Journal of Convex Analysis
Volume 24 (2017), No. 4, 1313–1339

Hypomonotonicity of the Normal Cone and Proximal Smoothness

Grigory M. Ivanov*

DCG, FSB, École Polytechnique de Lausanne,
Route Cantonale, 1015 Lausanne, Switzerland
grimivanov@gmail.com

Received: November 14, 2015
Revised manuscript received: October 1, 2016
Accepted: October 10, 2016

We study the properties of the normal cone to a proximally smooth set. We give a complete characterization of a proximally smooth set through the monotonicity properties of its normal cone in an arbitrary uniformly convex and uniformly smooth Banach space. We also give the exact bounds for the right-hand side in the monotonicity inequality for the normal cone in terms of the moduli of smoothness and convexity of a Banach space.

Keywords: Proximal smoothness, hypomonotonicity, variational inequality, Fréchet normal cone.

2010 Mathematics Subject Classification: 52A05, 52A40.

1. Introduction

Let X be a real Banach space. For a set $A \subset X$, by ∂A and $\operatorname{int} A$ we denote the boundary and the interior of A, respectively. We use $\langle p, x\rangle$ to denote the *value* of the functional $p \in X^*$ at vector $x \in X$. For $R > 0$ and $c \in X$ we denote by $\mathfrak{B}_R(c)$ the *closed ball with center c and radius R*, and by $\mathfrak{B}^*_R(p)$ we denote the *ball in the conjugate space.*

The *distance from a point $x \in X$ to a set $A \subset X$* is defined as

$$\rho(x, A) = \inf_{a\in A} \|x - a\|.$$

The *metric projection* of a point x onto a set A is defined as any element of the set

$$P_A(x) = \{a \in A : \|a - x\| = \rho(x, A)\}.$$

Let f and g be two non-negative functions, each one defined on a segment $[0, \varepsilon]$. We shall consider f and g as *equivalent at zero,* denoted by $f(t) \asymp g(t)$ as $t \to 0$,

*Research partially supported by the Swiss National Science Foundation, grants 200020-165977 and 20002. Supported also by the Russian Foundation for Basic Research, project 16-01-00259.

ISSN 0944-6532 / $ 2.50

if there exist positive constants a, b, c, d, e such that $af(bt) \leqslant g(t) \leqslant cf(dt)$ for $t \in [0, e]$.

Clarke, Stern and Wolenski [1] introduced and studied proximally smooth sets in a Hilbert space H in the following sense.

Definition 1.1. A set $A \subset X$ is called *proximally smooth* with constant R if the distance function $x \to \rho(x, A)$ is continuously differentiable on the set

$$U(R, A) = \{x \in X : 0 < \rho(x, A) < R\}.$$

We denote by $\Omega_{PS}(R)$ the set of all closed proximally smooth sets with constant R in X.

Properties of proximally smooth sets in a Banach space and the relation between such sets and similar classes of sets in a Banach space were investigated in [1]–[10]. Such sets are usually called weakly convex sets. Some notes about the history of these concepts can be found in [3] and in [8]. Probably, the most complete description of proximally smooth sets and their properties in a Hilbert space can be found in [5].

Definition 1.2. The *Fréchet normal cone* to a set $A \subset X$ at a point $a_0 \in A$ is defined as follows

$$N(a_0, A) = \Big\{p \in X^* \colon \forall \varepsilon > 0\ \exists \delta > 0\ \forall a \in A \cap \mathfrak{B}_\delta(a_0)\ \langle p, a - a_0\rangle \leq \varepsilon \|a - a_0\|\Big\}.$$

For brevity, in this paper we will refer to the Fréchet normal cone simply as the normal cone.

Remark 1.3. It is easily seen that the normal cone $N(a_0, A)$ does not change when the norm of the space X changes in an equivalent way.

The problem of characterizing proximally smooth sets in Hilbert spaces through the monotonicity properties of the normal cone turns out to be very important for applications. Thus, Theorem 1.9.1 of monograph [5] may be reformulated in the following way (see also [3], Cor. 2.2).

Theorem I. *Let A be a closed set in a Hilbert space H and $R > 0$. The following conditions are equivalent:*

(1) *the set A is proximally smooth with constant $R > 0$;*
(2) *for any vectors $x_1, x_2 \in A$, $p_1 \in N(x_1, A)$, $p_2 \in N(x_2, A)$ such that $\|p_1\| = \|p_2\| = 1$, the following inequality holds*

$$\langle p_2 - p_1, x_2 - x_1\rangle \geqslant -\frac{\|x_2 - x_1\|^2}{R}. \tag{1}$$

In a Hilbert space, equivalent properties with the normal cone were studied earlier in [1] and in [2]. Inequality (1) is often called the *variational inequality.*

So, Theorem I characterizes proximally smooth sets in terms of the variational inequality.

Afterwards, of course, attempts were made to extend this result to Banach spaces. Here we must note the works of Thibault and his colleagues, for example, [6], [7]. Usually, in these works other approaches to defining the normal cone are examined, and then the coincidence of different normal cones for proximally smooth sets in special Banach spaces (usually, uniformly smooth and uniformly convex) is proved.

Probably, the strongest results in this domain may be formulated in the following two theorems.

Theorem II. *Let X be a uniformly convex and uniformly smooth Banach space. Let $\rho_X(\tau) \asymp \tau^2$ as $\tau \to 0$. Then a proximally smooth set $A \subset X$ with constant $r > 0$ satisfies condition* (2) *of Theorem I for some (may be another) constant $R > 0$.*

Theorem III. *Let the convexity and smoothness moduli be of power order at zero in a Banach space X. Let $\delta_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$. If a set A satisfies condition* (2) *of Theorem I, then it is proximally smooth with some (may be another) constant $r > 0$.*

In the previous statements, we use $\rho_X(\cdot)$ and $\delta_X(\cdot)$ to denote modulus of smoothness and modulus of convexity of X, respectively (see definitions below).

In this paper, using the geometrical properties of the sets from the class $\Omega_{PS}(R)$, the connection between the smoothness of a set and the properties of its normal cone is studied.

2. Main results

In the sequel we shall need some additional notation.

We will use the notation xy for the *segment* with the (distinct) endpoints x and y, for the *line* passing through these points, as well as for the *vector* from x to y (the respective meaning will always be clear by the context).

Define the set $J_1(x) = \{p \in \partial\mathfrak{B}_1^*(o) : \langle p, x\rangle = \|x\|\}$ for $x \neq o$. Let

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} :\ x, y \in \mathfrak{B}_1(o),\ \|x-y\| \geq \varepsilon\right\}$$

and

$$\rho_X(\tau) = \sup\left\{\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1 :\ \|x\| = 1, \|y\| = \tau\right\}.$$

The functions $\delta_X(\cdot) : [0,2] \to [0,1]$ and $\rho_X(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ are referred to as the *moduli of convexity* and *smoothness* of X, respectively. A normed space X is called *uniformly convex*, if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A normed space X is said to be *uniformly smooth*, if $\lim\limits_{\tau \to +0} \frac{\rho_X(\tau)}{\tau} = 0$.

First we generalize condition (2) of Theorem I.

Definition 2.1. Let a function $\psi : [0, +\infty) \to [0, +\infty)$ be given. A set $A \subset X$ satisfies the *ψ-hypomonotonicity condition* of the normal cone with constant $R > 0$ if for some $\varepsilon > 0$ and for any $x_1, x_2 \in A$, $p_1 \in N(x_1, A)$, $p_2 \in N(x_2, A)$, $\|p_1\| = \|p_2\| = 1$ such that $\|x_1 - x_2\| \leqslant \varepsilon$, the inequality

$$\langle p_2 - p_1, x_2 - x_1 \rangle \geqslant -R\psi\left(\frac{\|x_2 - x_1\|}{R}\right) \tag{2}$$

holds. We denote by $\Omega_N^{\psi}(R)$ the class of all closed sets $A \subset X$ that satisfy the ψ-hypomonotonicity condition with constant $R > 0$.

Naturally, when studying the classes of sets that satisfy the ψ-hypomonotonicity condition, we suppose some restrictions on the function ψ.

Definition 2.2. We denote by $\mathfrak{M}$ the class of convex and Lipschitz functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$.

We say that a function $N : [0, +\infty) \to [0, +\infty)$ such that $N(0) = 0$, satisfies the *Figiel condition* if there exists a constant K such that the function $N(\cdot)$ on some interval $(0, \varepsilon)$ satisfies the condition

$$\frac{N(s)}{s^2} \leqslant K \frac{N(t)}{t^2} \quad \forall\, 0 < t \leqslant s < \varepsilon. \tag{3}$$

Remark 2.3. The modulus of smoothness of an arbitrary Banach space satisfies the Figiel condition (see[13], Proposition 1.e.5).)

Definition 2.4. We denote by $\mathfrak{M}_2$ the class of functions from $\mathfrak{M}$ that satisfy the Figiel condition.

It is easy to see that an arbitrary function $f(\cdot) \in \mathfrak{M}_2$ that is not identically zero satisfies the condition $t^2 = \mathrm{O}(f(t))$ as $t \to 0$.

Obviously, for the function $\psi(t) = t^2$ the assertion of Theorem I is equivalent to the equality $\Omega_N^{\psi}(R) = \Omega_{PS}(R)$ in the case of a Hilbert space H.

In the author's opinion, the condition $\psi \in \mathfrak{M}_2$ is rather natural. Moreover, it would be interesting to prove or contradict the following hypothesis.

Conjecture 2.5. *If in a uniformly convex and uniformly smooth Banach space X a set $A \subset X$ belongs to the class $\Omega_{PS}(R)$ and to the class $\Omega_N^{\psi}(r)$ for some function $\psi \in \mathfrak{M} \setminus \mathfrak{M}_2$ and constants $R > 0, r > 0$, then it belongs to the class $\Omega_N^{\psi_1}(r)$, where $\psi_1(t) = ct^2$ for some $c \geqslant 0$.*

The following two theorems provide a complete description of the connection between the classes $\Omega_N^{\psi}(R)$ and $\Omega_{PS}(R)$.

