Journal of Convex Analysis
Volume 24 (2017), No. 4, 1263–1279

Uniform Strong Proximinality and Continuity of Metric Projection

Sudipta Dutta
Department of Mathematics and Statistics,
Indian Institute of Technology, Kanpur 208016, India
sudipta@iitk.ac.in

P. Shunmugaraj
Department of Mathematics and Statistics,
Indian Institute of Technology, Kanpur 208016, India
psraj@iitk.ac.in

Vamsinadh Thota
Department of Mathematics and Statistics,
Indian Institute of Technology, Kanpur 208016, India
vamst@iitk.ac.in

Received: January 2, 2016
Accepted: September 14, 2016

We present a sufficient condition for the uniform continuity of metric projection. This condition is a natural strengthening of the notion of strong proximinality, appearing in literature for the past few years. We show that this condition is equivalent to the sufficient condition of continuity of metric projection introduced by K.-S. Lau back in 1979. A characterization of uniform convexity through proximinality is presented and we also relate quantitatively the power type estimate of modulus of uniform strong proximinality to the power type estimate of modulus of uniform convexity.

Keywords: Uniformly strongly proximinal, U-proximinal, metric projection, uniform convexity.

2000 Mathematics Subject Classification: 41A65, 46B20.

1. Introduction

Let X be a real Banach space and M a closed subspace of X. For $x \in X$ we define $P_M(x) = \{y \in M : \|x - y\| = d(x, M)\}$ the *set of best approximation of x in M*. Here $d(x, M)$ denotes the distance between x and M. For $\delta > 0$, we define $P_M(x, \delta) = \{y \in M : \|x - y\| \leq d(x, M) + \delta\}$. If for every $x \in X$, $P_M(x) \neq \emptyset$, we say that M is a *proximinal subspace* of X and if $P_M(x)$ is a singleton then M is called *Chebyshev*. The set-valued map $x \mapsto P_M(x)$ is called the *metric projection* of X onto M.

ISSN 0944-6532 / \$ 2.50

A sequence (x_n) in M is called a *minimizing sequence* for $x \in X$ in M if $\|x - x_n\| \to d(x, M)$. For a non-empty set A of X, let $B_\varepsilon(A) = A + \varepsilon B_X$ where B_X denotes the closed unit ball of X. The unit sphere of X is denoted by S_X.

All subspaces we consider in this paper are assumed to be closed.

One of the pertinent questions which received a lot of attention is under what geometric properties on X or proximinality properties on M, the metric projection is continuous or it has stronger continuity properties, such as, uniform continuity or Lipschitz continuity in terms of Hausdorff metric.

It is easy to find an example of a proximinal (even Chebyshev) subspace but the metric projection is not continuous, see for examples in the book by Holmes [12]. In case M is Chebyshev, a natural sufficient condition on M for the continuity of P_M is that the minimum norm problem is well-posed; i.e., every minimizing sequence in M converges. We note that if M is Chebyshev and a minimizing sequence (x_n) for $x \in X$ in M converges then it converges to $P_M(x)$ and the sequence of sets $(P_M(x, \frac{1}{n}))$ converges to $P_M(x)$ in the Hausdorff metric.

Attempts were made to generalize the notion of well posed-ness in case M is not Chebyshev and study the continuity properties of P_M which is a set-valued map. One natural generalization of the well posed-ness is generalized well posed-ness or approximative compactness [18, page 469]. We say that a proximinal subspace M of X is *approximatively compact* if every minimizing sequence has a convergent subsequence. Observe that a proximinal subspace M is approximatively compact if and only if $(P_M(x, \frac{1}{n}))$ converges to $P_M(x)$ in the Hausdorff metric and $P_M(x)$ is compact for all $x \in X$.

Further, by dropping the compactness condition on $P_M(x)$, a generalized notion called *strong proximinality* was introduced. We say that a proximinal subspace M is strongly proximinal if for every $x \in X$, $(P_M(x, \frac{1}{n}))$ converges to $P_M(x)$ in the Hausdorff metric. Locally the notion is defined as follows.

Definition 1.1. A proximinal subspace M is said to be *strongly proximinal* at $x \in X$ if for given $\varepsilon > 0$ there exists $\delta > 0$ such that $P_M(x, \delta) \subseteq B_\varepsilon(P_M(x))$.

This generalized notion received considerable interests recently (see [3, 4, 5, 6, 9, 14, 15], and the recent [7]). Though it is a natural generalization of the well posed-ness of the minimum norm problem, it is well known that strong proximinality alone is not a sufficient condition for the continuity of the metric projection. However, it is a simple consequence of the definition that strong proximinality implies the upper semi-continuity of the metric projection with respect to the Hausdorff metric. In some cases [4, 7, 14, 15] strong proximinality also gives the lower semi-continuity of the metric projection.

It is natural to look for a sufficient condition which is a stronger condition compared to strong proximinality for obtaining continuity properties of the metric projection. In this paper we introduce an obvious strengthening of the notion of strong proximinality, namely the following.

Definition 1.2. Let M be a proximinal subspace of Banach space X and $A \subseteq X$. M is said to be *uniformly strongly proximinal* on A if for $\varepsilon > 0$, there exists $\delta > 0$ such that $P_M(x, \delta) \subseteq B_\varepsilon(P_M(x))$ for every $x \in A$.

For a subspace M of X and for $0 < \rho < \beta$ we denote $D_M = \{x \in S_X : d(x, M) = 1\}$ and the metric annulus $D_\rho^\beta(M) = \{x \in X : \rho \leq d(x, M) \leq \beta\}$.

In Section 2 we show that if a subspace M is uniformly strongly proximinal on D_M then M is uniformly strongly proximinal on every metric annulus and, in particular, the metric projection is locally uniformly continuous on $X \setminus M$. We note that if a subspace M is uniformly strongly proximinal on a subset A then P_M is uniformly continuous on A. We present several examples of subspaces M which are uniformly strongly proximinal either on D_M or on whole of X. One of the natural class of subspaces which turns out to be uniformly strongly proximinal on whole of X is the subspaces with the $1\frac{1}{2}-$ball property (see Definition 2.9) in X. It is known [6] that if a subspace M has the $1\frac{1}{2}-$ball property in X then the metric projection P_M is 2-Lipschitz continuous.

In [17], way back in 1979, Lau introduced a sufficient condition for the existence of best approximation. We recall the notion of U-proximinality introduced by Lau.

Definition 1.3. A subspace M of X is said to be *U-proximinal* if there exists a function $\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\epsilon(\cdot)$ is continuous, increasing, $\epsilon(\rho) \to 0$ as $\rho \to 0$ and $(1 + \rho)B_X \cap [B_X + M] \subseteq B_X + \epsilon(\rho)B_M$.

