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On Weakly Sequentially Complete Banach Spaces

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We provide sufficient conditions for a Banach space Y to be weakly sequentially complete. These conditions are expressed in terms of the existence of directional derivatives for cone convex mappings with values in Y.

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1. Introduction

Weakly sequentially complete Banach spaces were introduced in [2]. Since then, properties of weakly sequentially complete spaces were investigated in a number of papers, see e.g. [5] and [22] and the references therein. An important result on weakly sequentially complete Banach spaces is given by Rosenthal in [22].

Theorem 1.1 ([22], Theorem 1). *If Y is weakly sequentially complete, then Y is either reflexive or contains a subspace isomorphic to l^1.*

In the present paper we prove sufficient conditions for a Banach space Y to be weakly sequentially complete. These sufficient conditions are expressed in terms of existence of directional derivatives for cone convex mappings.

Cone convex mappings appear in variational analysis, in the theory of topological vector spaces and in constructions of efficient iterative schemes for solving

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vector optimization problems. For instance, Newton method for solving some classes of unconstrained convex vector optimization problems under partial order induced by closed convex pointed cones can be found in [8].

The organization of the paper is as follows. In Section 2 we discuss basic facts concerning weakly sequentially complete Banach spaces. In Section 3 we prove necessary properties of cone convex mappings. Section 4 is devoted to constructions of convex functions that take given values at given points. Section 5 contains the main result, namely, the proof of the fact that the existence of directional derivatives at any point and any direction for any cone convex mapping with values in Y implies that the Banach space Y is weakly sequentially complete. In Section 6 we discuss properties of the mapping F constructed in Theorem 5.2.

2. Preliminary facts

Let us present some known facts about weakly sequentially complete Banach spaces.

Definition 2.1. A sequence $\{y_n\}$ in a Banach space Y is *weak Cauchy* if $\lim_{n\to\infty} y^*(y_n)$ exists for every $y^* \in Y^*$. A Banach space Y is *weakly sequentially complete* if every weak Cauchy sequence weakly converges in Y. A weak Cauchy sequence $\{y_n\}$ is called *nontrivial* if it is not weakly convergent.

Nontrivial weak Cauchy sequences were investigated in [9, 14, 19]. By the Uniform Boundedness principle, any weak Cauchy sequence $\{y_n\}$ in a Banach space Y is norm-bounded ([1], p. 38). In [6] we find the following fact about reflexive spaces.

Proposition 2.2 ([6], Corollary 4.4). *If Y is reflexive, then Y is weakly sequentially complete.*

However, in some nonreflexive spaces, sequences can be found which are weak Cauchy but not weakly convergent. Let us present some examples, c.f. [6, 17].

Example 2.3.

(1) Let $Y = c_0$. Consider $y_n := \sum_{k=1}^{n} e_k$, where $\{e_k\}$ is the standard basis in c_0. We have $(c_0)^* = l^1$ and $\{y_n\}$ is a weak Cauchy sequence. Since $\{y_n\}$ converges weak* to the element $(1, 1, \dots)$, it is not weakly convergent.

(2) Let K be a compact Hausdorff space. Then $C(K)$ is weakly sequentially complete if and only if K is finite.

(3) For every measure μ, the space $L^1(\mu)$ is weakly sequentially complete.

Remark 2.4. Any closed subspace $Z \subset Y$ of a weakly sequentially complete Banach space Y is weakly sequentially complete.

Remark 2.5. Let $\{y_n\} \in Y$ be a nontrivial weak Cauchy sequence. Then each subsequence of $\{y_n\}$ is also nontrivial weak Cauchy.

In the present paper we consider cone ordering relations in Y generated by convex closed cones. Assuming that Y is a Banach lattice, we get the following characterization.

