} = \nu_{--+},$$

then ν is symmetric.

Proof. Let us set $\nu_{---} = \beta$ and $\nu_{+--} = \nu_{-+-} = \nu_{--+} = \alpha$ (cf. Figure 2.1). Observe that, since $X \mapsto \pm \det X$ is rank-one convex, any laminate ν supported on K and with barycenter $\bar{\nu} = 0$ satisfies the equations (10) with $N = 3$. The first equation in (10) amounts to

$$\begin{aligned}
\nu_{+++} + \nu_{++-} + \nu_{+-+} + \beta &= 2\beta + \alpha + \nu_{-++} \\
\nu_{+++} + \nu_{-++} + \nu_{++-} + \beta &= 2\beta + \alpha + \nu_{+-+} \\
\nu_{+++} + \nu_{-++} + \nu_{+-+} + \beta &= 2\beta + \alpha + \nu_{++-}.
\end{aligned}$$

These three equations quickly lead to

$$\nu_{++-} = \nu_{+-+} = \nu_{-++} =: \gamma.$$

Then, the second equation in (10) gives

$$(a + b + c)(\alpha + 2\beta - \gamma) + (\alpha + \gamma)(-a - b - c) = 0,$$

hence $(a + b + c)(\beta - \gamma) = 0$. Since $a, b, c > 0$, this implies $\gamma = \beta$. Back-substitution then yields $\nu_{+++} = \alpha$. $\square$

Next, we look for special laminates supported on minimal subsets of K. It will be convenient from now on to switch the notation and write $X_0 = (1,1,1)$, $X_1 = (-1,1,1)$, $X_2 = (1,-1,1)$ and $X_3 = (1,1,-1)$. For later use we record

$$\det(X_1 - X_2) = -4a,\ \det(X_1 - X_3) = -4b,\ \det(X_2 - X_3) = -4c$$
$$\det(X_1 + X_0) = 4c,\ \det(X_2 + X_0) = 4b,\ \det(X_3 + X_0) = 4a.$$

We start with the following observation.

Lemma 4.4. *Suppose* $a, b, c > 0$ *and in addition* $c < a + b$, *i.e.* $\det X_1 < 0$. *Then there exists* $P \in [-X_0, -X_1]$ *such that* $\det(P - X_1) = 0$, *and furthermore*

$$0 \in \{X_0, X_1, X_2, X_3, P\}^{lc}.$$

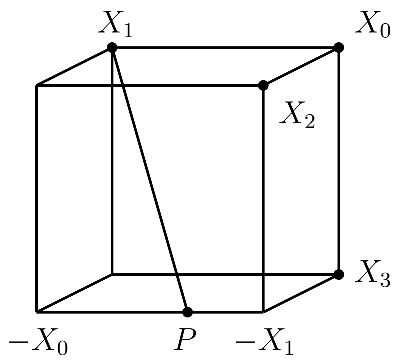


Figure 4.2: The point $P \in [-X_0, -X_1]$.

Proof. Observe that the statement is symmetric with respect to swapping X_2 and X_3, i.e. with respect to swapping a, b. Therefore we may assume without loss of generality that $a \le b$. Let $P = \lambda(-X_0) + (1-\lambda)(-X_1)$. Then

$$0 = \lambda \det(-X_0 - X_1) + (1-\lambda)\det(-X_1 - X_1) = 4c\lambda + 4(c - a - b)(1-\lambda)$$

hence

$$\lambda = \frac{a + b - c}{a + b}. \tag{15}$$

See Figure 4.2. Observe that $\det(P - X_0) = 4(a+b)$, $\det(X_1 - X_2) = -4a < 0$ and

$$\det(P - X_2) = \lambda \det(-X_0 - X_2) + (1-\lambda)\det(-X_1 - X_2) = 4\lambda b > 0.$$