Theorem 2.6. *In a uniformly convex and uniformly smooth Banach space X the following statements are equivalent for the function $\psi \in \mathfrak{M}$:*
(1) *there exists $k_1 > 0$ such that $\Omega_{PS}(R) \subset \Omega_N^{k_1\psi}(R)$ for any $R > 0$;*
(2) *$\rho_X(\tau) = \mathrm{O}(\psi(\tau))$ as $\tau \to 0$.*

Theorem 2.7. *In a uniformly convex and uniformly smooth Banach space X the following statements are equivalent for the function $\psi \in \mathfrak{M}_2$:*
(1) *there exists $k_2 > 0$ such that $\Omega_N^{k_2\psi}(R) \subset \Omega_{PS}(R)$ for any $R > 0$;*
(2) *$\psi(\varepsilon) = \mathrm{O}(\delta_X(\varepsilon))$ as $\varepsilon \to 0$.*

Clearly, $A \in \Omega_{PS}(1)$ iff $RA \in \Omega_{PS}(R)$ for some $R > 0$. A similar property holds for the class of ψ-hypomonotonous sets. Therefore, Theorems 2.6, 2.7 can be proved assuming that R is a fixed positive number. Using Theorems 2.7, 2.6 we will show that the following statement, which gives a complete answer to the question about the possibility of the inclusion $\Omega_N^{k_1\psi}(R) \subset \Omega_{PS}(R) \subset \Omega_N^{k_2\psi}(R)$, holds.

Theorem 2.8. *If in a uniformly smooth and uniformly convex Banach space X for some function $\psi \in \mathfrak{M}$ there exist $k_1 > 0$, $k_2 > 0$ such that the inclusions $\Omega_N^{k_1\psi}(R) \subset \Omega_{PS}(R) \subset \Omega_N^{k_2\psi}(R)$ hold, then $\delta_X(\varepsilon) \asymp \rho_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$, and, therefore, the space X is isomorphic to a Hilbert space.*

Conjecture 2.9. *The equality $\Omega_{PS}(R) = \Omega_N^{\psi}(R)$ holds only in a Hilbert space provided that $\psi(t) = t^2$.*

3. Preliminaries

Geometrical properties of the class $\Omega_{PS}(R)$ are hidden in its definition. To clarify the geometrical properties of $\Omega_{PS}(R)$, which are very useful in this paper, we introduce two equivalent (in certain spaces!) definitions of the class $\Omega_{PS}(R)$.

Definition 3.1. We say that a closed set A in a Banach space X satisfies the *P-supporting condition of weak convexity with constant* $R > 0$ if $u \in U(R, A)$ and $x \in P_A(u)$ imply the inequality

$$\rho\left(x + \frac{R}{\|u - x\|}(u - x), A\right) \geqslant R.$$

We denote by $\Omega_P(R)$ the set of all such sets.

Definition 3.2. We say that a closed set A in a Banach space X satisfies the *N-supporting condition of weak convexity with constant* $R > 0$ if $x \in A$, $p \in N(x, A) \cap \partial\mathfrak{B}_1^*(o)$, and $u \in \partial\mathfrak{B}_1(o)$ such that $p \in J_1(u)$, imply the inequality $\rho(x + Ru, A) \geqslant R$. The set of all such sets we denote by $\Omega_N(R)$.

The equality $\Omega_{PS}(R) = \Omega_P(R)$ was proved in [10] in case of a uniformly convex and uniformly smooth Banach space. The equality $\Omega_P(R) = \Omega_N(R)$ was proved in [9] in case of a uniformly convex Banach space. So we can formulate the following theorem.

Theorem IV. *Let X be a uniformly smooth and uniformly convex Banach space. Then* $\Omega_{PS}(R) = \Omega_P(R) = \Omega_N(R)$.

Further we will use the class $\Omega_N(R)$ instead of $\Omega_{PS}(R)$; as mentioned above, these classes coincide under the conditions of Theorems 2.6, 2.7, and 2.8.

Theorem 3.3. *Let X be a reflexive Banach space, $R > 0$, $A \in \Omega_N(R)$, $a_0 \in A$, $p \in N(a_0, A) \cap \partial\mathfrak{B}_1^*(o)$, $\delta > 0$, $\varepsilon = \frac{2R}{\delta}\rho_X\left(\frac{\delta}{R}\right)$. Then*

$$\langle p, a - a_0\rangle \le \varepsilon \, \|a - a_0\| \quad \forall a \in A \cap \mathfrak{B}_\delta(a_0).$$

Proof. Fix a vector $a \in A \cap \mathfrak{B}_\delta(a_0)$. By reflexivity of X, there exists a vector $u \in \partial\mathfrak{B}_1(o)$ such that $p \in J_1(u)$. Since $A \in \Omega_N(R)$, we have that $\rho(a_0 + Ru, A) \geqslant R$. Therefore, $\|a_0 + Ru - a\| \geqslant R$. Denote $w = \frac{a - a_0}{R}$. Then $\|u - w\| \geqslant 1$. By the definition of the modulus of smoothness we get that $2\rho_X(\|w\|) \geqslant \|u + w\| + \|u - w\| - 2$. Consequently,

$$2\rho_X(\|w\|) \geqslant \|u + w\| - 1 \geqslant \langle p, u + w\rangle - 1 = \langle p, w\rangle.$$

And now we have

$$\langle p, a - a_0\rangle = R\langle p, w\rangle \leqslant 2R\rho_X(\|w\|) = 2R\rho_X\left(\frac{\|a - a_0\|}{R}\right).$$

Since the function $\tau \to \frac{\rho_X(\tau)}{\tau}$ is increasing and $\|a - a_0\| \leqslant \delta$, we get

$$\rho_X\left(\frac{\|a - a_0\|}{R}\right) \leqslant \frac{\|a - a_0\|}{\delta}\rho_X\left(\frac{\delta}{R}\right).$$

Consequently, $\langle p, a - a_0\rangle \le \varepsilon \, \|a - a_0\|$. $\square$

We need to formulate some properties of the moduli of smoothness and convexity. It is well known ([21] Ch.3, §4, Lemma 1) that in a Banach space the following inequalities hold

$$2 \leqslant \limsup_{\tau \to 0} \frac{\rho_X(2\tau)}{\rho_X(\tau)} \leqslant 4. \tag{4}$$

The following two lemmas are technical and obvious. The first one can be found in [22].

Lemma 3.4. *Let $x, y \in X$, $x \neq 0$, $p \in J_1(x)$. Then*

$$\|x + y\| \leqslant \|x\| + \langle p, y\rangle + 2\,\|x\|\,\rho_X\left(\frac{\|y\|}{\|x\|}\right). \tag{5}$$

Lemma 3.5. *Let X be a uniformly convex Banach space, $x_0 \in \partial\mathfrak{B}_R(o)$ and $p_0 \in \partial\mathfrak{B}_1^*(o)$ be a functional dual to the vector $-x_0$. Then for any vector $z \in \operatorname{int}\mathfrak{B}_R(o)$ we have*

$$\langle p_0, z - x_0\rangle > 2R\delta_X\left(\frac{\|z - x_0\|}{R}\right). \tag{6}$$

The proof of the next proposition can be found in [9].

Theorem V. *Assume $A \subset X$, $x_1 \in X \setminus A$, $x_0 \in P_A(x_1)$, and that the norm is Fréchet differentiable at the point $x_1 - x_0$. Then $J_1(x_1 - x_0) \subset N(x_0, A)$.*

We will denote by $\mathfrak{B}_1^p(o)$ the intersection of the unit ball and the hyperplane

$$H_p = \{x \in X|\ \langle p, x\rangle = 0\},$$

i.e. the set $\mathfrak{B}_1(o) \cap H_p$. Let $\mathfrak{B}_1^p(a) = \mathfrak{B}_1^p(o) + oa$.

Lemma 3.6. *Let $x \in \partial\mathfrak{B}_1(o)$, $p \in J_1(ox)$, $z \in \mathfrak{B}_1^p(x)$. Take a point y such that the segment yz is parallel to ox and intersects the unit sphere at the unique point y. Then $2\|zx\| \geqslant \|xy\|$.*

Proof. By the triangle inequality, it suffices to show that $\|zx\| \geqslant \|zy\|$. By d denote a point on unit sphere such that vector od is collinear to the vector xz. Let line l be parallel to ox and $d \in l$. By constructions, we have that points x, y, z, o, d and line l lies at the same plane – linear span of the vectors ox and xz. So lines l and xz intersect, by c we denote their intersection point. Note that $odcx$ is a parallelogram and $\|dc\| = 1$; the segment dx belongs to the unit ball and does not intersect the interior of the segment zy. Let $y' = zy \cap dx$. By similarity, we have

$$\|zy\| \leqslant \|zy'\| = \frac{\|xz\|}{\|xc\|}\|dc\| = \|xz\|. \qquad \square$$

It is worth noticing, that in the conditions of Lemma 3.6 we have that z is a projection along vector ox of the point y on some supporting hyperplane to the unit ball at x. Moreover, z belongs to the metric projection of the point y on this hyperplane. That is to say, Lemma 3.6 shows us that if one projects along vector ox segment xy on the hyperplane, which is supporting to the unit ball at point x, then the length of the segment decrease no more than a factor of 2.

In this paper we will use the notion of the moduli of supporting convexity and smoothness, which were introduced and studied in [22]. This technique is very convenient in questions concerning the geometry of the unit ball and its supporting hyperplanes. Let us introduce the following definitions.

We say that y is *quasiorthogonal* to a vector $x \in X \setminus \{o\}$, and write $y\urcorner x$, if there exists a functional $p \in J_1(x)$ such that $\langle p, y\rangle = 0$. Note that the following conditions are equivalent:

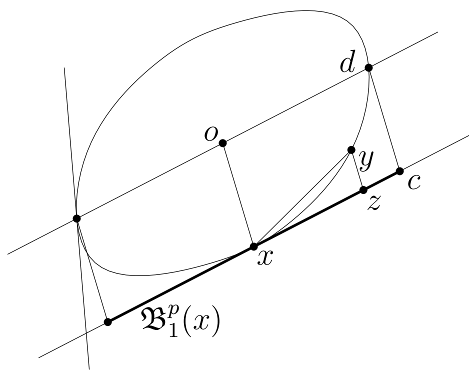

Figure 3.1: Under the conditions of Lemma 3.6 we have $2\|zx\| \geqslant \|xy\|$.

(i) y is quasiorthogonal to x;
(ii) for any $\lambda \in \mathbb{R}$ the vector $x + \lambda y$ lies in the supporting hyperplane to the ball $\mathfrak{B}_{\|x\|}(o)$ at x;
(iii) for any $\lambda \in \mathbb{R}$ the following inequality holds: $\|x + \lambda y\| \geqslant \|x\|$;
(iv) x is orthogonal to y in Birkhoff–James' sense ([21], Ch. 2, §1).