At first appearance it is difficult to understand as to what U-proximinal means geometrically and although it is introduced for the existence of best approximation, it is shown in [17] that it works as a sufficient condition for the continuity of metric projection. In Section 3 of this paper, we present some characterizations of U-proximinality which provide more clarity for understanding the notion of U-proximinality geometrically and the characterizations explain that the U-proximinality is a natual sufficient condition for the continuity of the metric projection. One of the characterizations is that a subspace M is U-proximinal if and only if it is uniformly strongly proximinal (USP) on D_M. This fits well with the local version of the result proved in [3] that M is locally U-proximinal if and only if it is strongly proximinal. As a consequence of the characterization, we present some new examples for U-proximinal subspaces and we also obtain the fact that if M is U-proximinal then the metric projection P_M is not only continuous on X, it is locally uniformly continuous on $X \setminus M$.

Further, we observe that the notion of *Property*-(A) introduced in [19] (see Definition 3.4) is actually equivalent to the property USP on D_M.

Section 4 deals with a characterization of uniform convexity in terms of uniform strong proximinality. We show that a Banach space is uniformly convex space if and only if the class of all closed subspaces of X is uniformly USP on any bounded subset of X. As a corollary of this result we obtain a result of Holmes [12, page 165].

In Section 5, we introduce the notion of modulus of uniform strong proximinality and present some quantitative results which relate the modulus of uniform strong proximinality and modulus of uniform convexity. In particular we show that in a uniformly convex space with modulus of uniform convexity of power type p, every subspace has modulus of uniform strong proximinality of power type $\frac{1}{p}$. Conversely if every subspace (every hyperplane) in X has modulus of uniform strong proximinality of power type $\frac{1}{p}$ then X is uniformly convex with modulus of uniform convexity of power type p. This gives rise to some natural questions which we present in Remark 5.4.

As a corollary to our results in Section 5, it follows that in every Banach space there is a hyperplane which fails to have the $1\frac{1}{2}-$ball property.

2. Uniform continuity of metric projection and examples

In the entire paper, unless specified otherwise, we assume M to be a proximinal subspace of a real Banach space X.

Definition 2.1. P_M is said to be *continuous at* $x_0 \in X$ if for $\varepsilon > 0$ there exists $\delta > 0$ such that $P_M(x) \subseteq B_\varepsilon(P_M(x_0))$ and $P_M(x_0) \subseteq B_\varepsilon(P_M(x))$ for all $x \in X$ satisfying $\|x_0 - x\| < \delta$.

One of our main results in this section is that uniform strong proximinality of M on D_M is a sufficient condition for the metric projection P_M to be uniformly continuous on each metric annulus. Recall that for $0 < \rho < \beta$ we denote the metric annulus $\{x \in X : \rho \leq d(x, M) \leq \beta\}$ (with respect to M) by D_ρ^β.

Theorem 2.2. *Let M be uniformly strongly proximinal on D_M. Then for any β and ρ satisfying $0 < \rho < \beta$, we have*

(a) *M is uniformly strongly proximinal on D_ρ^β;*
(b) *P_M is uniformly continuous on D_ρ^β.*

Proof. Suppose M is uniformly strongly proximinal on D_M and $0 < \rho < \beta$ for some $\rho, \beta \in \mathbb{R}$. Let $\epsilon > 0$.

(a) By definition, there exists $\delta > 0$ such that $P_M(x, \delta) \subseteq B_\varepsilon(P_M(x))$ for all $x \in D_M$. We show that for any $x \in D_\rho^\beta$, $P_M(x, \rho\delta) \subseteq B_{\varepsilon\beta}(P_M(x))$. Let $x \in D_\rho^\beta$ and $y \in P_M(x)$. Denote $x_0 = \frac{x-y}{\|x-y\|}$. Then $x_0 \in D_M$ and $P_M(x_0, \delta) \subseteq B_\varepsilon(P_M(x_0))$. Denote $d(x, M) = \alpha$ and note that

$$P_M(x_0, \delta) = \frac{1}{\alpha}(P_M(x, \alpha\delta) - y) \quad \text{and} \quad B_\varepsilon(P_M(x_0)) = \frac{1}{\alpha}B_{\alpha\varepsilon}(P_M(x) - y).$$

Since $P_M(x_0, \delta) \subseteq B_\varepsilon(P_M(x_0))$ the conclusion follows.

(b) Due to part (a), M is uniformly strongly proximinal on D^{β}_{ρ}. Hence there exists $\delta > 0$ such that $P_M(x,\delta) \subseteq B_{\epsilon}(P_M(x))$ for every $x \in D^{\beta}_{\rho}$. Then for any $x, y \in D^{\beta}_{\rho}$ satisfying $\|x - y\| < \frac{\delta}{4}$, we have $P_M(x) \subseteq P_M(y,\delta) \subseteq B_{\epsilon}(P_M(y))$. By interchanging x and y we get $P_M(y) \subseteq P_M(x,\delta) \subseteq B_{\epsilon}(P_M(x))$. This proves that P_M is uniformly continuous on D^{β}_{ρ}. □

Remark 2.3. The proof of part (b) of Theorem 2.2 actually shows more. It follows that if M is uniformly strongly proximinal on a subset A of X then the metric projection P_M is uniformly continuous on A. The converse is not true. For example, the metric projection on every closed subspace of a Hilbert space is uniformly continuous but even a one dimensional subspace of $\mathbb{R}^2$ with the usual norm is not uniformly strongly proximinal on $\mathbb{R}^2$.

As P_M is continuous on M, the following result is an immediate consequence of Theorem 2.2. Recall that for $x \in X$ and $r > 0$, $B_r(x) = \{y \in X : \|x - y\| \leq r\}$.

Corollary 2.4. *If M is uniformly strongly proximinal on D_M, then*

(a) *P_M is locally uniformly continuous on $X \backslash M$;*

(b) *P_M is continuous on X.*

In case X is strictly convex and if M is uniformly strongly proximinal on D_M then P_M is actually uniformly continuous on all bounded subsets of X. To see this it is enough to prove the following.

Corollary 2.5. *Suppose X is strictly convex and M is uniformly strongly proximinal on D_M. Let $\beta > 0$ and $D^{\beta} = \{x \in X : d(x, M) \leq \beta\}$. Then*

(a) *M is uniformly strongly proximinal on D^{β};*

(b) *P_M is uniformly continuous on D^{β}.*

Proof. (a) Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $P_M(x,\delta) \subseteq B_{\varepsilon}(P_M(x))$ for all $x \in D_M$. Assume that $\delta < \epsilon$. Let $0 \leq r < 1$ and $x_0 \in X$ be such that $d(x_0, M) = r$. Denote $P_M(x_0) = y_0$. We show that $P_M(x_0,\delta) \subseteq B_{\varepsilon}(y_0)$ which proves the result because of Theorem 2.2.