The point $P_1 = \lambda_1 X_1 + (1 - \lambda_1)P$ defined by

$$\lambda_1 = \frac{\lambda b}{a + \lambda b}$$

is a point on the segment $[X_1, P]$ with $\det(P_1 - X_2) = 0$. Similarly $P_2 = \lambda_2 X_1 + (1 - \lambda_2)P$, where

$$\lambda_2 = \frac{\lambda a}{b + \lambda a}.$$

Since we assumed that $a \leq b$, we have $\lambda_1 \geq \lambda_2$. Then

$$\det(X_2 - P_2) = \frac{4\lambda(b^2 - a^2)}{b + \lambda a} > 0 \quad \text{and} \quad \det(X_2 - X_3) = -4c$$

so that there exists $P_3 \in [P_2, X_3]$ with $\det(P_3 - X_2) = 0$. In fact a simple calculation shows that $P_3 = \lambda_3 X_3 + (1 - \lambda_3)P_2$, where

$$\lambda_3 = \frac{(a + b - c)(b - a)}{b^2 - a^2 + (1 + \lambda)ac}.$$

In particular, if $a = b$, then $P_1 = P_2 = P_3$. Let us summarize so far: By construction we have $P_1, P_2 \in K^{lc,1}$, $P_3 \in K^{lc,2}$ and furthermore

$$\det(X_0 - P_k) > 0 \quad \text{and} \quad \det(X_i - X_j) < 0,$$

and

$$\det(X_2 - P_2) > 0 \quad \text{and} \quad \det(P_1 - P_3) < 0;$$

we obtain these inequalities by using repeatedly that the determinant is linear when restricted to rank-one lines).

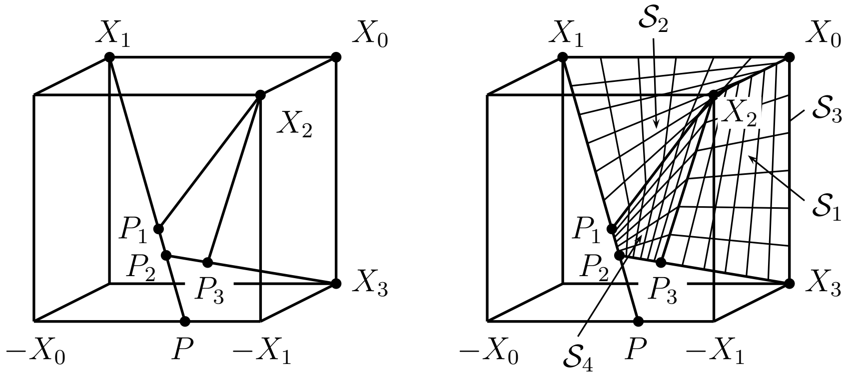


Figure 4.3: The surface $\mathcal{S} = \bigcup_i \mathcal{S}_i$ and its 2-cell embedding.

Now we can find the third lamination hull using Lemma 3.3. Indeed, for any of the following 4-tuples

$$\{X_0, X_2, P_3, X_3\},\ \{X_0, X_1, P_1, X_2\},\ \{X_0, X_1, P_2, X_3\},\ \{X_2, P_1, P_2, P_3\}$$

case 3 of Lemma 3.3 applies and yields a "filling" of the corresponding rank-one square with doubly ruled surfaces $\mathcal{S}_1, \ldots, \mathcal{S}_4$, which are contained in the lamination-convex hull. See Figure 4.3. Observe that any two such surfaces may intersect only along the common rank-one edge. To see this, consider for definiteness $X \in \mathcal{S}_1 \cap \mathcal{S}_2$ and assume that $X \notin [X_0, X_2]$. In Lemma 3.3 we showed that there exists a unique $R \in [X_0, X_2]$ with $\det(X - R) = 0$, and there exist $Q_1 \in [P_3, X_3]$, $Q_2 \in [P_1, X_1]$ such that $X \in [Q_1, R] \cap [Q_2, R]$. But then either $Q_1 \in [Q_2, R] \subset \mathcal{S}_2$ or $Q_2 \in [Q_1, R] \subset \mathcal{S}_1$. Either case leads to a contradiction, as $[P_3, X_3] \cap [P_1, X_1] = \emptyset$. The intersection of any other pair $\mathcal{S}_i \cap \mathcal{S}_j$ can be handled in a similar fashion.