Let $x, y \in \partial\mathfrak{B}_1(o)$ be such that $y \urcorner x$. By definition, put

$$\lambda_X(x, y, r) = \min\{\lambda \in \mathbb{R} : \|x + ry - \lambda x\| = 1\}$$

for any $r \in [0, 1]$. Define the *modulus of supporting convexity* as

$$\lambda_X^-(r) = \inf\{\lambda_X(x, y, r) : \|x\| = \|y\| = 1,\ y \urcorner x\}.$$

Define the *modulus of supporting smoothness* as

$$\lambda_X^+(r) = \sup\{\lambda_X(x, y, r) : \|x\| = \|y\| = 1,\ y \urcorner x\}.$$

In [22] the inequalities (7) – (9) are proved for any $r \in [0, 1]$:

$$\rho_X\left(\frac{r}{2}\right) \leqslant \lambda_X^+(r) \leqslant \rho_X(2r). \tag{7}$$

The last inequality was proved for $r \in [0, \frac{1}{2}]$.

$$\lambda_X^-(r) \leqslant \delta_X(2r) \leqslant 1 - \sqrt{1 - r^2}. \tag{8}$$

$$0 \leqslant \lambda_X^-(r) \leqslant \lambda_X^+(r) \leqslant r. \tag{9}$$

4. Proof of Theorem 2.6

For a set $A \subset X$ we introduce the following function

$$\Gamma(A, \varepsilon, X) = -\inf\langle p_1 - p_2, x_1 - x_2\rangle,$$

where the infimum is considered for all x_1, x_2 in X and p_1, p_2 in X^* that satisfy the following condition

$$x_i \in A,\ p_i \in N(A, x_i) \cap \partial \mathfrak{B}_1^*(o),\ i = 1, 2, \tag{10}$$

and such that $\|x_1 - x_2\| = \varepsilon$.

The function $-\Gamma(A, \varepsilon, X)$ is the maximum value that the right hand side can take in the hypomonotonicity inequality (2), meaning that if $A \in \Omega_N^{\psi}(R)$, then

$$-\Gamma(A, \varepsilon, X) \geqslant -R\psi\left(\frac{\varepsilon}{R}\right). \tag{11}$$

Lemma 4.1. *Let X be a Banach space with a Fréchet differentiable norm. Let $A = X \setminus \operatorname{int} \mathfrak{B}_1(o)$. Then*

$$\Gamma(A, \varepsilon, X) \asymp \lambda_X^+(\varepsilon) \asymp \rho_X(\varepsilon), \quad \varepsilon \to 0. \tag{12}$$

Moreover, if $\varepsilon \in [0, \frac{1}{2}]$ then the inequality

$$\Gamma(A, \varepsilon, X) \geqslant \rho_X\left(\frac{\varepsilon}{4}\right) \tag{13}$$

holds.

Proof. (1) The equivalence of $\lambda_X^+(\varepsilon) \asymp \rho_X(\varepsilon)$ as $\varepsilon \to 0$ is the consequence of (7).
(2) Fix any $\varepsilon \in (0, \frac{1}{2}]$. As $\partial A = \partial \mathfrak{B}_1(o)$, we have for any $x \in \partial A$

$$N(A, x) \cap \partial \mathfrak{B}_1^*(o) = -J_1(x). \tag{14}$$

Fix any x_1, x_2, p_1, p_2, satisfying the inclusions (10) and $\|x_1 - x_2\| \leqslant \varepsilon \leqslant \frac{1}{2}$. Clearly, we can assume that x_1 and x_2 are not collinear. Let ℓ_1 be a line in the plane $\operatorname{Lin}\{x_1, x_2\}$, supporting to the ball $\mathfrak{B}_1(o)$ at the point x_1 (its uniqueness follows from the differentiability of the norm). Let z be the projection of point x_2 on this line. Then $-\langle p_1, x_1 - x_2 \rangle = \|z - x_2\|$. From the triangle inequality it follows that $\|z - x_1\| \leqslant \|z - x_2\| + \|x_2 - x_1\| \leqslant 2\|x_2 - x_1\|$. Therefore,

$$\|z - x_2\| \leqslant \lambda_X^+(\|x_1 - z\|) \leqslant \lambda_X^+(2\|x_1 - x_2\|).$$

From the last inequality in this series we get $\langle p_1, x_1 - x_2 \rangle \geqslant -\lambda_X^+(2\|x_1 - x_2\|)$. Similarly, $\langle p_2, x_2 - x_1 \rangle \geqslant -\lambda_X^+(2\|x_1 - x_2\|)$. Summing the last two inequalities, we obtain

$$\langle p_1 - p_2, x_1 - x_2 \rangle \geqslant -2\lambda_X^+(2\|x_1 - x_2\|),$$

and, so, for any $\varepsilon \in (0, \frac{1}{2}]$ we have

$$\Gamma(A, \varepsilon, X) \leqslant 2\lambda_X^+(2\varepsilon). \tag{15}$$

The equality (14) implies that for any x_1, x_2, p_1, p_2, that satisfy the condition (10) the inequality

$$\langle p_1 - p_2, x_1 - x_2\rangle \leqslant \langle p_1, x_1 - x_2\rangle \tag{16}$$

holds. According to the definition of the convexity modulus, for any $\gamma > 0$ there exist $x_1, x_2 \in \partial A$, $p_1 \in N(x_1, A_1)$, and $z \in H_{p_1} = \{x \in X|\ \langle p_1, x\rangle = 1\}$ such that

(i) z is the projection of x_2 on the hyperplane H_{p_1};

(ii) $\|x_1 - z\| = \frac{\varepsilon}{2}$;

(iii) $\|z - x_2\| \geqslant \lambda_X^+(\|x_1 - z\|) - \gamma = \lambda_X^+\left(\frac{\varepsilon}{2}\right) - \gamma$.

This implies that

$$-\langle p_1, x_1 - x_2\rangle = \|z - x_2\| \geqslant \lambda_X^+\left(\frac{\varepsilon}{2}\right) - \gamma\,. \tag{17}$$

On a unit sphere in the plane x_1ox_2 we fix a point x_ε such that $\|x_1 - x_\varepsilon\| = \varepsilon$ and in the plane x_1ox_2 the triangle $x_1x_2x_\varepsilon$ lies on one side of line ox_1. Lemma 3.6 implies that $\varepsilon = 2\|x_1 - z\| \geqslant \|x_1 - x_2\|$. Using the inequality (16), we obtain

$$\Gamma(A, \varepsilon, X) \geqslant -\langle p_1, x_1 - x_\varepsilon\rangle \geqslant -\langle p_1, x_1 - x_2\rangle.$$

This and inequality (17) imply that

$$\Gamma(A, \varepsilon, X) \geqslant \lambda_X^+\left(\frac{\varepsilon}{2}\right) - \gamma.$$

Passing to the limit in the last inequality as $\gamma \to 0$, we obtain that

$$\Gamma(A, \varepsilon, X) \geqslant \lambda_X^+\left(\frac{\varepsilon}{2}\right). \tag{18}$$

The inequalities (15) and (18) imply that

$$2\lambda_X^+(2\varepsilon) \geqslant \Gamma(A, \varepsilon, X) \geqslant \lambda_X^+\left(\frac{\varepsilon}{2}\right) \tag{19}$$

Using this and (7) we obtain inequality (13). □

Proof (2)⇒(1) in Theorem 2.6

Fix a set $A \in \Omega_N(R)$. Theorem 3.3 implies that for any $x_1, x_2 \in A$ and $p_i \in N(x_i, A)$, $i = 1, 2$, the inequality

$$\langle p_1 - p_2, x_1 - x_2\rangle \geqslant -4R\rho_X\left(\frac{\|x_1 - x_2\|}{R}\right) \tag{20}$$

holds. Condition 2 implies that for some $\varepsilon > 0$ exists a positive constant k such that for any $\tau \in [0, \varepsilon]$ the inequality

$$\rho_X(\tau) \leqslant k\psi(\tau) \tag{21}$$

holds. The inequalities (21), (20) imply that $A \in \Omega_N^{k_1\psi}(R)$, where $k_1 = 4k$. $\square$

Proof (1)⇒(2) in Theorem 2.6

Let the constant k_1 be such that $\Omega_N(1) \subset \Omega_N^{k_1\psi}(1)$. Let us show that assertion 2 holds. Consider the set $A_1 = \{x \in X|\ \|x\| \geqslant 1\}$. Clearly, $A_1 \in \Omega_P(1)$, and by Theorem IV the inclusion $A_1 \in \Omega_N(1)$ holds. From the definition of $\Gamma(A, \varepsilon, X)$, inequalities (13), (11) and the relations (4) we obtain for some constant $\gamma > 0$ the following series of inequalities

$$k_1\psi(\varepsilon) \geqslant \Gamma(A, \varepsilon, X) \geqslant \rho_X\left(\frac{\varepsilon}{4}\right) \geqslant \frac{1}{17}\rho_X(\varepsilon),$$

where $\varepsilon \in [0, \gamma]$. $\square$

Particularly, we have shown the following statement.

Corollary 4.2. *In a uniformly smooth and uniformly convex Banach space X the inclusion $(X \setminus \operatorname{int} \mathfrak{B}_1(o)) \in \Omega_N^{\frac{1}{17}\rho_X(\cdot)}(1)$ holds.*

Remark 4.3. The implication $2 \Rightarrow 1$ in Theorem 2.6 is an elementary corollary of the smoothness modulus definition (see Theorem 3.3). But to prove the reverse implication we had to show the equivalence of smoothness and supporting smoothness moduli at zero.

5. Proof (2)⇒(1) in Theorem 2.7

Define the function $\omega_X : [0, +\infty) \to [0, +\infty)$ in the following way

$$\omega_X(\tau) = \frac{\rho_X(\tau)}{\tau}.$$

As $\omega_X(\cdot)$ is an increasing monotone function, the inverse function $\omega_X^{-1}(\cdot)$ is also increasing. Note that $\omega_X(1) = \rho_X(1) \geqslant \rho_H(1) = \sqrt{2} - 1$, and, therefore, for any $t \in [0, 1]$ the inequality

$$\omega_X^{-1}\Big((\sqrt{2} - 1)t\Big) \leqslant 1, \tag{22}$$

holds.

The next Lemma is a generalization of Lemma 1.9.1 from G. E. Ivanov's monograph [5], in which an analogous result was obtained for Hilbert spaces.

For a set $A \subset X$ denote by $T(A)$ the set of points $x \in X$ such that $P_A(x)$ is a single point. It is well known, that in a uniformly convex Banach space X for any closed set A the set $T(A)$ is dense in X (see [23]).