Theorem 2.6 ([18], Theorem 1.c.13). *If Y is a Banach lattice, then Y is weakly sequentially complete if and only if it does not contain any subspace isomorphic to c_0.*

Let us note that if we consider the concept of weak completeness in terms of nets (see [10] for the definition), then we cannot expect weak completeness in the infinite dimensional case. A net $S := \{x_\sigma, \sigma \in \sum\}$ in a topological vector space Y is Cauchy if for every neighbourhood U of zero there exists $\sigma_0 \in \sum$ such that $x_{\sigma_1} - x_{\sigma_2} \in U$ for every $\sigma_1, \sigma_2 \geq \sigma_0$. A topological vector space Y is complete if each Cauchy net converges (see [13] p.356).

First countable spaces, including metric spaces, have topologies that are determined by their convergent sequences. In an arbitrary first countable space, a point x is in the closure of a subset A if and only if there is a sequence in A converging to x.

If Y is an infinite dimensional Banach space, regardless of whether or not it is separable in the norm topology, then the weak topology on Y is not first countable, and is not characterized by its convergent sequences alone.

For the weak topology in infinite-dimensional spaces, there exist weak Cauchy nets which are not weakly convergent. This result can be found in [7, 9] and is deeply tied to the axiom of choice and Helly's Theorem.

Proposition 2.7 ([9] p.14). *The weak topology on an infinite-dimensional normed space is never complete.*

3. Cone convex mappings

Let X, Y be real linear vector spaces and let $K \subset Y$ be a convex cone inducing the standard ordering relation

$$x \leq_K y \quad \Leftrightarrow \quad y - x \in K \quad \text{for any} \quad x, y \in Y.$$

Analogously, we write $y \geq_K x$ if and only if $y - x \in K$. We use the notation $\geq$ if the ordering cone is clear from the context.

Let $F : X \to Y$. The mapping F is *K-convex* on a convex set $A \subset X$ if

$$F(\lambda x+(1-\lambda)y) \leq_K \lambda F(x)+(1-\lambda)F(y) \quad \text{for all } x, y \in A \text{ and } \lambda \in [0,1]. \quad (1)$$

As in the scalar case [24] we have the following characterization of cone convexity of F in terms of the *epigraph of* F defined as $\text{epi}F := \{(x, y) \in A \times Y : y \geq_K F(x)\}$. Namely, a mapping $F : X \to Y$ is K-convex on a convex set $A \subset X$ if and only if $\text{epi}F$ is a convex set in $A \times Y$.

Let Y be a Banach space with the dual space Y^*. Let K be a closed convex cone in Y. The *dual cone* K^* of K is defined as $K^* := \{y^* \in Y^* \colon y^*(y) \geq 0 \ \forall y \in K\}$.

Proposition 3.1 ([12], Lemma 3.21). *Let $K \subset Y$ be a closed convex cone in Y. Then the cone K can be represented as*

$$K = \{y \in Y \ : \ y^*(y) \geq 0 \ \forall y^* \in K^*\}. \tag{2}$$

The following lemma provides a characterization of K-convex mappings. This lemma is the main tool in Section 5 (for the finite-dimensional case of this lemma see [20]).

Lemma 3.2. *Let $A \subset X$ be a convex subset of X. Let $K \subset Y$ be a closed convex cone and let $F : X \to Y$ be a mapping. The following conditions are equivalent.*

(a) *The mapping F is K-convex on A.*

(b) *For any $u^* \in K^*$ the composite function $u^*(F) \colon A \to \boldsymbol{R}$ is convex.*

Proof. The implication (a) $\Rightarrow$ (b) follows directly from the definition of K^*. To prove the converse implication suppose on the contrary that for some $\lambda \in [0,1]$, $x, y \in A$ we have $\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y) \notin K$. By (2), there exists $u^* \in K^*$ such that $u^*(\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y)) < 0$, which contradicts the convexity of $u^*(F)$ on A. $\square$

4. Construction of convex functions

In this section we construct convex functions which take given values at a countable number of given points. Some constructions of convex functions are presented in the literature. For example, in [16] we can find the proof that the continuous convex function f defined on l_p, $p \geq 1$,

$$f(x) := \|(|x_1|^{1+1}, |x_2|^{1+1/2}, ...)\|_p,$$

is everywhere compactly differentiable but not Fréchet differentiable at zero. In [3] we can find an extension of a function $\Phi : X \to \bar{\mathbf{R}}$ to some convex function $\eta : C \to \bar{\mathbf{R}}$, $X \subset C$, C is a convex set, such that $\eta(x) = \Phi(x)$ for all $x \in X$.

