

Maximal Monotonicity of the Subdifferential of a Convex Function: a Direct Proof

Milen Ivanov*

*Faculty of Mathematics and Informatics, Sofia University,
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
milen@fmi.uni-sofia.bg*

Nadia Zlateva†

*Faculty of Mathematics and Informatics, Sofia University,
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
zlateva@fmi.uni-sofia.bg*

Received: September 9, 2016
Accepted: September 14, 2016

We provide a new proof for maximal monotonicity of the subdifferential of a convex function.

Keywords: Subdifferential, convex function, maximal monotonicity.

2010 Mathematics Subject Classification: 49J52, 47H05, 49J45.

1. Introduction

Let X be a Banach space with dual X^* . Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function.

Recall that for $x_0 \in \text{dom } f := \{x : f(x) \neq +\infty\}$ the *subdifferential* of f at x_0 in the sense of convex analysis is the set

$$\partial f(x_0) := \{p \in X^* : f(x) \geq f(x_0) + p(x - x_0), \forall x \in X\}, \quad (1)$$

while the ε -*subdifferential* of f at x_0 is the set

$$\partial_\varepsilon f(x_0) := \{p \in X^* : f(x) \geq f(x_0) + p(x - x_0) - \varepsilon, \forall x \in X\}.$$

If $x_0 \notin \text{dom } f$, then $\partial f(x_0) = \partial_\varepsilon f(x_0) = \emptyset$.

For $x_0 \in \text{dom } f$ the set $\partial f(x_0)$ might be empty but the set $\partial_\varepsilon f(x_0)$ is always non-empty for any $\varepsilon > 0$.

*Partially supported by Bulgarian National Scientific Fund under Grant DFNI-I02/10 and by Grant 239/2016 of the Science Fund of Sofia University.

†Partially supported by Bulgarian National Scientific Fund under Grant DFNI-I02/10.

It is clear that ∂f is a *monotone multivalued* mapping from X to X^* , in the sense that for all $x, y \in X$ and all $p \in \partial f(x), q \in \partial f(y)$ it holds

$$\langle q - p, y - x \rangle \geq 0,$$

which can be written also as

$$\langle \partial f(y) - \partial f(x), y - x \rangle \geq 0, \quad \forall x, y \in X.$$

It is a well-known classical result due to Rockafellar that ∂f is in fact *maximal* monotone (see Theorem 2.2). That is, ∂f can not be extended to strictly larger monotone mapping from X to X^* .

This statement goes back at least as far as [8], where Minty proves it for a continuous convex function on Hilbert space (see the discussion in [11]).

Moreau [5] gave a proof in Hilbert space using duality and Moreau-Yosida approximation.

A stumbling block to generalizing Minty's method is that for a lower semicontinuous convex function which is not continuous, ∂f might be empty at some points. Also controlling the norms of the subgradients appearing in the proof is not trivial (see the discussion in [12]). On the other hand, the method of Moreau relies heavily on the fact that Hilbert space is canonically isometric to its dual.

The method of reducing the considerations to a line, used by Minty in [8] and generalised by Rockafellar in [11], is prevalent in the subsequent proofs, like those of Taylor [15], Borwein [1], Thibault [16], the unpublished proof of Zagrodny, using [17], and the recent one of Jules and Lasonde [6]. A textbook following this line of proof is [9].

The first complete proof in Banach space: that of Rockafellar [12], is a methodological break through showing that Fenchel conjugate and duality can be used in non-reflexive case as well. The recent proofs of Marques Alves and Svaiter [7] and Simons [14] also use duality. It is mentioned in [18] that in many textbooks authors prefer proving only the reflexive case (where duality techniques are easier due to symmetry), e.g. [18, p. 278].

The famous proof of Simons [13] (see also [10]) shows how one can pick a subgradient with controlled norm.

Like some others, our proof starts with

$$\langle \partial f(x), x \rangle \geq 0, \quad \forall x \in \text{dom } \partial f.$$

which is a sufficient condition of minimality of Minty type (see [2]).