Lemma 5.1. *Let X be a uniformly smooth Banach space, A be a closed set such that the set $T(A)$ is dense in X. Let there be given $z_0 \notin A$, $z_1 \in A$ and*

$\varepsilon \in (0, 1)$. *Then there exist* $\lambda \in [0, 1]$, *a point* $y \in \partial A$ *and* $p \in N(y, A)$, $\|p\| = 1$ *such that*

$$\|(1-\lambda)z_0 + \lambda z_1 - y\| < \varepsilon \|z_1 - z_0\|, \tag{23}$$

$$\langle p, z_1 - z_0\rangle < \varepsilon \|z_1 - z_0\|. \tag{24}$$

Proof. Define $\delta = \frac{1}{2}\min\{\varepsilon \|z_1 - z_0\|, \rho(z_0, A)\}$. As the set A is closed and $z_0 \notin A$, we have that $\rho(z_0, A) > 0$, and, therefore $\delta > 0$. Since $2\delta \leqslant \rho(z_0, A)$, we obtain that $\operatorname{int}\mathfrak{B}_{2\delta}(z_0) \subset X \setminus A$.

Define the number λ as follows

$$\lambda = \inf\left\{t > 0 : \mathfrak{B}_\delta(z_0 + t(z_1 - z_0)) \bigcap A \neq \emptyset\right\}.$$

As $z_1 \in A$, we have that $\mathfrak{B}_\delta(z_1) \cap A \neq \emptyset$, and, therefore, $\lambda \leqslant 1$. Note that for $|t| < \frac{\delta}{\|z_0 - z_1\|}$ the inclusions

$$\mathfrak{B}_\delta(z_0 + t(z_1 - z_0)) \subset \operatorname{int}\mathfrak{B}_{2\delta}(z_0) \subset X \setminus A,$$

hold and, therefore, $\mathfrak{B}_\delta(z_0 + t(z_1 - z_0)) \cap A = \emptyset$ as $|t| < \frac{\delta}{\|z_0 - z_1\|}$. This and the definition of λ imply that

$$\lambda \geqslant \frac{\delta}{\|z_0 - z_1\|}, \tag{25}$$

$$\mathfrak{B}_\delta(z_0 + t(z_1 - z_0)) \cap A = \emptyset, \quad \forall t \in \left(-\frac{\delta}{\|z_0 - z_1\|}, \lambda\right). \tag{26}$$

Define the vector $z_\lambda = (1-\lambda)z_0 + \lambda z_1$. Since X is a uniformly smooth Banach space, there exists $\xi > 0$ such that

$$(\sqrt{2} + 1)\frac{\xi}{\rho_X^{-1}(\xi)} < \varepsilon \tag{27}$$

Define the number $\beta = \frac{\sqrt{2}-1}{2}\delta\min\{\xi, \frac{1}{2}\}$.

As the set $T(A)$ is dense in X, there exists $y_0 \in T(A)$ such that the inequality $\|z_\lambda - y_0\| < \beta$ holds. Let $\{y\} = P_A(y_0)$. Obviously,

$$|\rho(y_0, A) - \rho(z_\lambda, A)| \leqslant \|y_0 - z_\lambda\| < \beta.$$

By the definition of λ we have $\rho(z_\lambda, A) = \delta$. Therefore,

$$\delta - \beta < \|y_0 - y\| = \rho(y_0, A) < \delta + \beta. \tag{28}$$

From this and the inequalities $\|y_0 - z_\lambda\| < \beta$, $\beta \leqslant \frac{\delta}{2}$, $\delta \leqslant \frac{\varepsilon\|z_1 - z_0\|}{2}$ we get

$$\|z_\lambda - y\| \leqslant \|y_0 - y\| + \|y_0 - z_\lambda\| < \delta + 2\beta \leqslant 2\delta \leqslant \varepsilon \|z_1 - z_0\|.$$

Thus, inequality (23) holds.

Let the functional $p \in \partial\mathfrak{B}_1^*(o)$ be dual to the vector $y_0 - y$. Theorem V implies that $p \in N(y,\, A)$.

If $\langle p, z_1 - z_0\rangle \leqslant 0$, then inequality (24) holds. So, let us consider the case $\langle p, z_1 - z_0\rangle > 0$. Let $\gamma = \frac{\langle p, z_1 - z_0\rangle}{\|z_1 - z_0\|} \in (0, 1]$. We have to prove that $\gamma < \varepsilon$. Define the number

$$\mu = \omega_X^{-1}\Big((\sqrt{2}-1)\gamma\Big)\,\frac{\|y - y_0\|}{\|z_1 - z_0\|}. \tag{29}$$

Applying inequalities (22), (25) and (28) to (29) we obtain

$$0 < \mu \leqslant \frac{\|y - y_0\|}{\|z_1 - z_0\|} \leqslant \frac{\delta + \beta}{\|z_1 - z_0\|} < \frac{2\delta}{\|z_1 - z_0\|} \leqslant \frac{\delta}{\|z_1 - z_0\|} + \lambda\,.$$

Thus, $-\frac{\delta}{\|z_1 - z_0\|} < \lambda - \mu < \lambda$. This and equality (26) imply that

$$\mathfrak{B}_\delta(z_0 + (\lambda - \mu)(z_1 - z_0)) \cap A = \emptyset\,.$$

As $y \in A$, we have that $y \notin \mathfrak{B}_\delta(z_0 + (\lambda - \mu)(z_1 - z_0)) = \mathfrak{B}_\delta(z_\lambda - \mu(z_1 - z_0))$, and so $\|z_\lambda - y - \mu(z_1 - z_0)\| > \delta$. This and the inequality $\|y_0 - z_\lambda\| < \beta$ imply that $\|y_0 - y - \mu(z_1 - z_0)\| > \delta - \beta$. Applying Lemma 3.4, we obtain that

$$\frac{\delta - \beta}{\|y_0 - y\|} < 1 - \mu\frac{\langle p, z_1 - z_0\rangle}{\|y_0 - y\|} + 2\rho_X\left(\mu\frac{\|z_1 - z_0\|}{\|y_0 - y\|}\right).$$

Thus, considering inequalities (29), (28) we get

$$\frac{\delta - \beta}{\delta + \beta} < 1 - \gamma\omega_X^{-1}\Big((\sqrt{2}-1)\gamma\Big) + 2\rho_X\left(\omega_X^{-1}\Big((\sqrt{2}-1)\gamma\Big)\right).$$

Note that for any $k > 0$ the equalities

$$\gamma\omega_X^{-1}(k\gamma) = \frac{\omega_X^{-1}(k\gamma)\cdot\omega_X\left(\omega_X^{-1}(k\gamma)\right)}{k} = \frac{\rho_X\left(\omega_X^{-1}(k\gamma)\right)}{k}$$

hold, and, therefore, we get

$$\frac{\delta - \beta}{\delta + \beta} < 1 - \left(\frac{1}{\sqrt{2}-1} - 2\right)\rho_X\left(\omega_X^{-1}\Big((\sqrt{2}-1)\gamma\Big)\right).$$

Modifying the last inequality and using the definition of β, we obtain

$$\rho_X\left(\omega_X^{-1}\Big((\sqrt{2}-1)\gamma\Big)\right) < \frac{2(\sqrt{2}-1)}{(3-2\sqrt{2})}\frac{\beta}{(\delta+\beta)} = \frac{2}{(\sqrt{2}-1)}\frac{\beta}{(\delta+\beta)} \leqslant \xi,$$

and thus, considering inequality (27), we finally get

$$\gamma < \frac{1}{\sqrt{2}-1}\omega_X\left(\rho_X^{-1}(\xi)\right) = \frac{1}{\sqrt{2}-1}\frac{\rho_X\left(\rho_X^{-1}(\xi)\right)}{\rho_X^{-1}(\xi)} = (\sqrt{2}+1)\frac{\xi}{\rho_X^{-1}(\xi)} < \varepsilon. \qquad \square$$

Proof (2)⇒(1) in Theorem 2.7. As the functions $\delta_X(\cdot)$ and $\psi(\cdot)$ are non-negative and monotone, condition (2) of Theorem 2.7 implies that there exists a number $k > 0$ such that

$$R \cdot k\psi\left(\frac{t}{R}\right) \leqslant 2R \cdot \delta_X\left(\frac{t}{R}\right) \quad \forall t \in [0, 2R]. \tag{30}$$

Let us prove that $\Omega_N^{k\psi}(R) \subset \Omega_N(R)$.

Suppose the contrary. Then there exists a closed set A such that $A \in \Omega_N^{k\psi}(R)$ and $A \notin \Omega_N(R)$. In consequence, there exist a vector $x_0 \in A$, a functional $p_0 \in N(x_0, A) \cap \partial\mathfrak{B}_1^*(o)$, the unit vector u_0 dual to x_0, and also a vector $z_1 \in A$ such that the inequality

$$\|x_0 + Ru_0 - z_1\| < R \tag{31}$$

holds. According to Lemma 3.5, we obtain that

$$\langle p_0, z_1 - x_0\rangle > 2R\delta_X\left(\frac{\|z_1 - x_0\|}{R}\right) > 0. \tag{32}$$

Therefore, due to the inclusion $p_0 \in N(x_0, A)$ and according to Definition 1.2, there exists a number $\mu \in (0, 1)$ such that the vector

$$z_0 = x_0 + \mu(z_1 - x_0) \tag{33}$$

satisfies the condition $z_0 \notin A$.

Let L be the Lipschitz constant of function $\psi(\cdot)$. Denote

$$\varepsilon = \frac{\mu}{(6 + 2kL)R}\left(\langle p_0, z_1 - x_0\rangle - 2R\delta_X\left(\frac{\|z_1 - x_0\|}{R}\right)\right). \tag{34}$$

Inequality (32) implies that $\varepsilon > 0$.