Consequently $\mathcal{S} = \bigcup_{i=1}^4 \mathcal{S}_i$ is a regular (i.e. embedded) compact, piecewise smooth surface without boundary, with the rank-one edges of the surfaces $\mathcal{S}_i$ forming a canonical 2-cell embedding. It follows that therefore $\mathcal{S}$ is a topological sphere. By the Jordan-Brouwer theorem $\mathbb{R}^3 \setminus \mathcal{S}$ consists of precisely two connected components, an "inside" $\mathcal{U}$ and an "outside" . Obviously any point $Q \in \mathcal{U}$ is contained on a rank-one segment connecting points on $\mathcal{S} = \partial\mathcal{U}$, therefore $\overline{\mathcal{U}} \subset K^{lc}$. Hence, in order to complete the proof, we need to show that $0 \in \mathcal{U}$.

Note that, by construction, the surface $\mathcal{S}$ depends continuously on the parameters a, b, c in the set

$$a, b, c > 0, \quad c < a + b, \quad a \leq b. \tag{16}$$

Furthermore, it is easy to check that in the case $a = b = c$, we have $P_1 = P_2 = P_3 = -\frac{1}{3}(1, 1, 1)$. Hence in this case the line (t, t, t) $(t \in \mathbb{R})$ intersects $\mathcal{S}$ in precisely two points: $X_0 = (1, 1, 1)$ and P_1, and consequently $0 \in \mathcal{U}$. In fact it is not difficult to check that in this case $\overline{\mathcal{U}} = \{X_0, X_1, X_2, X_3, P_2\}^{pc}$, and we have

$$\begin{aligned} \lambda_0 X_0 + \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 P_2 = 0 \\ \lambda_0 \det X_0 + \lambda_1 \det X_1 + \lambda_2 \det X_2 + \lambda_3 \det X_3 + \lambda_4 \det P_2 = 0 \end{aligned}$$

with $\lambda_0 = \frac{9}{16}$, $\lambda_1 = \frac{1}{8}$, $\lambda_2 = \frac{1}{16}$, $\lambda_3 = \frac{1}{8}$, $\lambda_4 = \frac{1}{8}$, showing that $0 \in \mathcal{U} = \operatorname{int}\{X_0, X_1, X_2, X_3, P_2\}^{pc}$ (this should be compared with calculations made in [14]).

Now let us find the instances when 0 is on the boundary (in other words $0 \in \mathcal{S}_k$ for some $k = 1, 2, 3, 4$). We may assume that $\lambda > 0$, otherwise $0 \in [X_1, P]$. But then, as $\mathcal{S}_2$ lies above the plane $z = y$ (more precisely, $z - y \geq 0$ for all $(x, y, z) \in \mathcal{S}_2$ with equality $z = y$ only if $(x, y, z) \in [X_0, X_1] \cup [X_1, P_1]$), it cannot contain 0. Similarly, $0 \notin \mathcal{S}_3$.

Suppose $0 \in \mathcal{S}_4$. Then there exists, by definition of $\mathcal{S}_4$ (c.f. Lemma 3.3) $Q \in [P_1, P_2]$ with $\det(Q) = 0$, and moreover, Q is the unique point on the line segment $[X_1, P]$ with this property (since $\det(X_1 - P) = 0$). On the other hand we can calculate that $\frac{X_1+P}{2} = \frac{\lambda}{2}(X_1 - X_0)$, so that $\det(\frac{X_1+P}{2}) = 0$. Therefore $Q = \frac{X_1+P}{2}$. From Lemma 3.3 we also know that the rank-one line containing the segment $[\frac{X_1+P}{2}, 0]$ also needs to intersect $[X_2, P_3]$. In particular the orthogonal projection of $[X_2, P_3]$ onto the $\{x = 0\}$ plane contains the origin. It is a simple matter to check that this is only possible if either $P_3 = X_3$ (in which case $\det(X_2 - X_3) = 0$, i.e. $c = 0$), or $P_2 = \frac{X_1+P}{2}$. In the latter case $\lambda_2 = 1/2$, hence – since we assumed $a \leq b$ –, $\lambda = 1$ and $a+b = c$, contradicting our assumptions above.

Finally, let us look at what happens if $0 \in \mathcal{S}_1$. Then 0 is on a rank-one segment $[Q_1, Q_2]$ connecting $[X_0, X_2]$ to $[X_3, P_2]$, see Figure 4.4.