Suppose $r > 0$. We may assume (and we assume) that $y_0 = 0$. Then $\frac{x_0}{r} \in D_M$, $P_M(\frac{x_0}{r}) = 0$ and by assumption $P_M(\frac{x_0}{r}, \delta) \subseteq B_{\epsilon}(0)$. It is easy to verify that $P_M(x_0,\delta) \subseteq P_M(\frac{x_0}{r}, \delta)$. Therefore $P_M(x_0,\delta) \subseteq B_{\epsilon}(0) = B_{\epsilon}(y_0)$.

In case $r = 0$ then $y_0 = x_0$ and $P_M(x_0,\delta) \subseteq P_M(x_0,\epsilon) \subseteq B_{\epsilon}(x_0)$.

(b) This follows from Remark 2.3. □

In the rest of the section we give various examples related to uniform strong proximinality.

Proposition 2.6. *If M is a strongly proximinal hyperplane of X, then it is uniformly strongly proximinal on D_M.*

Proof. Suppose $M = \{y \in X : x^*(y) = 0\}$ for some $x^* \in S_{X^*}$ and M is strongly proximinal. Let $\epsilon > 0$ and $x_0 \in D_M$. Then $x^*(x_0) = \pm 1$. Suppose $x^*(x_0) = 1$. Denote $H = \{y \in X : x^*(y) = 1\}$.

We observe that $P_H(0) = x_0 - P_M(x_0)$ and $P_H(0,\delta) = x_0 - P_M(x_0,\delta)$. Since M is strongly proximinal, there exists $\delta > 0$ such that $P_M(x_0,\delta) \subseteq B_\varepsilon(P_M(x_0))$. This implies that

$$x_0 - P_M(x_0,\delta) \subseteq B_\varepsilon(x_0 - P_M(x_0)),$$

and hence $P_H(0,\delta) \subseteq B_\varepsilon(P_H(0))$. Again by translation we see that $P_M(x,\delta) \subseteq B_\varepsilon(P_M(x))$ for all $x \in D_M$ satisfying $x^*(x) = 1$.

Finally, $P_M(x,\delta) \subseteq B_\varepsilon(P_M(x))$ for all $x \in D_M$ satisfying $x^*(x) = -1$ can be shown in a similar way. $\square$

The following corollary is immediate from Corollary 2.4, Corollary 2.5 and Proposition 2.6.

Corollary 2.7. *Let M be a strongly proximinal hyperplane of X. Then P_M is uniformly continuous on every metric annulus and in particular it is locally uniformly continuous on $X \backslash M$. If X is strictly convex then P_M is uniformly continuous on bounded subsets of X.*

We refer to [6, 9] for a necessary and sufficient condition for a proximinal hyperplane to be strongly proximinal (see Proposition 4.3).

Remark 2.8. (1) Uniform strong proximinality is not a necessary condition for the continuity of the metric projection. If we consider any proximinal hyperplane M, then P_M is continuous whereas M need not be even strongly proximinal.

(2) It is known [2] that there is a one dimensional subspace M of a three dimensional space X such that P_M is not continuous. As M is finite dimensional, it is strongly proximinal however by Corollary 2.4 it is not uniformly strongly proximinal on D_M. Moreover, the same example reveals that Proposition 2.6 is not valid for M which is of co-dimension 2.

For the next class of examples we need the following definition.

Definition 2.9. A closed subspace M is said to have the $1\frac{1}{2}-$*ball property* in X if $x \in X, y \in M,\ r, s > 0,\ B_r(x) \cap M \neq \emptyset$ and $\|x - y\| < r + s$ then the intersection $M \cap B_r(x) \cap B_s(y) \neq \emptyset$.

We will use the fact [10] that every closed subspace which has the $1\frac{1}{2}-$ball property is proximinal. We also need the following lemma.

Lemma 2.10. *Let M be a proximinal subspace of X. Then the following assertions are equivalent.*

(a) *M has the $1\frac{1}{2}-$ball property;*
(b) *For any $x \in X$ and $y \in M$ we have $\|x - y\| = d(x, M) + d(y, P_M(x))$.*
(c) *For any $x \in x_0 + M, x_0 \in D_M$, we have $\|x\| = 1 + d(x, P_{x_0+M}(0))$.*

Proof. $(a) \Leftrightarrow (b)$ is proved in [10] and the proof of $(b) \Leftrightarrow (c)$ follows easily by the translation. □

The following simple result unifies some of the results of [13, 17] which is explained in Remark 2.12.

Proposition 2.11. *Let M have the $1\frac{1}{2}-$ball property in X. Then M is uniformly strongly proximinal on D_M. In fact more, it is uniformly strongly proximinal on entire X.*

Proof. Let $\varepsilon > 0$ and $x \in X \setminus M$. We show that $P_M(x, \frac{\varepsilon}{2}) \subseteq B_\varepsilon(P_M(x))$. Let $y \in P_M(x, \frac{\varepsilon}{2})$. Then by the definition of the $1\frac{1}{2}-$ball property $d(x, M) + d(y, P_M(x)) = \|x - y\| \leq d(x, M) + \frac{\varepsilon}{2}$. Therefore $d(y, P_M(x)) \leq \frac{\varepsilon}{2}$. □

Remark 2.12. (1) In the next section we will show that a proximinal subspace M is U-proximinal if and only if it is uniformly strongly proximinal on D_M. In [13, 17] it is proved that M-ideals, semi-L-summands and semi M-ideals are U-proximinal subspaces. However it is known [11] that these classes have the $1\frac{1}{2}-$ball property and hence by Proposition 2.11 they are U-proximinal.

(2) It is known [6] that if M has the $1\frac{1}{2}-$ball property then P_M is 2-Lipschitz continuous.

In [17] it is shown that for a compact Hausdorff space K, the space of continuous functions $C(K)$ is U-proximinal in the space of all bounded functions $B(K)$ with the supremum norm. We show below that in this case also $C(K)$ in $B(K)$ has the $1\frac{1}{2}-$ball property.