Let us start with the following lemma.

Lemma 4.1. *For any decreasing and convergent sequence $\{a_m\}$ of nonnegative reals, there exist a subsequence $\{a_{m_k}\}_{k \in \boldsymbol{N}} \subset \{a_m\}$ and a convex nonincreasing function $g : \boldsymbol{R} \to \boldsymbol{R}$ such that $g(m_k) = a_{m_k}$, $k \in \boldsymbol{N}$.*

Proof. Let $a_{m_1} := a_1$, $a_{m_2} := a_2$. Let $f_1 : \mathbf{R} \to \mathbf{R}$ be defined as

$$f_1(x) := a_{m_1} + \frac{x-1}{m_2 - 1}(a_{m_2} - a_{m_1}).$$

Let x_2 be such that $f_1(x_2) = 0$. We have $x_2 > m_2$. By taking $m_3 := [x_2] + 1$, where $[x]$ is the largest integer not greater than x, we define $f_2 : \mathbf{R} \to \mathbf{R}$ as

$$f_2(x) := a_{m_2} + \frac{x - m_2}{m_3 - m_2}(a_{m_3} - a_{m_2}).$$

Suppose that we have already defined $f_1, \ldots, f_k$, $k \geq 2$. We define f_{k+1} as follows. Let x_k be such that $f_k(x_k) = 0$. We have $x_k > m_k$. By taking $m_{k+2} := [x_k] + 1$, we define $f_{k+1} : \mathbf{R} \to \mathbf{R}$ as

$$f_{k+1}(x) := a_{m_{k+1}} + \frac{x - m_{k+1}}{m_{k+2} - m_{k+1}}(a_{m_{k+2}} - a_{m_{k+1}}).$$

We define $g : \mathbf{R} \to \mathbf{R}$ by the formula $g(x) := \sup_k f_k(x)$, $x \in \mathbf{R}$. Clearly, g is convex on $\mathbf{R}$. Let us show that

$$\begin{array}{ll} f_{k+1}(x) \geq f_k(x) & \text{for } x \geq m_{k+1} \\ f_{k+1}(x) \leq f_k(x) & \text{for } x \leq m_{k+1}. \end{array} \tag{3}$$

By construction of m_{k+1}, we have $a_{m_{k+2}} = f_k(m_{k+2})$ and

$$a_{m_{k+2}} = a_{m_k} + \frac{m_{k+2} - m_k}{m_{k+1} - m_k}(a_{m_{k+1}} - a_{m_k}) \geq a_{m_k} + \frac{m_{k+2} - m_{k+1}}{m_{k+1} - m_k}(a_{m_{k+1}} - a_{m_k}).$$

From the monotonicity of $\{a_m\}$ we get

$$\frac{a_{m_{k+1}} - a_{m_{k+2}}}{m_{k+2} - m_k} \leq \frac{a_{m_k} - a_{m_{k+2}}}{m_{k+2} - m_k} \leq \frac{a_{m_k} - a_{m_{k+1}}}{m_{k+1} - m_k}$$

and

$$\frac{a_{m_k} - a_{m_{k+1}}}{m_{k+1} - m_k} \geq \frac{a_{m_{k+1}} - a_{m_{k+2}}}{m_{k+2} - m_{k+1}}\,. \tag{4}$$

To get (3) we multiply the above inequality by $m_{k+1} - x$. If $m_{k+1} \geq x$, then from (4)

$$\frac{a_{m_k} - a_{m_{k+1}}}{m_{k+1} - m_k}\,(m_{k+1} - x) + \frac{a_{m_{k+2}} - a_{m_{k+1}}}{m_{k+2} - m_{k+1}}\,(m_{k+1} - x) \geq 0\,,$$

$$(a_{m_k} - a_{m_{k+1}})\left(1 - \frac{x - m_k}{m_{k+1} - m_k}\right) + \frac{a_{m_{k+2}} - a_{m_{k+1}}}{m_{k+2} - m_{k+1}}\,(m_{k+1} - x) \geq 0\,.$$

We get $f_k(x) \geq f_{k+1}(x)$. Analogously we prove the first inequality in (3) when $x \geq m_{k+1}$.