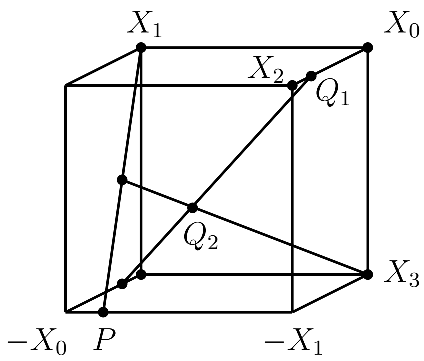


Figure 4.4: The case $0 \in \mathcal{S}_1$.

Here $Q_1 = \tilde{\lambda} X_0 + (1 - \tilde{\lambda}) X_2$ with $\tilde{\lambda} = \frac{a+c-b}{2(a+c)}$, and

$$Q_2 = tX_3 + (1 - t)P_2 = -sQ_1.$$

Solving first for the first and third coordinate of Q_2 we obtain

$$s = \frac{b}{2a + b} \text{ and } t = \frac{(a + b)(a - c)}{(a + b - c)(2a + b)}$$

and substituting into the second coordinate we obtain

$$\frac{a^2 - b^2 - c^2}{(a + c)(a + b - c)} = 0.$$

But we assumed that $a \leq b$, hence this forces $c = 0$ and $a = b$ again.

We can conclude that under the assumptions (16) the origin 0 cannot lie on any one of the quadratic surfaces $\mathcal{S}_1, \ldots, \mathcal{S}_4$. Consequently, for any a, b, c satisfying (16), 0 is contained in the "inside" $\mathcal{U}$, hence in the interior of the lamination-convex hull of $\{X_0, X_1, X_2, X_3, P\}$. This concludes the proof. □

We are now ready to prove our main theorem, which we restate for the convenience of the reader. In the following it will be convenient to use the notation, that, given a probability measure ν supported on

$$K = \{X_\epsilon : \epsilon \in \{-1, +1\}^3\},$$

we denote the mass of each point X_ϵ by $\nu(X_\epsilon)$.

Theorem 4.5. *The set K supports symmetric laminates with barycenter 0 and ratio*

$$\frac{\nu(X_0)}{\nu(X_1)} \leq \frac{1}{3}.$$

In particular, given any C_1, C_2, C_3 rank-1 matrices in $\mathbb{R}^{2\times 2}$, for any rank-one convex function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ we have

$$\begin{aligned} f(0) \leq & \frac{1}{16}\big(f(X_{+++}) + f(X_{+--}) + f(X_{-+-}) + f(X_{--+})\big) + \\ & \frac{3}{16}\big(f(X_{---}) + f(X_{++-}) + f(X_{+-+}) + f(X_{-++})\big), \end{aligned}$$

where $X_{\pm\pm\pm}$ denotes the matrix $\pm C_1 \pm C_2 \pm C_3$.

Proof. As in the beginning of this section, we may assume without loss of generality that $a, b, c > 0$. In particular this means that $\det X_0 > 0$, and then there are two cases depending on the signs of $\det X_i$ for $i = 1, 2, 3$.

1. $\det X_i < 0$ for all i,
2. $\det X_i > 0$ for some i.

Case 1. If $\det X_1, \det X_2, \det X_3 < 0$, corresponding to

$$a + b > c, \ a + c > b \ \text{and} \ b + c > a,$$

then Lemma 4.4 implies the existence of P_1, P_2, P_3 on the segments $[-X_0, -X_1]$, $[-X_0, -X_2]$, $[-X_0, -X_3]$ respectively, with the property that

$$0 \in \{X_0, X_1, X_2, X_3, P_k\}^{lc}.$$

Let $\tilde{\nu}_1$ be a laminate supported on $\{X_0, X_1, X_2, X_3, P_1\}$ with barycenter 0. As P_1 lies on the rank-one segment $[-X_0, -X_1]$, more precisely (c.f. (15))

$$P_1 = \lambda(-X_0) + (1 - \lambda)(-X_1)$$

with $\lambda = \frac{a+b-c}{a+b}$, we can split along this segment to obtain the laminate ν_1 supported on the set

$$K_1 = \{X_0, X_1, X_2, X_3, -X_0, -X_1\}$$

with barycenter 0. Furthermore,

$$\frac{\nu_1(-X_0)}{\nu_1(-X_1)} = \frac{\lambda}{1 - \lambda} = \frac{a + b - c}{c}.$$