Proposition 2.13. *Let K be a compact Hausdorff space. Then $C(K)$ has the $1\frac{1}{2}-$ball property in $B(K)$.*

Proof. It is known [16] that $C(K)$ is proximinal in $B(K)$. Denote $M = C(K)$. Let $f_0 \in D_M$ and $f \in f_0 + M$. By Lemma 2.10, it is enough to show that $\|f\| = 1 + d(f, P_{f_0+M}(0))$. If $\|f\| = 1$ then f satisfies this condition trivially. Therefore let $\|f\| = 1+\rho$ where $\rho > 0$. Define $h' = (f \wedge \rho) \vee (-\rho)$ and $g' = f - h'$. Then $h' \in M$, $\|h'\| \leq \rho$ and it follows from the proof of Proposition 4.4 of [17] that $\|g'\| \leq 1$. Note that $g' = f - h' \in f_0 + M$ and hence $g' \in P_{f_0+M}(0)$. It follows that $\|f - g'\| \geq d(f, P_{f_0+M}(0))$. Thus $\rho \geq \|h'\| \geq d(f, P_{f_0+M}(0))$. Hence $\|f\| \geq 1 + d(f, P_{f_0+M}(0))$. The other inequality is obvious. □

Remark 2.14. (1) In [17] it is also proved that $K(L_1(\mu), \ell_1)$ - the space of all compact operators from $L_1(\mu)$ to ℓ_1, μ non atomic, is U-proximinal in $L(L_1(\mu), \ell_1)$ - the space of all bounded operators from $L_1(\mu)$ to ℓ_1. However, in [21] it is shown that $K(L_1(\mu), \ell_1)$ has the $1\frac{1}{2}-$ball property in $L(L_1(\mu), \ell_1)$.

(2) By Proposition 2.13 and Remark 2.12 it follows that for a compact Hausdorff space K, the metric projection $P_{C(K)}$ is 2-Lipschitz continuous on $B(K)$. We refer to [12, page 173] for a variant of this result.

The examples of subspaces of classical spaces which are U-proximinal given above or given in [17] have the $1\frac{1}{2}-$ball property. Taking cue from an example in [20], in the following we show that there exists a subspace M of c_0 which does not have the $1\frac{1}{2}-$ball property in c_0 but it is uniformly strongly proximinal on D_M; that is, M is U-proximinal.

Example 2.15. Take the space c_0 with sup norm and $f = (\frac{2}{3}, \frac{-1}{3}, 0, 0, 0, \dots) \in c_0^*$. Let $M = Ker(f) = \{x(x_i) \in c_0 : 2x_1 = x_2\}$. Since f has a finite support, M is proximinal and hence it follows from [15] that M is strongly proximinal in c_0. Therefore, by Proposition 2.6, M is uniformly strongly proximinal on D_M. We show that M does not have the $1\frac{1}{2}-$ball property.

Consider $x = (-1, 2, 0, 0, \dots) \in X$ and $y = (2, 4, 0, 0, \dots) \in M$. First we show $d(x, M) = \frac{4}{3}$. Suppose $(z_1, z_2, z_3, \dots) \in B[x, r] \cap M$ for some $r > 0$ then $-r - 1 \leq z_1 \leq r - 1$ and $-r + 2 \leq z_2 \leq r + 2$. If $r < \frac{4}{3}$ then $2z_1 < z_2$ which implies that $B[x, r] \cap M = \emptyset$. Since $(\frac{1}{3}, \frac{2}{3}, 0, 0, \dots) \in M \cap B[x, \frac{4}{3}]$ we have $d(x, M) = \frac{4}{3}$.

If $(z_1, z_2, z_3, \dots) \in B[x, \frac{4}{3}] \cap M = P_M(x)$ then $z_1 = \frac{1}{3}$. Therefore for any $z \in P_M(x)$ we have $\|y - z\| \geq \frac{10}{3}$ and hence $d(y, P_M(x)) \geq \frac{10}{3}$. This implies that

$$d(x, M) + d(y, P_M(x)) \geq \frac{4}{3} + \frac{10}{3} > 3 = \|x - y\|.$$

Therefore by Lemma 2.10 the subspace M fails to have the $1\frac{1}{2}-$ball property.

Remark 2.16. In Section 5 we show a more general result (Corollary 5.5) which provides several examples of M which are uniformly strongly proximinal on D_M but they do not have the $1\frac{1}{2}-$ball property.

Another natural class of examples of uniformly strongly proximinal subspaces are subspaces of uniformly convex spaces. We will deal with this in Section 4.

3. U-proximinality and uniform strong proximinality

In this section we present some characterizations of U-proximinality. In particular we show that M is a U-proximinal subspace of X if and only if it is uniformly strongly proximinal on D_M.

Observe that a subspace M of X is U-proximinal if for $\varepsilon > 0$ there exists $\delta > 0$ such that $(1+\delta)B_X \cap [B_X + M] \subseteq B_X + \varepsilon B_M$. It is known [17] that U-proximinal subspaces are proximinal.

The following characterizations provide more clarity for understanding the notion U-proximinality geometrically and also show that U-proximinality is a natural sufficient condition for the (uniform) continuity of the metric projection.

Theorem 3.1. *Let M be a proximinal subspace of a Banach space X. Then the following statements are equivalent.*

(a) *M is U-proximinal in X;*

(b) *Given $\varepsilon > 0$ there exists $\delta > 0$ such that $(1+\delta)B_X \cap [x+M] \subseteq B_\varepsilon(B_X \cap [x+M])$ for all $x \in B_X$;*

(c) *Given $\varepsilon > 0$ there exists $\delta > 0$ such that $(1+\delta)B_X \cap [x+M] \subseteq B_\varepsilon(B_X \cap [x+M])$ for all $x \in D_M$;*

(d) *M is uniformly strongly proximinal on D_M.*

Proof. (a) $\Rightarrow$ (b) Suppose M is U-proximinal in X and $\varepsilon > 0$. Then there exists $\delta > 0$ such that $(1+\delta)B_X \cap [B_X + M] \subseteq B_X + \varepsilon B_M$. Let $y \in (1+\delta)B_X \cap [x+M]$ for some $x \in B_X$. Then $y \in (1+\delta)B_X \cap [B_X+M]$ and $y = x+m$ for some $m \in M$. By the assumption $y \in B_X + \varepsilon B_M$. Therefore $y = x_0 + \varepsilon m_0$ for some $x_0 \in B_X$ and $m_0 \in B_M$. Moreover $\|y - x_0\| \leq \varepsilon$. Since $x_0 = y - \varepsilon m_0 = x + m - \varepsilon m_0$, $x_0 \in x+M$. Therefore $x_0 \in B_X \cap [x+M]$ and hence $y \in B_\varepsilon(B_X \cap [x+M])$ which proves the implication.

(b) $\Rightarrow$ (c) This implication is obvious.

(c) $\Leftrightarrow$ (d) These implications follow from the observation that for any $\delta > 0$, $x \in D_M$, $(1+\delta)B_X \cap (x+M) = P_{x+M}(0,\delta) = x - P_M(x,\delta)$ and $B_X \cap (x+M) = P_{x+M}(0) = x - P_M(x)$.

(c) $\Rightarrow$ (b) Let $x \in B_X \setminus D_M$ and $\varepsilon > 0$.