According to Lemma 5.1 there exists a number $\lambda \in [0, 1]$, a vector $y \in \partial A$ and a unit functional $p \in N(y, A)$ such that

$$\|z_\lambda - y\| < \varepsilon \|z_1 - z_0\|, \tag{35}$$

$$\langle p, z_1 - z_0\rangle < \varepsilon \|z_1 - z_0\|, \tag{36}$$

where

$$z_\lambda = (1 - \lambda)z_0 + \lambda z_1. \tag{37}$$

The definition of vector z_0 (equality (33)) implies

$$z_1 - z_0 = (1 - \mu)(z_1 - x_0), \quad z_0 - x_0 = \mu(z_1 - x_0).$$

From this and equality (37) we get

$$z_\lambda - x_0 = z_\lambda - z_0 + z_0 - x_0 = \lambda(z_1 - z_0) + z_0 - x_0 = (\lambda(1-\mu)+\mu)(z_1 - x_0). \quad (38)$$

Therefore

$$\frac{z_\lambda - x_0}{\|z_\lambda - x_0\|} = \frac{z_1 - x_0}{\|z_1 - x_0\|} = \frac{z_1 - z_0}{\|z_1 - z_0\|} \,.$$

This and inequality (36) imply that

$$\langle p, z_\lambda - x_0 \rangle < \varepsilon \|z_\lambda - x_0\| \leqslant \varepsilon \|z_1 - x_0\| \,.$$

Thus, according to inequality (35), we have that $\langle p, y - x_0 \rangle \leqslant 2\varepsilon \|z_1 - x_0\|$. Inequality (31) implies that $\|z_1 - x_0\| \leqslant 2R$, and, therefore,

$$\langle p, y - x_0 \rangle \leqslant 4\varepsilon R. \quad (39)$$

As $A \in \Omega_N^{k\psi}(R)$, the inequality

$$\langle p - p_0, y - x_0 \rangle \geqslant -Rk\psi\left(\frac{\|y - x_0\|}{R}\right)$$

holds. From this and the inequality (39) we get

$$\langle p_0, y - x_0 \rangle \leqslant Rk\psi\left(\frac{\|y - x_0\|}{R}\right) + 4\varepsilon R\,. \quad (40)$$

The relations (38) imply that $\|z_\lambda - x_0\| \leqslant \|z_1 - x_0\| < 2R$. Therefore, by inequality (35), we have

$$\|y - x_0\| \leqslant \|z_\lambda - x_0\| + \varepsilon \|z_1 - z_0\| < \|z_\lambda - x_0\| + 2\varepsilon R\,.$$

Thus, considering the Lipschitzness of function $\psi(\cdot)$, we get

$$Rk\psi\left(\frac{\|y - x_0\|}{R}\right) \leqslant Rk\psi\left(\frac{\|z_\lambda - x_0\|}{R}\right) + 2\varepsilon LkR\,.$$

Substituting the last inequality in (40), we obtain that

$$\langle p_0, y - x_0 \rangle \leqslant Rk\psi\left(\frac{\|z_\lambda - x_0\|}{R}\right) + \varepsilon R(4 + 2kL).$$

Thus, using inequality (35), we get that

$$\langle p_0, z_\lambda - x_0 \rangle < Rk\psi\left(\frac{\|z_\lambda - x_0\|}{R}\right) + \varepsilon R(6 + 2kL).$$

Denote $t = \lambda(1-\mu) + \mu$. Then $0 < \mu \leqslant t \leqslant 1$ and, according to equality (38), we get that $z_\lambda - x_0 = t(z_1 - x_0)$. Thus,

$$t\langle p_0, z_1 - x_0\rangle < Rk\psi\left(t\frac{\|z_1 - x_0\|}{R}\right) + \varepsilon R(6 + 2kL).$$

Due to the convexity of function $\psi(\cdot)$, we obtain

$$\begin{aligned}\langle p_0, z_1 - x_0\rangle &< Rk\frac{\psi\left(t\frac{\|z_1-x_0\|}{R}\right)}{t} + \frac{\varepsilon R(6+2kL)}{t}\\ &\leqslant Rk\psi\left(\frac{\|z_1 - x_0\|}{R}\right) + \frac{\varepsilon R(6+2kL)}{\mu}.\end{aligned}$$

Considering inequality (30), we get

$$\langle p_0, z_1 - x_0\rangle < 2R\delta_X\left(\frac{\|z_1 - x_0\|}{R}\right) + \frac{\varepsilon R(6+2kL)}{\mu},$$

which contradicts equality (34). □

6. Proof (1)⇒(2) in Theorem 2.7

The proof consists in constructing examples of sets in the class $\Omega_N^{k_2\psi}(R)$ that do not belong to the class $\Omega_N(R)$, and consists in number of steps. The main idea is building in the plane $L \subset X$ a set A_1 that belongs to the class $\Omega_N^{k\psi}(R)$ in space L. Then, with the help of some technical lemmas, the set A_1 is extended to the set A, which already is in the class $\Omega_N^{K\psi}(K_1 R)$ in X, where the constants k, K, K_1 do not depend on the initial choice of the plane L and space X. Using the fact that $\delta_X(t) \asymp \lambda_X^-(t) = o(\psi(t))$ at zero, we choose such vectors that their existence contradicts the N-supporting condition.

Let us first describe the functions ψ from the class $\mathfrak{M}$, such that there exists a normed space whose smoothness modulus is equivalent to the function ψ. In [14] the following theorem was proved.

Theorem VI. *For a function $N : [0, +\infty) \to [0, +\infty)$, that satisfies the Figiel condition and such that $N(0) = 0$ there exists a two-dimensional space X_2 whose smoothness modulus $\rho_{X_2}(t)$ is equivalent to $N(t)$ at zero.*

The Day-Nordlander Theorem (see [21]) implies that if for function ψ there exists a Banach space with the smoothness modulus, equivalent to ψ at zero, then $t^2 = \mathrm{O}(\psi(t))$. However, the function $\psi(\cdot)$ from $\mathfrak{M}$ such that $t^2 = \mathrm{O}(\psi(t))$ may not satisfy the Figiel condition (see (3)), but the following Lemma holds.

Lemma 6.1. *For any function $\psi \in \mathfrak{M}$ such that $t^2 = o(\psi(t))$ at zero there exists a function $\psi_1(\cdot) : [0, +\infty) \to [0, +\infty)$ that satisfies the Figiel condition and such that $t^2 = o(\psi_1(t))$ at zero and $\psi_1(t) = o(\psi(t))$ at zero.*

Proof. Clearly, it is sufficient to examine the case $\psi(t) = o(t)$ at zero, otherwise $\psi_1(t) = t^{\frac{3}{2}}$.
Consider the function $h(t) = t^2/\psi(t)$ on the interval $[0,1]$. The function is continuous on this interval and for some constant $k > 0$ the inequality $h(t) \geqslant kt$ holds. The set $K = \{(t,y)|\ t \in [0,1],\ 0 \le y \le h(t)\}$ is compact, its convex hull $\operatorname{co} K$ is also a compact set.

We define a continuous function $\gamma(t) = \max\{y \in \mathbb{R}|\ (t,y) \in \operatorname{co} K\}$, $t \in [0,1]$. This function is concave on the interval $[0,1]$, $\gamma(0) = 0$. Therefore, in some neighborhood of zero, $\gamma(t)$ strictly increases.

Define the function $\psi_1(t) = \frac{t^2}{\sqrt{\gamma(t)}}$. As $\frac{\psi_1(t)}{\psi(t)} = \frac{h(t)}{\sqrt{\gamma(t)}} \le \sqrt{\gamma(t)} \to 0$ and since $\frac{t^2}{\psi_1(t)} = \sqrt{\gamma(t)}$ approaches zero monotonically as $t \to 0$, the function $\psi_1(\cdot)$ satisfies the required conditions. $\square$

Now let us prove the technical lemmas, using which we will extend the set A_1, which lies in the class $\Omega_N^{k\psi}(R)$ on a plane, to the set A, from the class $\Omega_N^{K\psi}(K_1 R)$ in the space, which contains the plane, where the constants k, K, K_1 do not depend on the choice of the initial plane and the space containing it.

If the Banach space can be represented as the direct sum of two of its closed subspaces $Z = Z_1 \oplus Z_2$, then any vector $z \in Z$ is uniquely expressible as the sum of two vectors $z_1 \in Z_1$ and $z_2 \in Z_2$, and in this case we will write $z = (z_1, z_2)$. Hereafter, when we speak about the representation of a Banach space as the direct sum of its subspaces we will consider that the subspaces are closed.

Lemma 6.2. *In the Banach space X fix any $x \in \partial\mathfrak{B}_1(o)$, $p \in J_1(x)$, $y \in \operatorname{Ker} p \cap \partial\mathfrak{B}_1(o)$ (i.e. $y \urcorner x$). Denote $L = \operatorname{lin}\{x, y\}, l = \operatorname{lin} y$. Then there exists a projector $P : X \to L$ such that $P(\operatorname{Ker} p) = l$ and $\|P\| \leqslant 3$.*

Proof. If $\dim X = 2$, then the identity transformation satisfies the required condition.
Let $\dim X > 2$. According to the paper [16], in the subspace $\operatorname{Ker} p$ of co-dimension 1 there exists a projector $P_1 : \operatorname{Ker} p \to l$ with unit norm. Clearly, X is the direct sum of the subspaces $\operatorname{Lin} x$ and $\operatorname{Ker} p$.
Define the projector $P : X \to L$ in the following way

$$Pz = P(z_1, z_2) = (z_1, P_1 z_2), \quad \forall z = (z_1, z_2) \in X,\ z_1 \in \operatorname{Lin} x,\ z_2 \in \operatorname{Ker} p.$$

Let us estimate its norm. Fix an arbitrary vector $u = (u_1, u_2) \in \partial\mathfrak{B}_1(o)$, where $u_1 \in \operatorname{Lin} x$, $u_2 \in \operatorname{Ker} p$. As $u_2 \urcorner u_1$, then $\|u_1\| \leqslant \|u_1 + u_2\| = \|u\| = 1$. From this and the triangle inequality we get that $\|u_2\| \leqslant \|u_1 + u_2\| + \|u_1\| \leqslant 2$. Therefore,

$$\|Pu\| = \|(u_1, P_1 u_2)\| = \|u_1 + P_1 u_2\| \leqslant \|u_1\| + \|P_1 u_2\| \leqslant \|u_1\| + \|u_2\| \leqslant 3. \quad \square$$

Lemma 6.3. *Let X be a linear vector space with two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$:*

$$\|x\|_1 \leqslant \|x\|_2 \leqslant c_1 \|x\|_1. \tag{41}$$

Denote $X_1 = (X, \|\cdot\|_1)$, $X_2 = (X, \|\cdot\|_2)$. *Let* $A = X_2 \setminus \operatorname{int} \mathfrak{B}_1(o)$ *and* $A \in \Omega_N^{\psi}(R)$ *in the space* X_2, *then* $A \in \Omega_N^{c_1\psi}\left(\frac{R}{c_1}\right)$ *in the space* X_1.