In a similar manner we obtain the laminates ν_2 and ν_3 with barycenter 0, supported on K_2 and K_3 respectively, where

$$K_i = \{X_0, X_1, X_2, X_3, -X_0, -X_i\},$$

and such that

$$\frac{\nu_2(-X_0)}{\nu_2(-X_2)} = \frac{a+c-b}{b} \quad \text{and} \quad \frac{\nu_3(-X_0)}{\nu_3(-X_3)} = \frac{b+c-a}{a}.$$

To obtain a symmetric laminate, we form the convex combination:

$$\nu = C\big(\frac{1}{\nu_1(-X_1)}\nu_1 + \frac{1}{\nu_2(-X_2)}\nu_2 + \frac{1}{\nu_3(-X_3)}\nu_3\big)$$

where $C > 0$ is the normalizing factor

$$C = \frac{1}{\frac{1}{\nu_1(-X_1)} + \frac{1}{\nu_2(-X_2)} + \frac{1}{\nu_3(-X_3)}}.$$

Note that $\nu(-X_i) = C$ for all $i = 1, 2, 3$, hence ν is symmetric in light of Lemma 4.3. Furthermore

$$\begin{aligned}\frac{\nu(-X_0)}{\nu(-X_k)} &= \sum_{i=1}^{3} \frac{\nu_i(-X_0)}{\nu_i(-X_i)} = \frac{a+b}{c} + \frac{a+c}{b} + \frac{b+c}{a} - 3 \\ &= (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 6 \geq 9 - 6 = 3,\end{aligned}$$

where in the last line we used the harmonic-arithmetic mean inequality. In particular we see that

$$\min_{a,b,c>0} \frac{\nu(-X_0)}{\nu(-X_k)} = \frac{\nu(-X_0)}{\nu(-X_k)}\bigg|_{a=b=c} = 3,$$

so that

$$\nu(-X_0) \geq 3\nu(-X_k).$$

Case 2. Suppose $\det X_2 > 0$, i.e. $b > a + c$. Then in particular, as $a, b, c > 0$, necessarily $a + b > c$ and $b + c > a$, so that $\det X_1 < 0$ and $\det X_3 < 0$. Because $\det X_2 > 0$, there is no rank-one line from X_2 hitting the segment $[-X_0, -X_2]$, but instead there is one hitting the segment $[X_1, -X_2]$. Indeed, setting $P = \lambda X_1 + (1-\lambda)(-X_2)$ and requiring $\det(P - X_2) = 0$, we obtain

$$0 = 4(b - a - c) - 4\lambda(b - c),$$

hence $\lambda = \frac{b-a-c}{b-c}$. Note that $b - c > a$ by assumption, so that $\lambda \in (0, 1)$.

Then splitting first along the line $(0, 0, 1)$, then along the parallel segments $[P, X_2]$ and $[-X_2, -P]$ and finally on the segments $[-X_2, X_1]$ and $[-X_1, X_2]$ (see Figure 4.5) we obtain the laminate ν_2' with barycenter 0 and supported on

$$\{X_1, X_2, -X_1, -X_2\}.$$

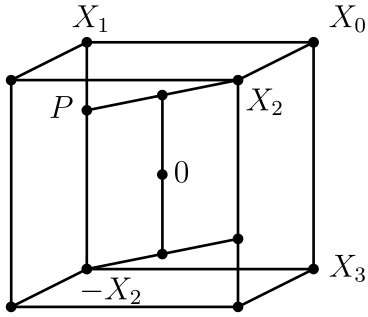


Figure 4.5: Obtaining ν_2' in Case 2.