We prove it by adding a slack function to ensure that the sum is bounded below.

Finally, all tools we use had been well-known by 1970.

2. Proof of the main result

The following is proved in [6] for general lower semicontinuous function through mean value inequality. We give a simple proof for the convex case.

Proposition 2.1. *Let X be a Banach space and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. If $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfies*

$$\langle \partial f(x), x \rangle \geq 0, \quad \forall x \in \text{dom } \partial f \quad (2)$$

then $0 \in \partial f(0)$.

Proof. For $a > 0$ let $g_a(x) := a\|x\|^2$.

If $p \in \partial g_a(x)$ then by definition $p(0 - x) \leq g_a(0) - g_a(x)$, that is,

$$p(x) \geq a\|x\|^2. \quad (3)$$

Let

$$f_a(x) := f(x) + g_a(x). \quad (4)$$

Since g_a is continuous, the Sum Theorem, see for example [18], implies that $\text{dom } \partial f_a = \text{dom } \partial f$ and

$$\partial f_a(x) = \partial f(x) + \partial g_a(x), \quad \forall x \in \text{dom } \partial f. \quad (5)$$

From (2), (3) and (5) it follows that

$$\langle \partial f_a(x), x \rangle \geq a\|x\|^2, \quad \forall x \in \text{dom } \partial f.$$

Consequently,

$$\forall p \in \partial f_a(x) \Rightarrow \|p\| \geq a\|x\|. \quad (6)$$

On the other hand, f_a is bounded below for each $a > 0$. Indeed, take $x_0 \in \text{dom } f$. Since f is lower semicontinuous there is $\delta > 0$ such that $\inf f(x_0 + \delta B_X) > -\infty$, where B_X is the closed unit ball. Take $c \in \mathbb{R}$ such that $c < \inf f(x_0 + \delta B_X)$. By Hahn-Banach Theorem, see for example [4], we can separate the epigraph of f , that is the set $\{(x, t) : f(x) \leq t\}$, from the set $(x_0 + \delta B_X) \times (-\infty, c]$. So, there is $(p, r) \in X^* \times \mathbb{R} \setminus (0, 0)$ such that

$$p(x) + rt \geq p(y) + rs, \quad \forall x \in \text{dom } f, \forall t \geq f(x), \forall y \in x_0 + \delta B_X, \forall s \leq c.$$

Setting $x = x_0$ we see that $\sup\{rs : s \leq c\} < \infty$ which is only possible if $r \geq 0$. But if $r = 0$ then setting $x = x_0$ and $y = x_0 + h$ we see that $p(x_0) \geq p(x_0) + p(h)$ for all $h \in \delta B_X$ which implies $p = 0$, contradiction. Therefore, $r > 0$ and we can divide the above inequality by r and use $t = f(x)$ and $s = c$ to obtain for $q = p/r$ that

$$q(x) + f(x) \geq q(y) + c, \quad \forall x \in \text{dom } f, \forall y \in x_0 + \delta B_X.$$

Set $y = x_0$ and rearrange to get for $r = q(x_0) + c$

$$f(x) \geq -q(x) + r \geq -\|q\|\|x\| + r, \quad \forall x \in X.$$

Therefore, $f_a(x) \geq r + \|x\|(a\|x\| - \|q\|)$ and f_a is bounded below.

Let x_n be a minimising sequence for f_a . So, $f_a(x_n) < f_a(x_n) + \varepsilon_n$ for some $\varepsilon_n \rightarrow 0$. Equivalently, $0 \in \partial_{\varepsilon_n} f_a(x_n)$.