Proof. According to Remark 1.3, at every point of set A the normal cone does not change, but the norms of the corresponding functionals may change. Fix arbitrary x_1, x_2, p_1^1, p_2^1, which satisfy the condition (10) for the space X_1. For $j = 1, 2$ denote $p_j^2 = \frac{p_j^1}{\|p_j^1\|_2}$. As $\|p_j^1\|_1 = 1$, then by inequalities (41) we have $\frac{1}{c_1} \leqslant \|p_j^1\|_2 \leqslant 1$. As the set A is a complement of a convex set, then $\langle p_1^1, x_1 - x_2 \rangle \leqslant 0$. Therefore, $\langle p_1^2, x_1 - x_2 \rangle \leqslant \langle p_1^1, x_1 - x_2 \rangle$. Similarly, $\langle p_2^2, x_2 - x_1 \rangle \leqslant \langle p_2^1, x_2 - x_1 \rangle$. Summing the last two inequalities, we obtain

$$\langle p_1^2 - p_2^2, x_1 - x_2 \rangle \leqslant \langle p_1^1 - p_2^1, x_1 - x_2 \rangle. \tag{42}$$

From the fact that $A \in \Omega_N^{\psi}(R)$ in the space X_2 and from the inequalities (41) we have

$$\langle p_1^2 - p_2^2, x_1 - x_2 \rangle \geqslant -R\psi\left(\frac{\|x_1 - x_2\|_2}{R}\right) \geqslant -R\psi\left(\frac{c_1 \|x_1 - x_2\|_1}{R}\right).$$

This and the last inequality in sequence (42) imply

$$\langle p_1^1 - p_2^1, x_1 - x_2 \rangle \geqslant -c_1 \frac{R}{c_1} \psi\left(\frac{c_1 \|x_1 - x_2\|_1}{R}\right).$$

And, thus, $A \in \Omega_N^{c_1\psi}\left(\frac{R}{c_1}\right)$ in the space X_1. $\square$

Let $X = X_1 \oplus X_2$, and, besides, there exists a projector $P : X \to X_1$ with the norm $\|P\| < +\infty$ and $X_2 = \operatorname{Ker} P$. Then $X^* = X_1^{\perp} \oplus X_2^{\perp}$, where the spaces $X_1^{\perp}, X_2^{\perp}$ are the right annihilators to the spaces $X_2 \subset X$ and $X_1 \subset X$ accordingly. Extending any functional $p \in X_1^*$ with zero on the space X_2, we obtain a natural isomorphism $X_1^* = X_2^{\perp}$. Similarly, $X_2^* = X_1^{\perp}$.

Lemma 6.4. *Let* $X = X_1 \oplus X_2$,. *Then there exists a projector* $P \colon X \to X_1$ *with the norm* $\|P\| < +\infty$ *and* $X_2 = \operatorname{Ker} P$. *Let the set* $A_1 \subset X_1$. *Define the set* $A = \{x \in X | Px \in A_1\}$. *Then, for any* $x \in A$:

$$N(x, A) = (N(Px, A_1), o) = \{(p, o) |\ p \in N(Px, A_1)\}.$$

Proof. Fix an arbitrary vector $x = (x_1, x_2) \in A$.
(1) Let $p = (p_1, p_2) \in N(x, A)$. We prove that $p_2 = o$. Suppose that $p_2 \neq o$. Then there exists a non-zero vector $v = (o, v_2)$ such that $\langle p, v \rangle = \langle p_2, v_2 \rangle = k \neq 0$. According to the definition of set A for any $\lambda \in \mathbb{R}$ the vector $x + \lambda v$ lies in A. But then $\langle p, \lambda v \rangle = \lambda k$ is a linear in λ function and, therefore, $p \notin N(x, A)$. Contradiction.
(2) Let $p = (p_1, o) \in N(x, A)$. Let us show that $p_1 \in N(Px, A_1) = N(x_1, A_1)$.

Suppose the contrary, that $p_1 \notin N(x_1, A_1)$. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists a vector $y_1^\delta \in A_1$ such that $\left\|y_1^\delta - x_1\right\| \leqslant \delta$ and also $\langle p_1, y_1^\delta - x_1\rangle \geqslant \varepsilon \left\|y_1^\delta - x_1\right\|$. Define $y_\delta = (y_1^\delta, x_2)$. Clearly, $y_\delta \in A$. Then from the last inequality we obtain

$$\langle p, y_\delta - x\rangle = \langle (p_1, o), (y_1^\delta - x_1, o)\rangle = \langle p_1, y_1^\delta - x_1\rangle \geqslant \varepsilon \left\|y_1^\delta - x_1\right\| = \varepsilon \left\|y_\delta - x\right\|.$$

Therefore, $p \notin N(x, A)$. A contradiction.
(3) Let $p_1 \in N(x_1, A_1)$. Let us show that the inclusion $p = (p_1, o) \in N(x, A)$ holds. Fix $\varepsilon > 0$. According to the definition of the normal cone, for $\varepsilon / \|P\|$ there exists $\delta > 0$ such that for any vector $d_1 \in \mathfrak{B}_\delta(x_1) \cap A_1$ the inequality $\langle p_1, d_1 - x_1\rangle \leqslant \frac{\varepsilon}{\|P\|} \|d_1 - x_1\|$ holds. For any $y = (y_1, y_2) \in \mathfrak{B}_{\frac{\delta}{\|P\|}}(x) \cap A$ we have $\|P(y - x)\| = \|y_1 - x_1\|_1 \leqslant \|P\| \|y - x\| \leqslant \delta$. This and the previous inequality imply

$$\langle p, y - x\rangle = \langle p_1, y_1 - x_1\rangle \leqslant \frac{\varepsilon}{\|P\|} \|y_1 - x_1\| \leqslant \varepsilon \|y - x\|. \qquad \square$$

Lemma 6.5. *If, additionally, in the assertion of Lemma 6.4 the set A_1 is a complement to a convex set in X_1 and $A_1 \in \Omega_N^{\psi}(R)$ in X_1, then $A \in \Omega_N^{\|P\|\psi}\left(\frac{R}{\|P\|}\right)$ in the space X.*

Proof. By $\|\cdot\|_1$ denote the norm of the space X_1^*. Clearly, for $p = (p_1, o) \in X^*$ the following inequality

$$\|p_1\|_1 \leqslant \|p\| \tag{43}$$

holds. Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in A$, $p^x \in N(x, A) \cap \partial\mathfrak{B}_1^*(o)$, and finally $p^y \in N(y, A) \cap \partial\mathfrak{B}_1^*(o)$. The inequality $\|x_1 - y_1\| \leqslant \varepsilon$ holds, where ε is taken from the definitions of the class $\Omega_N^{\psi}(R)$. Lemma 6.4 implies that $p^x = (p_1^x, o)$, $p^y = (p_1^y, o)$, $p_1^x \in N(x_1, A_1)$, and $p_1^y \in N(y_1, A_1)$ in the space X_1. As A_1 is a complement to a convex set in the space X_1, then the set A is a complement to a convex set in X. Then $\langle p^x, x - y\rangle \leqslant 0$. From this and the inequalities (43) we get

$$\langle p^x, x - y\rangle = \langle p_1^x, x_1 - y_1\rangle \geqslant \langle \frac{p_1^x}{\|p_1^x\|_1}, x_1 - y_1\rangle.$$

Similarly, $\langle p^y, y - x\rangle \geqslant \langle \frac{p_1^y}{\|p_1^y\|}, y_1 - x_1\rangle$. Summing the last two inequalities, we obtain that

$$\begin{aligned}\langle p^x - p^y, x - y\rangle &\geqslant \langle \frac{p_1^x}{\|p_1^x\|_1} - \frac{p_1^y}{\|p_1^y\|}, x_1 - y_1\rangle \geqslant -R\psi\left(\frac{\|x_1 - y_1\|}{R}\right) \\ &\geqslant -R\psi\left(\frac{\|P\| \|x - y\|}{R}\right). \qquad \square\end{aligned}$$

In the next lemma there are constructed examples of sets that belong to the class $\Omega_N^{k_2\psi}(R)$ and do not belong to the class $\Omega_N(R)$. To prove it we will need some additional results from the geometry of convex sets in Banach spaces.

Definition 6.6. For a convex set $K \subset \mathbb{R}^n$ the ellipsoid of maximal n-dimensional volume contained within K is called the *John ellipsoid*.

It is known that for any convex $K \subset \mathbb{R}^n$ such an ellipsoid exists and is unique. Moreover, the following theorem holds (see [17]).

Theorem VII. *Let B_E be the John ellipsoid of the unit ball B_n in the space X_n, $\dim X_n = n$. Then the following inclusions hold:*

$$B_E \subset B_n \subset \sqrt{n} B_E \tag{44}$$

According to [18], the convexity modulus of an arbitrary Banach space X satisfies the relation

$$\frac{\delta_X(s)}{s^2} \leqslant L \frac{\delta_X(t)}{t^2} \quad \forall\, 0 < s \leqslant t \leqslant 2 \tag{45}$$

for some constant $0 < L < 4$.

Lemma 6.7. *Let there be given a function $\psi(\cdot) \in \mathfrak{M}$ such that $t^2 = \mathrm{O}(\psi(t))$ as $t \to 0$. Let the convexity modulus $\delta_X(\cdot)$ in the Banach space X satisfy the equality $\delta_X(t) = o(\psi(t))$ as $t \to 0$. Then there does not exist a constant $k_2 > 0$ such that the inclusion $\Omega_N^{k_2\psi}(R) \subset \Omega_N(R)$ holds for any $R > 0$.*

Proof. Suppose the contrary, that is, there exists a constant $k_2 > 0$ such that $\Omega_N^{k_2\psi}(R) \subset \Omega_N(R)$. As the condition of Lemma 6.7 implies that $\delta_X(\varepsilon) = o(\psi(\varepsilon))$ as $\varepsilon \to 0$, it is sufficient to consider two cases of relations between the functions $t^2, \delta_X(t)\,, \psi(t)$ at zero:

(1) $t^2 = o(\psi(t))$ as $t \to 0$;

(2) $\delta_X(t) = o(t^2)$ as $t \to 0$.

Indeed, $t^2 = \mathrm{O}(\psi(t))$ and $t^2 \neq o(\psi(t))$ as $t \to 0$, imply $\liminf\limits_{t\to 0} \frac{\psi(t)}{t^2} = c \in (0, +\infty)$. Then the equality $\delta_X(t) = o(\psi(t))$ as $t \to 0$ implies $\liminf\limits_{t \to 0} \frac{\delta_X(t)}{t^2} = 0$. From inequality (45) we finally obtain $\delta_X(t) = o(t^2)$ as $t \to 0$.

We now pass to the consideration of the two cases, mentioned above.