If $x \in M$ then it is easy to see that $(1+\varepsilon)B_X \cap [x+M] \subseteq B_\varepsilon(B_X \cap [x+M])$. Suppose x is not in M, that is $x + M \neq M$. By (c), there exists δ such that $(1+\delta)B_X \cap [x_0+M] \subseteq B_{\frac{\varepsilon}{2}}(B_X \cap [x_0+M])$ for all $x_0 \in D_M$. We now use the argument used in the proof of Proposition 2.2 of [17].

Let $y \in (1+\delta)B_X \cap [x+M]$. If $y \in B_X$ then $y \in B_X \cap [x+M]$ hence there is nothing to prove. So let $y \in (1+\delta)B_X \setminus B_X$. Find $x_0 \in D_M \cap span\{x, M\}$ and $\alpha > 0$ such that $\alpha y \in x_0 + M$ and $[y, x_0] \cap M = \emptyset$. Let $y_1 = \alpha y$.

Choose $w_1 \in (x_0+M) \cap B_X$ such that $\|y_1 - w_1\| \leq d(y_1, (x_0+M)\cap B_X) + \dfrac{\varepsilon}{2}$.

Define x_1 as follows. If $y_1 \in (1+\delta)B_X$, let $x_1 = y_1$. Otherwise let

$$\{x_1\} = [w_1, y_1] \cap (1+\delta)S_X.$$

Then

$$\begin{aligned}\|y_1 - x_1\| + \|x_1 - w_1\| &\leq d(y_1, (x_0+M)\cap B_X) + \frac{\varepsilon}{2} \\ &\leq \|y_1 - x_1\| + d(x_1, (x_0+M)\cap B_X) + \frac{\varepsilon}{2}\,.\end{aligned}$$

Therefore $\|x_1-w_1\| \leq d(x_1,(x_0+M)\cap B_X)+\frac{\varepsilon}{2}$. Since $x_1 \in (1+\delta)B_X\cap(x_0+M)$, by the assumption, $x_1 \in B_{\frac{\varepsilon}{2}}(B_X \cap (x_0+M))$. Therefore $\|x_1-w_1\| \leq \varepsilon$.

Using Lemma 2.1 of [17] and the remark below this lemma in [17], find $z \in (x+M)\cap S_X$ such that $\|y-z\| \leq \|x_1-w_1\|$. Hence $y \in B_\varepsilon(B_X \cap [x+M])$.

(b) $\Rightarrow$ (a) Let $\varepsilon > 0$. By (b) there exists $\delta > 0$ such that

$$(1+\delta)B_X \cap [x+M] \subseteq B_\varepsilon(B_X \cap [x+M])$$

for all $x \in B_X$. We claim that

$$(1+\delta)B_X \cap [B_X+M] \subseteq B_X + \varepsilon B_M.$$

Suppose $y \in (1+\delta)B_X \cap [B_X+M]$. Then $y = x+m$ for some $x \in B_X$, $m \in M$ and $\|y\| \leq 1+\delta$. Observe that $y \in (1+\delta)B_X \cap [x+M]$. Therefore by the assumption $y \in B_\varepsilon(B_X \cap [x+M])$. Find $x_0 \in B_X \cap [x+M]$ such that $\|y-x_0\| \leq \varepsilon$. Since $y \in x+M$ and $x_0 \in x+M$, $y-x_0 \in M$. Therefore $y = x_0 + (y-x_0) \in x_0 + \varepsilon B_M \subseteq B_X + \varepsilon B_M$. □

The following corollary which sharpens Theorem 3.4 of [17] is a consequence of Theorem 3.1.

Corollary 3.2. *If M is a U-proximinal subspace of X, then the metric projection P_M is continuous on X. Moreover, it is uniformly continuous on every metric annulus and in particular it is locally uniformly continuous on $X \setminus M$.*

Remark 3.3. Given a paracompact Hausdorff space K and a subspace M of X, let $C_b(K,M)$ (resp., $C_b(K,X)$) denote the space of all bounded continuous functions $f : K \to M$ (resp., $f : K \to X$). It was observed in [3] that the strong proximinality of M in X is not sufficient to ensure the proximinality of $C_b(K,M)$ in $C_b(K,X)$. However, in [19], the following property, called Property-(A), is considered for studying the sufficiency part.

Definition 3.4. A subspace M of X is said to have *Property*-(A) if for every $\varepsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that, given $x \in X$ with $d(x,M) \leq R$ and $w \in M$ such that $\|x-w\| < R+\delta$, there exists $y \in M$ such that $\|x-y\| \leq R$ and $\|w-y\| \leq \varepsilon$.

It is proved in [19, Theorem 4] that if M has Property-(A) in X, then $C_b(K,M)$ is proximinal in $C_b(K,X)$.

It is straightforward to see that if M has Property-(A) in X then M is USP on D_M. On the other hand in [19, Example 6], it is shown that if M is U-proximinal in X then M has Property-(A). Hence, it follows from Theorem 3.1 that all three properties are equivalent. Therefore, if M is USP on D_M, then $C_b(K,M)$ is proximinal in $C_b(K,X)$.

4. Uniform strong proximinality and uniform convexity

In this section we present a characterization of uniform convexity in terms of uniform strong proximinality.

It is shown in [17] that in a uniformly convex space every closed subspace is U-proximinal. Therefore, by Theorem 3.1 and Corollary 2.5, it follows that in a uniformly convex space every closed subspace is uniformly strongly proximinal (USP) on bounded subsets. We see below that in a uniformly convex space the class of all closed subspaces is uniformly USP on bounded subsets and interestingly the converse is also true. This implies the result of Holmes [12, page, 165] that the family of maps $\{P_M : M$ is a closed subspace of a uniformly convex space $X\}$ is uniformly equi-continuous on any bounded subset of X.

The following is our main result of this section.

Theorem 4.1. *The following assertions are equivalent.*

(a) *X is uniformly convex;*

(b) *For a given $\varepsilon > 0$ there exists $\delta > 0$ such that $P_M(x, \delta) \subseteq B_\varepsilon(P_M(x))$ holds for every closed subspace M of X and $x \in D_M$.*

(c) *For a given $\varepsilon > 0$ there exists $\delta > 0$ such that $P_M(x, \delta) \subseteq B_\varepsilon(P_M(x))$ holds for every closed hyperplane M of X and $x \in D_M$.*

For the proof we need some definitions and notations.

It is known [18, page 483] that for a fixed $x \in S_X$ the function

$$t \to \frac{\|x + th\| - \|x\|}{t}$$

is increasing on $\mathbb{R} \setminus \{0\}$ for any $h \in X$ and hence the limit

$$d_+(x, h) = \lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t}$$

exists.

Definition 4.2. The norm $\|\cdot\|$ of X is said to be *strongly subdifferentiable* (in short ssd) at $x \in X$ if the one-sided limit

$$\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t}$$

exists uniformly for all $h \in S_X$. If the norm $\|\cdot\|$ of X is ssd at all points of S_X, we say that $\|\cdot\|$ is ssd or the space X is ssd.