By (3), g is decreasing and $g(m_k) = a_{m_k}$. □

In Lemma 4.1 we constructed the function g whenever $\{a_m\} \subset \mathbf{R}$ is nonnegative decreasing and $\{t_k\}$, $t_k = m_k$, $k \in \mathrm{N}$, is a subsequence of integers. Now we consider the case when $\{a_m\} \subset \mathbf{R}$ is arbitrary and $\{t_k\} \subset \mathbf{R}$ is a decreasing sequence of reals.

Lemma 4.2. *Let $\{a_m\}, \{t_m\} \subset \boldsymbol{R}$ be sequences with $\{t_m\}$ decreasing. If*

$$\frac{a_{m+1} - a_m}{t_{m+1} - t_m} \geq \frac{a_{m+2} - a_{m+1}}{t_{m+2} - t_{m+1}} \qquad \textit{for } m \in \boldsymbol{N}, \tag{5}$$

there exists a convex function $g : \boldsymbol{R} \to \boldsymbol{R}$ such that

$$g(t_m) = a_m, m \in \boldsymbol{N}.$$

Proof. For each $m \in \mathbf{N}$, let $f_m : \mathbf{R} \to \mathbf{R}$ be defined as

$$f_m(x) := a_m + \frac{x - t_m}{t_{m+1} - t_m}(a_{m+1} - a_m) \quad \text{for } x \in \mathbf{R}.$$

We show that for $m \in \mathbf{N}$ we have

$$\begin{aligned} f_{m+1}(x) \leq f_m(x) \quad &\text{for } x \geq t_{m+1} \\ f_{m+1}(x) \geq f_m(x) \quad &\text{for } x \leq t_{m+1}. \end{aligned} \tag{6}$$

If $x \geq t_{m+1}$ then multiplying both sides of (5) by $t_{m+1} - x$ we get

$$(a_{m+1} - a_m)\left(1 - \frac{x - t_m}{t_{m+1} - t_m}\right) + \frac{x - t_{m+1}}{t_{m+2} - t_{m+1}}(a_{m+2} - a_{m+1}) \leq 0$$

which is equivalent to $f_{m+1}(x) \leq f_m(x)$. Analogously, if $x \leq t_{m+1}$ we get $f_{m+1}(x) \geq f_m(x)$.

We show that the function $g : \mathbf{R} \to \mathbf{R}$ defined as

$$g(r) := \sup_m f_m(r), \quad r \in \mathbf{R}$$

satisfies the requirements of the Lemma 4.2. Indeed, g is convex as the supremum of convex functions. By the first inequality of (6) and the monotonicity of $\{t_m\}$, we get

$$f_{m+k}(t_m) \leq f_m(t_m) = a_m \ \text{ for } k \geq 1\,.$$

By the second inequality of (6) we get

$$f_{m-k}(t_m) \leq f_m(t_m) \ \text{ for } k = 1, \ldots, m-1\,.$$

This gives that $g(t_m) = a_m$, $m \in \mathbf{N}$. □

Observe that we have the following property:

$$g(x) = f_m(x) \quad \text{for} \quad t_{m+1} \leq x \leq t_m. \tag{7}$$

Indeed, by the first inequality of (6) and the monotonicity of $\{t_m\}$, we get

$$f_m(x) \geq f_{m-1}(x) \geq f_{m-2}(x) \geq \ldots f_1(x)\,.$$

By the second inequality of (6) and from the fact that $x \geq t_{m+k}, k \geq 1$, we get

$$f_{m+k}(x) \leq f_m(x) \quad \text{for } k \geq 1\,.$$

Now, starting from any converging sequence $\{z_m\}$ with positive limit $z > 0$ of nonnegative elements $z_m \geq 0$, $m \in \mathbf{N}$, we construct the sequences $\{a_k\}$, $\{t_k\}$ such that the function g constructed in Lemma 4.2 is convex.