Furthermore, a quick calculation shows that

$$\nu_2'(X_1) = \nu_2'(-X_1) = \frac{1}{4}\lambda,$$
$$\nu_2'(X_2) = \nu_2'(-X_2) = \frac{1}{4}(2-\lambda),$$

with $\lambda = \frac{b-a-c}{b-c}$ from above. In addition we have the laminates ν_1 and ν_3 as before. Our symmetric laminate this time will be

$$\nu' = C\left(\frac{1}{\nu_1(-X_1)}\left(1-\frac{\nu_2'(-X_1)}{\nu_2'(-X_2)}\right)\nu_1 + \frac{1}{\nu_2'(-X_2)}\nu_2' + \frac{1}{\nu_3(-X_3)}\nu_3\right),$$

where again $C > 0$ is a normalizing factor (so that ν' is a probability measure). Since $\nu_2'(-X_1)/\nu_2'(-X_2) = \lambda/(2-\lambda) \leq 1$, ν' is a probability measure. Note that

$$\nu'(-X_1) = \nu'(-X_2) = \nu'(-X_3) = C,$$

hence ν' is symmetric by Lemma 4.3. Moreover, for $k = 1, 2, 3$

$$\frac{\nu'(-X_0)}{\nu'(-X_k)} = \left(1-\frac{\nu_2'(-X_1)}{\nu_2'(-X_2)}\right)\frac{\nu_1(-X_0)}{\nu_1(-X_1)} + \frac{\nu_3(-X_0)}{\nu_3(-X_3)}$$

since $\nu_2'(-X_0) = 0$. Substituting and using that $b > a + c$ we get

$$\frac{\nu'(-X_0)}{\nu'(-X_k)} = \frac{2a}{c} + \frac{b+c-a}{a} > \frac{2a}{c} + \frac{2c}{a} \geq 4.$$

Therefore the symmetric laminate ν' we obtain in this case also satisfies $\nu'(-X_0) \geq 3\nu'(-X_k)$. $\square$

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References

[1] I. Al Baernstein, S. Montgomery-Smith: Some conjectures about integral means of ∂f and $\bar{\partial} f$, Acta Univ. Upsaliensis C 64 (1999) 92–109.

[2] K. Astala, T. Iwaniec, I. Prause, E. Saksman: Burkholder integrals, Morrey's problem and quasiconformal mappings, J. Amer. Math. Soc. 25(2) (2012) 507–531.

[3] L. Bandeira, A. Ornelas: On the characterization of a class of laminates for 2×2 symmetric gradients, J. Convex Analysis 18(1) (2011) 37–58.

[4] D. Faraco, L. Székelyhidi Jr.: Tartar's conjecture and localization of the quasiconvex hull in $R^{2\times 2}$, Acta Math. 200(2) (2008) 279–305.

[5] D. Kinderlehrer, P. Pedregal: Characterizations of Young measures generated by gradients, Arch. Rational Mech. Anal. 115(4) (1991) 329–365.

[6] B. Kirchheim: Rigidity and Geometry of Microstructures, Habilitation Thesis, Univ. Leipzig (2003).

[7] J. Matoušek, P. Plechac: On functional separately convex hulls, Discrete Comput. Geom. 19(1) (1998) 105–130.

[8] C. Morrey: Quasi-convexity and the lower semicontinuity of multiple integrals, Pacific J. Math. 2(1) (1952) 25–53.

[9] S. Müller: Variational models for microstructure and phase transitions, in: Calculus of Variations and Geometric Evolution Problems, Lectures given at the 2nd Session of the Centre Internazionale Matematico Estivo, Cetaro (1996).

[10] P. Pedregal: Laminates and microstructure, European J. Applied Mathematics 4(2) (1993) 121–149.

[11] P. Pedregal: Some remarks on quasiconvexity and rank-one convexity, Proc. Royal Society of Edinburgh, Section A 126(5) (1996) 1055–1065.

[12] P. Pedregal: Some evidence in favor of Morrey's conjecture, arXiv (June 2014).

[13] P. Pedregal, V. Sverak: The quasiconvexification of a cubic polynomial at the origin, Nonlinear Analysis 30(8) (1997) 5093–5100.

[14] P. Pedregal, V. Sverak: A note on quasiconvexity and rank-one convexity for 2×2 matrices, J. Convex Analysis 5(1) (1998) 107–117.

[15] V. Sverak: New examples of quasiconvex functions, Arch. Rational Mech. Anal. 119(4) (1992) 293–300.

[16] V. Sverak: Rank-one convexity does not imply quasiconvexity, Proc. Royal Society of Edinburgh, Section A 120(1-2) (1992) 185–189.