The norm is uniformly ssd on X (or X is uniformly ssd) if the above limit is uniform in $x \in S_X$.

The following observation is an immediate consequence of Proposition 2.6 and the characterization of strongly proximinal hyperplanes in terms of ssd of the dual norm given in [9] (see Proposition 2.1 of [6]).

Proposition 4.3. *Let $M = \{x \in X : x^*(x) = 0\}$ for some $x^* \in S_{X^*}$. Then the following assertions are equivalent.*

(a) *M is uniformly strongly proximinal on D_M;*

(b) *The norm of X^* is ssd at x^*.*

For $x \in S_X$, $x^* \in S_{X^*}$ and $0 \leq \delta < 1$ we define

$$S(X, x^*, \delta) = \{x \in B_X : x^*(x) \geq 1 - \delta\}$$

and

$$S(X^*, x, \delta) = \{x^* \in B_{X^*} : x^*(x) \geq 1 - \delta\}.$$

Proof of Theorem 4.1. (a) $\Rightarrow$ (b) Let X be uniformly convex and ε be such that $0 < \varepsilon < 2$. Then there exists δ such that $0 < \delta < 1$ and for every $x, y \in B_X$ with $\|x - y\| \geq \frac{\epsilon}{2}$ we have $\|\frac{x+y}{2}\| < 1 - 2\delta$. Let M be a closed subspace of X and $x_0 \in D_M$. Find $x^* \in S_{X^*}$ such that $x^*|_M = 0$ and $x^*(x_0) = 1$. We show that $diam(P_{x_0+M}(0, \delta)) \leq \epsilon$ which proves (b). Let $x, y \in P_{x_0+M}(0, \delta)$. Consider $\overline{x} = \frac{x}{1+\delta}$ and $\overline{y} = \frac{y}{1+\delta}$. Then $\overline{x}, \overline{y} \in B_X$. Moreover, $x^*(\overline{x}) > 1 - \delta$ and $y^*(\overline{y}) > 1 - \delta$. If $\|\overline{x} - \overline{y}\| \geq \frac{\epsilon}{2}$ then $x^*(\frac{\overline{x}+\overline{y}}{2}) \leq \|\frac{\overline{x}+\overline{y}}{2}\| < 1 - 2\delta$ which is a contradiction. Therefore $\|\overline{x} - \overline{y}\| < \frac{\epsilon}{2}$ and hence $\|x - y\| < \epsilon$.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) The proof of this implication will be done in several steps. Consider the following assertions.

(1) For each $\epsilon > 0$ there is $\delta > 0$ such that $P_H(0, \delta) \subseteq B_\epsilon(P_H(0))$ whenever H is a translate of a closed hyperplane satisfying $d(0, H) = 1$.

(2) For each $\epsilon > 0$ there is $\delta > 0$ such that $S(X, y^*, 0) \subseteq B_\epsilon(S(X, x^*, 0))$ for all $x^*, y^* \in S_{X^*}$ satisfying $\|x^* - y^*\| \leq \delta$.

(3) The dual norm is uniformly ssd on S_{X^*}.

(4) The dual norm is uniformly smooth.

We will show that (c) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) which proves (c) $\Rightarrow$ (a).

(c) $\Rightarrow$ (1) This is easy to verify.

(1) $\Rightarrow$ (2) Let $\epsilon > 0$. Then by (1) there exists $\delta > 0$ such that $\delta < \frac{\varepsilon}{2}$ and $P_H(0, \delta) \subseteq B_{\frac{\epsilon}{2}}(P_H(0))$ whenever H is a translate of a closed hyperplane satisfying $d(0, H) = 1$. Let $x^*, y^* \in S_{X^*}$ be such that $\|x^* - y^*\| \leq \delta$ and $x \in S(X, y^*, 0)$. Then

$$x^*(x) = y^*(x) - (y^* - x^*)(x) \geq 1 - \delta\,.$$

This implies that $x \in S(X, x^*, \delta)$. Define $H = \{y \in X : x^*(y) = 1\}$ and choose $\overline{x} \in P_H(x)$. Note that $\|\overline{x}\| \leq \|x - \overline{x}\| + \|x\| \leq d(x, H) + 1 \leq \delta + 1$. Since $\overline{x} \in P_H(0, \delta)$ choose $\overline{y} \in P_H(0) = S(X, x^*, 0)$ such that $\|\overline{x} - \overline{y}\| \leq \frac{\epsilon}{2}$. Therefore $\|x - \overline{y}\| \leq \|x - \overline{x}\| + \|\overline{x} - \overline{y}\| \leq \epsilon$. This shows that $x \in B_\epsilon(S(X, x^*, 0))$.

(2) $\Rightarrow$ (3) This follows from Theorem 2.1 of [1]. Since several concepts are involved in Theorem 2.1 of [1], we present here a direct proof following the proof of Theorem 1.2 of [8] for reader's convenience. Let $\varepsilon > 0$ and $x^* \in S_{X^*}$. Observe that (2) implies that the space X is reflexive.

By (2) there exists $\delta > 0$ which is independent of x^* such that $0 < \delta < 1$ and

$$S(X^{**}, y^*, 0) \subseteq B_\epsilon(S(X^{**}, x^*, 0))$$

whenever $y^* \in S_{X^*}$ and $\|x^* - y^*\| < \delta$. Let $0 < t < \frac{\delta}{4}$ and $h^* \in S_{X^*}$. We will show that

$$\frac{\|x^* + th^*\| - \|x^*\|}{t} - d_+(x^*, h^*) < \varepsilon.$$

Choose $x_t^{**} \in S_{X^{**}}$ such that $x_t^{**}(x^* + th^*) = \|x^* + th^*\|$. Note that $\|x^* + th^*\| \geq 1 - \frac{\delta}{4} > \frac{1}{2}$. Define $x_t^* = \frac{x^* + th^*}{\|x^* + th^*\|}$. Note that $x_t^{**} \in S(X^{**}, x_t^*, 0)$ and

$$\|x_t^* - x^*\| < 2[(1 - \|x^* + th^*\|)\|x^*\| + t\|h^*\|] \leq \delta.$$

Therefore, there exists $x^{**} \in S(X^{**}, x^*, 0)$ such that $\|x_t^{**} - x^{**}\| < \varepsilon$.

Since, for any $t > 0$, $x^{**}(x^* + th^*) - x^{**}(x^*) \leq \|x^* + th^*\| - \|x^*\|$, we have $x^{**}(h^*) \leq \frac{\|x^* + th^*\| - \|x^*\|}{t}$ and hence $x^{**}(h^*) \leq d_+(x^*, h^*)$. Therefore

$$\frac{\|x^* + th^*\| - 1}{t} - d_+(x^*, h^*) \leq \frac{x_t^{**}(x^* + th^*) - x_t^{**}(x^*)}{t} - x^{**}(h^*) < \varepsilon\,.$$

(3) $\Rightarrow$ (4) This was proved in [8].