Proposition 4.3. *Let $\{z_m\} \subset \boldsymbol{R}$ be a converging sequence of nonnegative reals with limit $z > 0$. Then there exists a subsequence $\{z_{m_k}\} \subset \{z_m\}$, a sequence $\{t_k\} \subset \boldsymbol{R}$, $t_k \downarrow 0$, and a convex function $g : \boldsymbol{R} \to \boldsymbol{R}$, such that*

$$g(t_k) = a_k,\ \ k \in \boldsymbol{N},\ \ \textit{where}$$

$$\begin{aligned}
&a_k := t_k z_{m_k} \ \textit{if } \{z_{m_k}\} \textit{ is nonincreasing and } 0 < z < 1,\\
&a_k := t_k z_{m_k}\tfrac{1}{q},\ \ q > z,\ \ \textit{if } \{z_{m_k}\} \textit{ is nonincreasing and } z > 1,\\
&a_k := -t_k z_{m_k},\ \textit{if } \{z_{m_k}\} \textit{ is increasing and } z > 1,\\
&a_k := -t_k z_{m_k}\tfrac{1}{q},\ \ q < z,\ \ \textit{if } \{z_{m_k}\} \textit{ is increasing and } 0 < z < 1.
\end{aligned}$$

Proof. We can assume that $z_m > 0$ for $m \in \mathbf{N}$. Two cases should be considered:

1. $\{z_m\}$ contains an infinite subsequence $\{z_{m_k}\} \subset \{z_m\}$, $\{z_{m_k}\}$ is nonincreasing, i.e. $z_{m_{k+1}} \leq z_{m_k}$ for $k \in \mathbf{N}$,

2. $\{z_m\}$ contains an infinite subsequence $\{z_{m_k}\} \subset \{z_m\}$, $\{z_{m_k}\}$ is increasing, i.e. $z_{m_{k+1}} > z_{m_k}$ for $k \in \mathbf{N}$.

To prove the assertion we show that in both cases condition (5) of Lemma 4.2 is satisfied.

Case 1. Without loss of generality we can assume that $z_{m_k} = z_m$. Moreover, we can assume that $\{z_m\}$ is decreasing (in case when $\{z_m\}$ is constant the existence of the function g satisfying the requirements of the proposition follows trivially from Lemma 4.2).

Case 1a. Consider first the case when $0 < z < 1$. By eliminating, if necessary, a finite number of z_m we can assume that $0 < z_m < 1$ for $m \in \mathbf{N}$. Let $t_m := (z_m)^m$ and $a_m := (z_m)^{m+1}$ for $m \in \mathbf{N}$. Both sequences are decreasing. Let $m_1 := 1$, $m_2 := 2$. Since $\{z_m\}$ and $\{a_m\}$ are decreasing, the number C defined below is positive, i.e.

$$C := \frac{t_1 t_2}{a_2 - a_1}(z_2 - z_1) > 0\,.$$

Since $t_m \to 0$, there exists $K \geq m_2$ such that $t_m < C$ for $m > K$. Now put $m_3 := K + 1$. We have

$$\begin{aligned} t_{m_3} &< \frac{t_1 t_2}{a_2 - a_1}\ (z_2 - z_1) = \frac{t_1 a_2 - t_2 a_1 + t_2 a_2 - t_2 a_2}{a_2 - a_1} \\ &= \frac{t_2(a_2 - a_1) - a_2(t_2 - t_1)}{a_2 - a_1} = t_2 - \frac{a_2(t_2 - t_1)}{a_2 - a_1} \\ &< t_2 - \frac{a_2(t_2 - t_1)}{a_2 - a_1} + \frac{a_{m_3}(t_2 - t_1)}{a_2 - a_1}\ . \end{aligned}$$