From Brøndsted-Rockafellar's Theorem, see [3], it follows that there are $y_n \in x_n + \sqrt{\varepsilon_n}B_X$ and $p_n \in \partial f_a(y_n)$ such that $\|p_n\| \leq \sqrt{\varepsilon_n}$. From this and (6) it follows that

$$\|y_n\| \leq \frac{\sqrt{\varepsilon_n}}{a}.$$

Therefore, $x_n \rightarrow 0$. Since f_a is lower semicontinuous, 0 is the global minimum of f_a . In other words,

$$f_a(x) \geq f_a(0) \iff f(x) \geq f(0) - a\|x\|^2, \quad \forall x \in X.$$

Since $a > 0$ was arbitrary, 0 is a global minimum of f , or, equivalently, $0 \in \partial f(0)$. $\square$

The Rockafellar's Theorem follows by an easy and well known argument (see for example [9], p. 59):

Theorem 2.2 (Rockafellar [12]). *Let X be a Banach space and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Then ∂f is a maximal monotone mapping from X to X^* .*

Proof. Let $(y, q) \in X \times X^*$ be in monotone relation to the graph of ∂f , that is

$$\langle \partial f(x) - q, x - y \rangle \geq 0, \quad \forall x \in \text{dom } \partial f. \quad (7)$$

Consider the function

$$\bar{f}(x) := f(x + y) - q(x).$$

It is immediate to check that (7) implies (2) for $\bar{f}$. By Proposition 2.1 we get $0 \in \partial \bar{f}(0)$ which easily translates to $q \in \partial f(y)$. Therefore, ∂f cannot be properly extended in a monotone way. $\square$

References

- [1] J. M. Borwein: A note on ε -subgradients and maximal monotonicity, *Pacific J. Math.* 103 (1982) 307–314.
- [2] M. W. Botsko: The teaching of mathematics: a first derivative test for functions of several variables, *Amer. Math. Monthly* 93 (1986) 558–561.
- [3] A. Brøndsted, R. T. Rockafellar: On the subdifferentiability of convex functions, *Proc. Amer. Math. Soc.* 16 (1965) 605–611.

- [4] R. B. Holmes: Geometric Functional Analysis and its Applications, Graduate Texts in Mathematics 24, Springer-Verlag, Berlin (1975).
- [5] J.-J. Moreau: Proximité et dualité dans un espace hilbertien, Bull. Soc. Math. France 93 (1965) 273–299.
- [6] F. Jules, M. Lassonde: Subdifferential test for optimality, J. Global Optim. 59 (2014) 101–106.
- [7] M. Marques Alves, B. F. Svaiter: A new proof for maximal monotonicity of subdifferential operators, J. Convex Analysis 15 (2008) 345–348.
- [8] G. J. Minty: On the monotonicity of the gradient of a convex function, Pacific J. Math. 14 (1964) 243–247.
- [9] R. R. Phelps: Convex functions, monotone operators and differentiability, Lecture Notes in Mathematics 1364, Springer-Verlag, Berlin (1989).
- [10] R. R. Phelps: Convex Functions, Monotone Operators and Differentiability, Lecture Notes in Mathematics 1364, second edition, Springer-Verlag, Berlin (1993).
- [11] R. T. Rockafellar: Characterization of the subdifferentials of convex functions, Pacific J. Math. 17 (1966) 497–510.
- [12] R. T. Rockafellar: On the maximal monotonicity of subdifferential mappings, Pacific J. Math. 33 (1970) 209–216.
- [13] S. Simons: The least slope of a convex function and the maximal monotonicity of its subdifferential, J. Optim. Theory Appl. 71 (1991) 127–136.
- [14] S. Simons: A new proof of the maximal monotonicity of subdifferentials, J. Convex Analysis 16 (2009) 165–168.
- [15] P. D. Taylor: Subgradients of a convex function obtained from a directional derivative, Pacific J. Math. 44 (1973) 739–747.
- [16] L. Thibault: Limiting convex subdifferential calculus with applications to integration and maximal monotonicity of subdifferential, in: Constructive, Experimental, and Nonlinear Analysis (Limoges, 1999), CMS Conf. Proc. 27, Amer. Math. Soc., Providence (2000) 279–289.
- [17] D. Zagrodny: Approximate mean value theorem for upper subderivatives, Nonlinear Analysis 12 (1988) 1413–1428.
- [18] C. Zălinescu: Convex analysis in general vector spaces, World Scientific, River Edge (2002).