The case (1): $t^2 = \boldsymbol{o}(\psi(t))$ as $t \to 0$.

Lemma 6.1 and Theorem VI imply that in a two-dimensional linear space Y there exists a norm $\|\cdot\|_s$ such that the smoothness modulus $\rho_{X_s}(\cdot)$ of the space $X_s = (Y, \|\cdot\|_s)$ satisfies the conditions $t^2 = o(\rho_{X_s}(t))$ and $\rho_{X_s}(t) = o(\psi(t))$ as $t \to 0$. Therefore, there exists a constant γ_0 such that for any $\gamma \in (0, \gamma_0]$ the inequality

$$\frac{\rho_{X_s}\left(\frac{\gamma}{4}\right)}{16\gamma} \geqslant 51\gamma \tag{46}$$

holds. Fix $0 < \varepsilon < \min\left\{\frac{1}{2}, \gamma_0\right\}$. By the definition of the modulus of supporting convexity and (8), in space X there exist vectors $u, v \in \partial\mathfrak{B}_1(o)$ such that $v \urcorner u$ and

$$\lambda_X(u, v, \varepsilon) \leqslant 2(1 - \sqrt{1-\varepsilon^2}) < 2\varepsilon^2. \tag{47}$$

Let $p \in -J_1(u)$ be such that $\langle p, v\rangle = 0$.

Consider the plane $X_2 = \operatorname{lin} u, v$. Note that the restriction of the functional p on X_2 does not change its norm. Denote this restriction by p_1. Through B_2 denote the unit ball in this space with the norm induced by the space X. Consider the John ellipsoid (ellipse) B_E for $\frac{B_2}{\sqrt{2}}$. Denote the norm, generated by the set B_E as a unit ball through $\|\cdot\|_E$. Inclusion (44) implies that for any $x \in X_2$ the inequality $2\|x\| \geqslant \|x\|_E \geqslant \sqrt{2}\|x\|$ holds. There exists an affine transformation that transforms the John ellipsoid (ellipse) of the space X_s into the ellipsoid B_E. Replacing the space X_s by the isometrically isomorphic to it space, which is obtained from the affine transformation, we consider that the set B_E is the John ellipsoid of the unit ball B_s in the space X_s. Moreover, the inequality (46) holds. Therefore, the norms $\|\cdot\|_E, \|\cdot\|, \|\cdot\|_s$ in a two-dimensional linear space are related by

$$\|x\| \leqslant \frac{\|x\|_E}{\sqrt{2}} \leqslant \|x\|_s \leqslant \|x\|_E \leqslant 2\|x\|. \tag{48}$$

Denote $\gamma = \dfrac{\varepsilon}{25}$. In X_s, according to the definition of $\Gamma(X_s \setminus \operatorname{int} B_s, \gamma, X_s)$ and the inequality (13), there exist d and f from the unit ball in X_s such that $\|d - f\|_s = \gamma$, and for $q \in -J_1(f)$ the inequality

$$\langle q, d - f\rangle \geqslant \frac{1}{2} \cdot \frac{1}{2}\Gamma(X_s \setminus \operatorname{int} B_s, \gamma, X_s) \geqslant \frac{1}{4}\rho_{X_s}\left(\frac{\gamma}{4}\right)$$

holds. Executing the affine transformation K, which, if necessary, consists in the symmetry with respect to the main axis of the ellipse B_E and rotation by some angle ϕ_0 with respect to the ellipse B_E (i.e. K is such an affine transformation of the plane X_2 that the ellipse B_E transforms into itself), we can achieve that

1. The line l, supporting to $K(B_s)$ at point Kf, is parallel to $\operatorname{Ker} p = \{\lambda v \mid \lambda \in \mathbb{R}\}$.
2. The projection Kd on l along u lies in ray $\{Kf + \lambda v \mid \lambda \geqslant 0\}$.

Denote $A_1 = X_s \setminus \operatorname{int} K(B_s)$, $f_1 = Kf$, $d_1 = Kd$. Using the fact that the linear functional on a two-dimensional space is defined up to a multiplicative constant by the line that is the kernel of the functional, the functional $q_1 \in -J_1(f_1)$ (unique due to the smoothness of B_s) is codirectional with the functional p_1 (see Fig. 6.1).

Comments to Figure 6.1 in the notations of Lemma 6.7:

– the boundaries of the ellipses $= \frac{B_E}{\sqrt{2}}, B_E, \sqrt{2}B_E$ are shown with dotted lines;

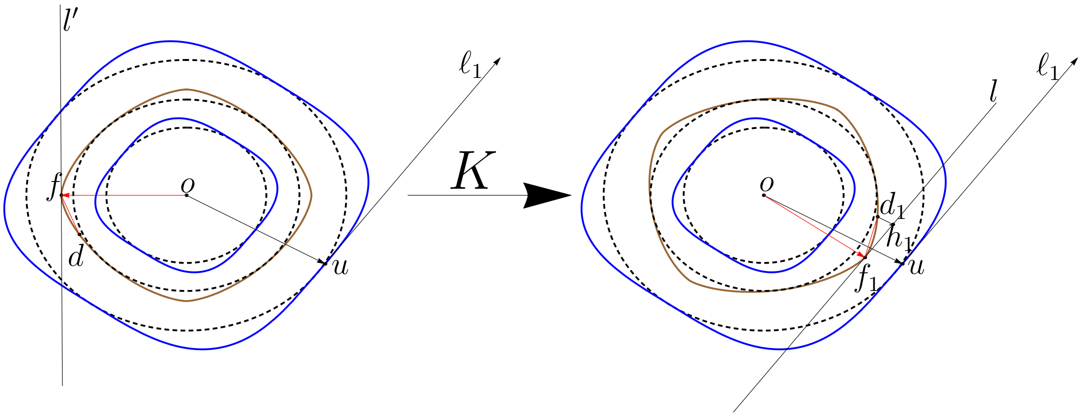

Figure 6.1: The projection K

- closed curves touching the boundaries of the ellipses $\sqrt{2}B_E$ and $\frac{B_E}{\sqrt{2}}$ are the boundaries of the balls B_2, $\frac{B_2}{2}$, respectively;
- closed curves touching the boundary of the B_E are the boundaries of B_s and KB_s;
- ℓ_1 is the ray $\{u+\lambda v | \lambda \geqslant 0\}$ (part of the supporting line $u + \operatorname{Ker} p_1$);
- l' is the preimage of line l (the supporting line to the ball B_s at point f).

Let $q_1 = kp_1$. Then, as the transformation K is affine, we obtain that

$$\langle kp_1, d_1 - f_1 \rangle \geqslant \frac{1}{4}\,\rho_{X_s}\left(\frac{\gamma}{4}\right). \tag{49}$$

On the other hand, it is clear that the norm $\|\cdot\|_s^{'}$ of the space X_s', generated by the set $K(B_s)$ (i.e. we consider $K(B_s)$ to be the unit ball), satisfies the following series of inequalities

$$\|x\| \leqslant \frac{\|x\|_E}{\sqrt{2}} \leqslant \|x\|_s^{'} \leqslant \|x\|_E \leqslant 2\,\|x\|\,. \tag{50}$$

This, particularly, implies that $1 \leqslant k \leqslant 2$. As the right side of the inequality (49) is positive, we obtain

$$\langle p_1, d_1 - f_1 \rangle \geqslant \frac{1}{8}\rho_{X_s'}\left(\frac{\gamma}{4}\right). \tag{51}$$

According to Lemma 6.2, in X there exists a projector $P : X \to X_2$ with the norm not greater than 3, that projects $\operatorname{Ker} p$ on $\operatorname{Ker} p_1$. Define the set $A = \{x \in X | \; Px \in A_1\}$. According to Corollary 4.2, $A_1 \in \Omega_N^{\frac{1}{17}\rho_{X_s'}(\cdot)}(1)$ in the space $X_s^{'}$. Then Lemma 6.3 and the series of inequalities (50) imply that $A_1 \in \Omega_N^{\frac{2}{17}\rho_{X_s'}(\cdot)}\left(\frac{1}{2}\right)$ in the space X_2. This, according to Lemma 6.5, implies that the inclusion $A \in \Omega_N^{\frac{6}{17}\rho_{X'_s}(\cdot)}\left(\frac{1}{6}\right)$ in space X holds. But by construction we have that $\rho_{X'_s}(t) = \rho_{X_s}(t) = o(\psi(t))$ as $t \to 0$. Therefore, for sufficiently small t the inequality $\frac{6}{17}\rho_{X'_s}(6t) \leqslant k_2\psi(6t)$ holds, which implies that $A \in \Omega_N^{k_2\psi(\cdot)}\left(\frac{1}{6}\right)$. According to the assumption made, we obtain that $A \in \Omega_N\left(\frac{1}{6}\right)$.

As a consequence of the construction of $f_1 \in X_2 \subset X$ the equality $p = (p_1, o)$ holds, and, therefore, $p = N(f_1, A) \cap \partial\mathfrak{B}_1^*(o)$. Let $D = \mathfrak{B}_{\frac{1}{6}}\left(f_1 - \frac{1}{6}u\right)$. According to the definition of the class $\Omega_N\left(\frac{1}{6}\right)$, the following relation is satisfied

$$\operatorname{int} D \cap A = \emptyset. \tag{52}$$

The inequality (47) implies that $\frac{\lambda_X(u,v,\varepsilon)}{6} < \frac{\varepsilon^2}{3} < \frac{1}{6}$. By the definition of $\lambda_X(u, v, \varepsilon)$ and the choice of vectors u and v we get $z_1 = f_1 + \frac{\varepsilon}{6}v - \frac{\varepsilon^2}{3}u \in \operatorname{int} D$. In the space X_2 consider the projection h_1 of the point d_1 on the line l. By the uniform convexity and uniform smoothness of the space X_2 the segment h_1d_1 is parallel to u. On the other hand, as the vector $d_1 \in X$ lies in A, then from (52) we obtain that $d_1 \notin \operatorname{int} D$. Therefore, the segment f_1d_1 intersects the sphere ∂D in two points. Then, by construction, the following series of inequalities hold $\|h_1 - f_1\| \leqslant \|f_1 - d_1\| + \|d_1 - h_1\| \leqslant 2\|f_1 - d_1\| \leqslant 2\|f_1 - d_1\|_s' = 2\gamma < \frac{\varepsilon}{6} < \frac{1}{6}$. From this, (9) and considering the segment $h_1d_1 \cap \operatorname{int} D = \emptyset$, $h_1d_1 \parallel u$, the point h_1 lies on the supporting line to the ball D of radius $\frac{1}{6}$ at the point f_1, we get

$$\|d_1 - h_1\| \leqslant \frac{1}{6}\lambda_{X_2}\left(f_1, \frac{h_1 - f_1}{\|h_1 - f_1\|}, 6\|h_1 - f_1\|\right) \leqslant \|h_1 - f_1\|,$$

and, thus, $\|f_1 - d_1\| \leqslant 2\|h_1 - f_1\|$. From this, the choice of f_1, d_1 and the inequality (50), we obtain

$$0 < \frac{\gamma}{2} = \frac{\|f_1 - d_1\|_s'}{2} \leqslant \|f_1 - d_1\| \leqslant 2\|h_1 - f_1\| \leqslant 4\gamma < \frac{\varepsilon}{6}. \tag{53}$$