The last inequality follows from the fact that $a_{m_3}(t_2 - t_1) > (a_2 - a_1)$. Thus,

$$t_{m_3} - t_2 < \frac{(a_{m_3} - a_2)(t_2 - t_1)}{a_2 - a_1}\ ,$$

which proves that $\dfrac{a_2 - a_1}{t_2 - t_1} \geq \dfrac{a_{m_3} - a_2}{t_{m_3} - t_2}$.

Assume that we have already defined $m_1, m_2, ..., m_k$, $k \geq 3$. By induction with respect to k, starting from m_{k-1} and m_k we choose m_{k+1} by repeating the above reasoning with m_1 and m_2 replaced by m_{k-1} and m_k, respectively, and with $K \geq m_k$. In this way we choose m_{k+1} and we prove condition (5) of Lemma 4.2.

Case 1b. When $z > 1$ it is enough to take $t_m := (z_m/q)^m$ and $a_m := t_m z_m \frac{1}{q}$ for arbitrary $q > z$ for $m \in \mathbf{N}$. Again both sequences are decreasing. Hence, we can repeat the reasoning from Case 1a with these sequences and get the existence of the function g satisfying the requirements of the proposition.

Case 2. As previously, without loss of generality, we can assume that $z_{m_k} = z_m$.

Case 2a. Consider the case $z > 1$. By eliminating, if necessary, a finite number of z_m we can assume that $z_m > 1$ for $m \in \mathbf{N}$.

Let $t_m := (z_m)^{-m}$ and $a_m := -t_m z_m = -(z_m)^{-m} z_m = -(z_m)^{-m+1}$.

Let $m_1 := 1$, $m_2 := 2$. Since $\{t_m\}$ is decreasing and $\{z_m\}$ is increasing, the number C defined below is negative, i.e.

$$C := \frac{t_1 t_2}{t_2 - t_1}\ (z_2 - z_1) < 0.$$

Since $a_m \to 0$ there exists $K \geq m_2$ such that $a_m > C$ for $m > K$.

Put $m_3 := K + 1$. We have

$$\begin{aligned} a_{m_3} &\geq \frac{t_1 t_2}{t_2 - t_1}\ (z_2 - z_1) = \frac{t_2 a_1 - t_1 a_2 + t_2 a_2 - t_2 a_2}{t_2 - t_1} \\ &= \frac{a_2(t_2 - t_1) - t_2(a_2 - a_1)}{t_2 - t_1} = a_2 - \frac{t_2(a_2 - a_1)}{t_2 - t_1} \\ &> a_2 - \frac{t_2(a_2 - a_1)}{t_2 - t_1} + \frac{t_{m_3}(a_2 - a_1)}{t_2 - t_1}. \end{aligned}$$

The latter inequality follows from the fact that $t_{m_3}(a_2 - a_1) < t_2 - t_1$. Thus,

$$a_{m_3} - a_2 > \frac{(t_{m_3} - t_2)(a_2 - a_1)}{t_2 - t_1}.$$

As previously, this proves that $\dfrac{a_2 - a_1}{t_2 - t_1} \geq \dfrac{a_{m_3} - a_2}{t_{m_3} - t_2}$.

Assume that we have already defined $m_1, m_2, ..., m_k$, $k \geq 3$. By induction with respect to k, starting from m_{k-1} and m_k, we choose m_{k+1} by repeating the above reasoning with m_1 and m_2 replaced by m_{k-1} and m_k, respectively, and with $K \geq m_k$. In this way we choose m_{k+1} and prove condition (5) of Lemma 4.2.

Case 2b. When $0 < z < 1$ it is enough to take $t_m := (\frac{q}{z_m})^m$ for some $q < z$ and $a_m := -t_m z_m \frac{1}{q}$. By applying the reasoning from Case 2a to these sequences we get the existence of the function g satisfying the requirements of the proposition.