(4) $\Rightarrow$ (a) This is well known (see [18, page 500]). $\square$

The following result follows from the proof of Corollary 2.5 and Theorem 4.1.

Corollary 4.4. *If X is uniformly convex then the family of maps*

$$\{P_M : M \text{ a closed subspace of } X\}$$

is uniformly equi-continuous on any bounded subset of X.

Remark 4.5. It is known [9] that if M is strongly proximinal subspace of finite codimension then $M^\perp$ is ssd. It is tempting to conjecture that if M is uniformly strongly proximinal on D_M then $M^\perp$ is uniformly ssd, that is by [8], uniformly smooth. However this is not true. For an easy example, consider $M = \{f \in C([0, 1]) : f(x_i) = 0, x_1, x_2, \cdots, x_n \in [0, 1]\}$. Then M is an M-ideal in $C([0, 1])$ [11] hence has the $1\frac{1}{2}-$ball property. Therefore M is uniformly strongly proximinal on D_M. However $M^\perp$ is isometric to ℓ_1^n and it cannot be uniformly smooth. Equally it is easy to see that if every closed hyperplane M in X is strongly proximinal (therefore uniformly strongly proximinal) on D_M then X need not be uniformly smooth - any non super reflexive space X with X^* having a Frechet smooth norm will serve as an example.

5. Modulus of uniform strong proximinality

For a proximinal subspace M of X, the modulus of strong proximinality ε : $X \setminus M \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined in [5] as follows:

$$\varepsilon_M(x,t) = \inf\{r > 0 : P_M(x,t) \subseteq B_r(P_M(x))\}.$$

We may consider the uniform version of this modulus as follows:

$$\varepsilon_M(t) = \sup_{x \in D_M} \varepsilon(x,t).$$

It then follows that M is uniformly strongly proximinal on D_M if and only if $\varepsilon_M(t) \to 0$ as $t \to 0$.

Similarly one may define the modulus of U-proximinality as follows:

$$\varepsilon_M^u(t) = \inf\{r > 0 : (1+t)B_X \cap [B_X + M] \subseteq B_X + rB_M\}.$$

The proof of Lemma 3.1 shows that we actually have $\varepsilon_M(t) = \varepsilon_M^u(t)$.

Theorem 4.1 can be stated in terms of the modulus of uniform strong proximinality as follows: X is uniformly convex if and only if $\varepsilon_M(t) \to 0$ uniformly for all closed subspaces M of X as $t \to 0$.

In the following quantitative result we see that one can relate the modulus of uniform convexity $\delta_X(t)$ to $\varepsilon_M(t)$.

Proposition 5.1. *Let X be uniformly convex with modulus of uniform convexity $\delta_X(t)$. Then for any closed subspace M of X, $\varepsilon_M(t) \leq 9\delta_X^{-1}(\frac{t}{2})$.*

Proof. Let $x \in D_M$ and M a closed subspace of X. Then $d(x,M) = 1$. It is shown in the proof of Proposition 3.5 of [5] that

$$\varepsilon_M(x,t) \leq (1 + 2\|x\|)t + (1+t)\delta_X^{-1}(\frac{t}{2}).$$

Hence for $x \in D_M$ and sufficiently small t

$$\varepsilon_M(x,t) \leq 3t + 3\delta_X^{-1}(\frac{t}{2}) \leq 6\delta_X^{-1}(\frac{t}{2}) + 3\delta_X^{-1}(\frac{t}{2}).$$

Hence the result follows. $\square$

If for every closed subspace M of X we have uniform estimate on $\varepsilon_M(t)$, then we can also get the other way inequality as the following result shows.

Proposition 5.2. *Suppose there exists a non-decreasing, continuous function $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varepsilon(t) \to 0$ as $t \to 0$ such that for every closed hyperplane M of X we have $P_M(x,t) \subseteq B_{\varepsilon(t)}(P_M(x))$ for all $x \in D_M$. Then X is uniformly convex and $\delta_X(t) \geq \varepsilon^{-1}(\frac{t}{6})$ for $0 < t \leq 1$.*

Proof. It follows from Theorem 4.1 that X is uniformly convex. Observe that $t \leq \varepsilon(t)$ for $t > 0$.

We first show that for $t \in (0,1)$, $S(X,x^*,t) \subseteq S(X,x^*,0) + 2\varepsilon(t)B_X$ for all $x^* \in S_{X^*}$. For $x^* \in S_{X^*}$ let $M = \{x \in X : x^*(x) = 0\}$ and $H = \{x \in X : x^*(x) = 1\}$. Let $x \in S(X,x^*,t), t \in (0,1)$. Denote $\overline{x} = P_H(x)$. Since $\overline{x} \in P_H(0,t)$ and $P_H(0,t) \subseteq B_{\varepsilon(t)}(P_H(0))$ we get $\|\overline{x} - P_H(0)\| \leq \varepsilon(t)$. Therefore $\|x - P_H(0)\| \leq 2\varepsilon(t)$. This shows that $S(X,x^*,t) \subseteq S(X,x^*,0) + 2\varepsilon(t)B_X$.

We finally show that if $x, y \in S_X$ and $\left\|\frac{x+y}{2}\right\| > 1 - \varepsilon^{-1}(t)$, then $\|x - y\| < 6t$. Suppose $x, y \in S_X$ and $\left\|\frac{x+y}{2}\right\| > 1 - \varepsilon^{-1}(t)$. Choose $x_0^* \in S_{X^*}$ such that $x_0^*(\frac{x+y}{2}) = \left\|\frac{x+y}{2}\right\|$. Now either $x_0^*(x) > 1-\varepsilon^{-1}(t)$ or $x_0^*(y) > 1-\varepsilon^{-1}(t)$. Suppose $x_0^*(x) > 1 - \varepsilon^{-1}(t)$, then $x \in S\left(X, x_0^*, \varepsilon^{-1}(t)\right) \subseteq S(X, x_0^*, 0) + 2tB_X$.

Since $S(X,x_0^*,0) = \{\frac{x+y}{\|x+y\|}\}$, $\left\|x - \frac{x+y}{\|x+y\|}\right\| \leq 2t$ and therefore

$$\|(\|x+y\| - 2)x + (x-y)\| \leq 4t.$$

This implies that $\|x - y\| < 4t + 2\varepsilon^{-1}(t) \leq 6t$. We have shown that whenever $x, y \in S_X$ and $\|x-y\| \geq 6t$, we have $1 - \|\frac{x+y}{2}\| \geq \varepsilon^{-1}(t)$. Hence $\delta_X(6t) \geq \varepsilon^{-1}(t)$. That is $\delta_X(t) \geq \varepsilon^{-1}(\frac{t}{6})$. □

The following result is a consequence of Proposition 5.1 and Proposition 5.2.