But $d_1 = f_1 + \|h_1 - f_1\| v - \langle p_1, d_1 - f_1\rangle u = f_1 + \|h_1 - f_1\| v - \langle p, d_1 - f_1\rangle u$. The inequalities (53), (51), (46) and the choice of constant γ imply

$$\frac{\langle p, d_1 - f_1\rangle}{\|h_1 - f_1\|} \geqslant \frac{\rho_{X_s}\left(\frac{\gamma}{4}\right)}{16\gamma} \geqslant 51\gamma > 2\varepsilon. \tag{54}$$

From $\|f_1 - h_1\| < \frac{\varepsilon}{6}$ and inequality (54) we obtain that the segments $d_1h_1 \subset A$ and $f_1z_1 \subset D$ intersect. Denote by y_1 their intersection point. Clearly, $y_1 \in \operatorname{int} D \cap A$ (see Fig. 6.2). This contradicts relation (52).

Comments to Figure 6.2 in the notations of Lemma 6.7:

– $o' = f_1 - \frac{1}{6}u$, $\quad b = o' + \frac{1}{6}v$, $\quad a_1 = f_1 + \frac{1}{6}v$, $\quad g_1 = f_1 + \frac{\varepsilon}{6}v$;

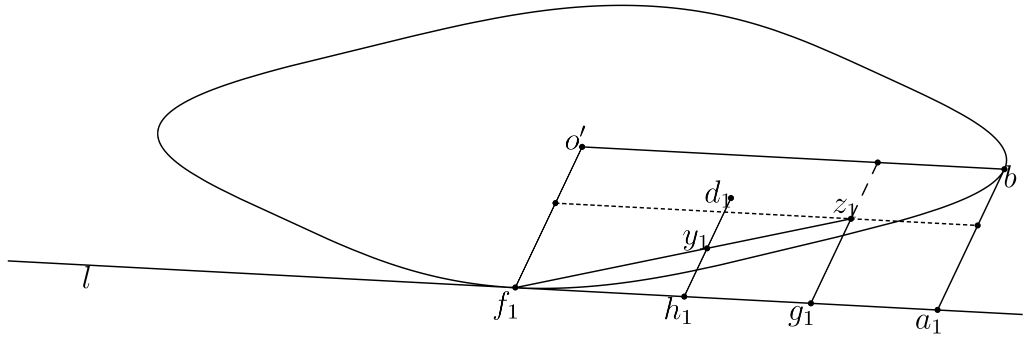

Figure 6.2: Proof of Lemma 6.7.

– as $\frac{\lambda_X(u,v,\varepsilon)}{6} < \|z_1 - g_1\| < \frac{1}{6}$, then z_1 lies inside the parallelogram $o'ba_1f_1$ and inside D;
– similarity implies that $\frac{\|z_1-g_1\|}{\|f_1-g_1\|} = \frac{\|y_1-h_1\|}{\|f_1-h_1\|} < \frac{\|d_1-h_1\|}{\|f_1-h_1\|}$.

The case (2): $\boldsymbol{\delta_X(t) = o(t^2)}$ as $\boldsymbol{t \to 0}$.

The consideration of this case does not differ much from the consideration of the previous one. The difference consists in the specific choice of constants. All the notations coincide with the notations in the previous case, if not mentioned otherwise. Let $k_3 > 0$ be such that $\psi(t) \geqslant k_3 t^2$ in some neighborhood of zero. Note that if $t \in [0,1]$ we get the inequalities:

$$\frac{t^2}{4} \leqslant \rho_H(t) = \sqrt{1+t^2} - 1 \leqslant t^2 \tag{55}$$

Let $R_1 = \frac{17}{36}k_2k_3$. By assumption we have that $\delta_X(t) = o(t^2)$ as $t \to 0$. Fix $\gamma_0 \in (0,1)$ such that for any $t \in (0,\gamma_0)$ the inequality $\delta_X(t) < \frac{1}{3} \cdot \frac{R_1}{2^{12}} t^2$ holds.

By the definition of the modulus of supporting convexity and (8) for

$$\varepsilon \in \left(0, \min\left\{\frac{\gamma_0}{2}, \frac{1}{\sqrt{R_1}}\right\}\right)$$

there exist $u, v \in \partial\mathfrak{B}_1(o)$ in the space X such that $v \urcorner u$ and

$$\lambda_X(u,v,\varepsilon) \leqslant 2\delta_X(2\varepsilon) < \frac{1}{3} \cdot \frac{R_1}{512}\varepsilon^2 < 1. \tag{56}$$

The space X_2 and the functionals p, p_1 are defined similarly to the previous case. We consider the unit ball in X_s' to be the set B_E. Fix $\gamma = \frac{R_1\varepsilon}{3}$. Similarly to the previous case, we take the points f_1, d_1, h_1, and construct the set A. As in the previous case, $A \in \Omega_N^{\frac{6}{17}\rho_{X_s'}(\cdot)}\left(\frac{1}{6}\right)$. As the ball in X_s' is Euclidean and considering the left inequality in series (55) we have

$$\langle p_1, d_1 - f_1\rangle \geqslant \frac{1}{4}\rho_{X_s'}\left(\frac{\gamma}{4}\right) \geqslant \frac{1}{256}\gamma^2. \tag{57}$$

From the fact that the norm in X_s' is Euclidean, the right hand side of inequality (55) and the choice of the constants k_3 and R_1 we obtain

$$R_1 k_2 \psi\left(\frac{t}{R_1}\right) \geqslant \frac{k_2 k_3 t^2}{R_1} = \frac{36t^2}{17} \geqslant \frac{1}{6} \cdot \frac{6}{17} \rho_{X_{s'}}(6t).$$

Thus, $A \in \Omega_N^{k_2\psi(\cdot)}(R_1)$. The assumption implies that $A \in \Omega_N(R_1)$.

Let $D = \mathfrak{B}_{R_1}(f_1 - R_1 u)$. Inequality (56) and the choice of constant ε imply that $R_1 \lambda_X(u, v, \varepsilon) < \frac{1}{3} \cdot \frac{R_1^2}{512} \varepsilon^2 < R_1$. By the definition of $\lambda_X(u, v, \varepsilon)$ and the choice of vectors u and v we get $z_1 = f_1 + R_1 \varepsilon v - \frac{1}{3} \cdot \frac{R_1^2}{512} \varepsilon^2 u \in \operatorname{int} D$.

The estimations for the norm $h_1 - f_1$ are made similarly as in the previous case:

$$\frac{\gamma}{4} \leqslant \|h_1 - f_1\| \leqslant 2\gamma < R_1 \varepsilon. \tag{58}$$

Furthermore, $d_1 = f_1 + \|h_1 - f_1\| \, v - \langle p, d_1 - f_1 \rangle u$. Inequalities (58), (57) and the choice of constant γ imply that

$$\frac{\langle p, d_1 - f_1 \rangle}{\|h_1 - f_1\|} \geqslant \frac{1}{512} \gamma = \frac{1}{3} \cdot \frac{1}{512} R_1 \varepsilon. \tag{59}$$

By the inequality $\|f_1 - h_1\| < R_1 \varepsilon$ and inequality (59) we get that the segments $d_1 h_1 \subset A$ and $f_1 z_1 \subset D$ intersect. Denote their intersection point by y_1. Clearly, $y_1 \in \operatorname{int} D \cap A$. This contradicts relation (52). □

Proof (1)⇒(2) in Theorem 2.7. The proof follows directly from Lemma 6.7. The inequalities (45), (3) imply that if for the function $\psi \in \mathfrak{M}_2$ the relation $\psi(\varepsilon) \neq \mathrm{O}(\delta_X(\varepsilon))$ holds as $\varepsilon \to 0$, then $\delta_X(\varepsilon) = o(\psi(\varepsilon))$ as $\varepsilon \to 0$. □

Lemma 6.8. *Let X be a uniformly smooth and uniformly convex Banach space and $\psi(\cdot) \in \mathfrak{M}$ a function such that $t^2 = \mathrm{O}(\psi(t))$ as $t \to 0$. If there exists a constant $k_2 > 0$ such that the inclusion $\Omega_N^{k_2\psi}(R) \subset \Omega_N(R)$ holds, then we have $\delta_X(\varepsilon) \asymp \varepsilon^2$ at zero and $\liminf\limits_{t \to 0} \frac{\psi(t)}{t^2} = c \in (0, +\infty)$.*

Proof. Applying Lemma 6.7, we get that $\delta_X(\varepsilon) \neq o(\psi(\varepsilon))$ as $\varepsilon \to 0$. But for the function $\psi(\cdot) \in \mathfrak{M}$ such that $t^2 = \mathrm{O}(\psi(t))$ as $t \to 0$, the relation $\delta_X(\varepsilon) \neq o(\psi(\varepsilon))$ as $\varepsilon \to 0$ may hold if and only if $\delta_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$ and $\liminf\limits_{t \to 0} \frac{\psi(t)}{t^2} = c \in (0, +\infty)$. □

Proof of Theorem 2.8. Applying Theorem 2.6, we obtain that in some neighborhood of zero for some constant $k > 0$ the inequality

$$\psi(\varepsilon) \geqslant k \rho_X(\varepsilon) \tag{60}$$

holds. This and the Day-Nordlander theorem (see [21]) imply that $\varepsilon^2 = \mathrm{O}(\psi(\varepsilon))$ as $\varepsilon \to 0$. Now, applying Lemma 6.8, we obtain that $\delta_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$

and $\liminf\limits_{t\to 0} \frac{\psi(t)}{t^2} = c \in (0,+\infty)$. This, inequality (60) and the fact that the smoothness modulus satisfies the Figiel condition (see Remark 2.3) imply that $\rho_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$. Since $\delta_X(\varepsilon) \asymp \rho_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$, X is isomorphic to a Hilbert space (see [19], §5.5.6). □

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