Corollary 5.3. *Suppose that X is a uniformly convex Banach space such that $\delta_X(t) \geq Kt^p$ for some constant $K > 0$ and $2 \leq p < \infty$. Then there exists a constant $C > 0$ such that for every closed subspace M, $\varepsilon_M(t) \leq Ct^{\frac{1}{p}}$.*

Conversely, let $\varepsilon(t)$ be the function considered in the statement of Proposition 5.2. If there exists $q > 0$ such that $\varepsilon(t) \leq Ct^q$ for some constant $C > 0$ then X is uniformly convex and there exists a constant $K > 0$ such that $\delta_X(t) \geq Kt^{\frac{1}{q}}$.

Remark 5.4. Since the power type p of any uniformly convex space satisfies $2 \leq p < \infty$, from Corollary 5.3 we see that the power type q of $\varepsilon(t)$ satisfies $0 < q \leq \frac{1}{2}$. Note that for any closed subspace M of a Banach space X the best possible power type of $\varepsilon_M(t)$ is 1 and that occurs when M has the $1\frac{1}{2}-$ball property in X. It will be interesting to construct an example where $\varepsilon_M(t)$ has power type q for $q \in (\frac{1}{2}, 1)$. Also can we have a space X and a subspace M such that ε_M has power type q for $q \in (1, \infty)$? Note that such a space must have the $1\frac{1}{2}-$ball property. Finally, can there be a space X such that for every $q \in (0,\infty)$ there exists a subspace $M_q \subseteq X$ such that ε_{M_q} has power type q?

We end with the following corollary which shows that in every Banach space we have a hyperplane which does not have the $1\frac{1}{2}-$ball property. This also provides several examples of subspaces which are U-proximinal but do not have the $1\frac{1}{2}-$ball property.

Corollary 5.5. *Let X be a Banach space. Then there exists a closed hyperplane M of X such that M does not have the $1\frac{1}{2}-$ball property.*

Proof. If X is not reflexive then there exists a closed hyperplane M which is not proximinal. In this case M cannot have the $1\frac{1}{2}-$ball property. Suppose now X is reflexive and every closed hyperplane M in X has the $1\frac{1}{2}-$ball property. Let M be a hyperplane in X and $\varepsilon > 0$. Let $y \in P_M(x, \frac{\varepsilon}{2})$ for some $x \in X$. Then by Lemma 2.10, $d(x,M) + d(y, P_M(x)) = \|x - y\| \leq d(x,M) + \frac{\varepsilon}{2}$. Therefore $d(y, P_M(x)) \leq \frac{\varepsilon}{2}$ which shows that $P_M(x, \frac{\varepsilon}{2}) \subseteq B_\varepsilon(P_M(x)$ for all $x \in X$. It follows from the definition that $\varepsilon_M(t) \leq 2t$ for every hyperplane M of X. Hence by Proposition 5.2 and Corollary 5.3, X is uniformly convex with $\delta_X(t) \geq Ct$ for some constant $C > 0$ which is absurd as power type of modulus of uniform convexity is always greater or equal to 2. □

References

[1] A. K. Chakrabarty, P. Shunmugaraj, C. Zălinescu: Continuity properties for the subdifferential and ε-subdifferential of a convex function and its conjugate, J. Convex Analysis 14(3) (2007) 479–514.

[2] F. Deutsch, P. Kenderov: Continuous selections and approximate selection for set-valued mappings and applications to metric projections, SIAM J. Math. Anal. 14(1) (1983) 185–194.

[3] S. Dutta, D. Narayana: Strongly proximinal subspaces in Banach spaces, Contemp. Math., Amer. Math. Soc. 435 (2007) 143–152.

[4] S. Dutta, D. Narayana: Strongly proximinal subspaces of finite codimension in $C(K)$, Colloq. Math. 109(1) (2007) 119–128.

[5] S. Dutta, P. Shunmugaraj: Modulus of strong proximinality and continuity of metric projection, Set-Valued Var. Analysis 19(2) (2011) 271–281.

[6] S. Dutta, P. Shunmugaraj: Strong proximinality of closed convex sets, J. Approx. Theory 163(4) (2011) 547–553.

[7] V. Fonf, J. Lindenstrauss, L. Vesely: Best approximation in polyhedral Banach spaces, J. Approx. Theory 163(11) (2011) 1748–1771.

[8] C. Franchetti, R. Payá: Banach spaces with strongly subdifferentiable norm, Boll. Un. Mat. Ital. B 7(1) (1993) 45–70.

[9] G. Godefroy, V. Indumathi: Strong proximinality and polyhedral spaces, Rev. Mat. Complut. 14(1) (2001) 105–125.

[10] G. Godini: Best approximation and intersections of balls, in: Banach Space Theory and its Applications, Lecture Notes in Mathematics 991, Springer-Verlag, Berlin (1983) 44–54.

[11] P. Harmand, D. Werner, W. Werner: M-ideals in Banach Spaces and Banach Algebras, Lecture Notes in Mathematics 1547, Springer-Verlag, Berlin (1993).

[12] R. Holmes: A course on optimization and best approximation, Lecture Notes in Mathematics 257, Springer-Verlag, Berlin-New York (1972).

[13] R. Holmes: M-ideals in Approximation Theory. Approximation theory II, Academic Press, New York (1976).

[14] V. Indumathi: Metric projections of closed subspaces of c_0 onto subspaces of finite codimension, Colloq. Math. 99(2) (2004) 231–252.

[15] V. Indumathi: Metric projections and polyhedral spaces, Set-Valued Anal. 15(3) (2007) 239–250.

[16] B. Kripke, R. Holmes: Approximation of bounded functions by continuous functions, Bull. Amer. Math. Soc. 71 (1965) 896–897.

[17] K.-S. Lau: On a sufficient condition for proximity, Trans. Amer. Math. Soc. 251 (1979) 343–356.

[18] R. E. Megginson: An Introduction to Banach Space Theory, Springer-Verlag, New York (1998).

[19] J. B. Prolla, A. O. Chiacchio: Proximinality of certain subspaces of $C_b(S; E)$, Rocky Mountain J. Math. 19(1) (1989) 335–344.

[20] T. S. S. R. K. Rao: The one and half ball property in spaces of vector-valued functions, J. Convex Analysis 20(1) (2013) 13–23.

[21] D. Yost: Approximation by compact operators between $C(X)$ spaces, J. Approx. Theory 49(2) (1987) 99–109.