In consequence, in both cases, by Lemma 4.2, we get the function $g : \mathbf{R} \to \mathbf{R}$ defined by the formula $g(r) = \sup_k f_k(r)$, $r \in \mathbf{R}$, such that $g(t_k) = a_k$, where f_k, $k \in \mathbf{N}$, are as in the proof of Lemma 4.2. $\square$

As a corollary we obtain the following proposition which is used in the proof of the main result (Theorem 5.2).

Proposition 4.4. *Let Y be a normed space and let $\{y_m\} \subset Y$ be such that the sequence $\{\bar{y}^*(y_m)\}$ converges to $z \in R \setminus \{0\}$ for a certain $\bar{y}^* \in Y^* \setminus \{0\}$. Then there exists a subsequence $\{y_{m_k}\}_{k \in \boldsymbol{N}} \subset \{y_m\}$, a functional $y^* \in Y^*$ and a convex function $g : \boldsymbol{R} \to \boldsymbol{R}$, such that*

$$g(t_k) = a_k, \ \ k \in \boldsymbol{N}, \ \ \textit{where}$$

(1) $t_k := (y^*(y_{m_k}))^{m_k}$, $a_k := t_k y^*(y_{m_k})$, *if* $\{y^*(y_{m_k})\}$ *is nonincreasing and* $0 < z < 1$,

(2) $t_k := (\dfrac{y^*(y_{m_k})}{q})^{m_k}$, $a_k := t_k y^*(y_{m_k})\frac{1}{q}$, *for some* $q > z$, *if* $\{y^*(y_{m_k})\}$ *is nonincreasing and* $z > 1$,

(3) $t_k := (\dfrac{1}{y^*(y_{m_k})})^{m_k}$, $a_k := -t_k y^*(y_{m_k})$, *if* $\{y^*(y_{m_k})\}$ *is increasing and* $z > 1$,

(4) $t_k := (\dfrac{q}{y^*(y_{m_k})})^{m_k}$, $a_k := -t_k y^*(y_{m_k})\frac{1}{q}$ *for some* $q < z$, *if* $\{y^*(y_{m_k})\}$ *is increasing and* $0 < z < 1$.

Proof. The proof follows directly from Proposition 4.3. Indeed, let us observe that we can always find some $y^* \in Y^*$, $y^* = c\bar{y}^*$ for a certain $c \neq 0$ such that

$\{y^*(y_m)\}$ is a converging sequence of nonnegative reals with the limit point $z > 0$. □

5. Main result

Let X be a vector space and Y be a real Banach space. Let $F : A \to Y$ be a mapping defined on a subset A of X.

Definition 5.1. We say that the mapping $F : A \to Y$ is *directionally differentiable* at $x_0 \in A$ in the direction $h \neq 0$ such that $x_0 + th \in A$ for all t sufficiently small, if the limit

$$F'(x_0; h) := \lim_{t \downarrow 0} \frac{F(x_0 + th) - F(x_0)}{t} \tag{8}$$

exists. The element $F'(x_0; h)$ is called the *directional derivative* of F at x_0 in the direction h.

Let $A \subset X$ be a convex subset of X. Let $F : A \to Y$ be a K-convex mapping on A, where $K \subset Y$ is a closed convex cone in Y. By elementary calculations ([23]), one can prove that for any K-convex mapping F on A, and any $x_0 \in A$, $h \in X$, $x_0 + th \in A$ for all t sufficiently small, the difference quotient $q(t) := [F(x_0 + th) - f(x_0)]/t$ is nondecreasing as a function of $t > 0$.

Now we are ready to prove our main theorem.

Theorem 5.2. *Let X be a real linear vector space and let Y be a Banach space. If for every closed convex cone $K \subset Y$ and every K-convex mapping $F : A \to Y$, where $A \subset X$ is a convex subset of X, $0 \in A$, the directional derivative $F'(0; h)$ exists for any $h \in X$, $h \neq 0$ such that $th \in A$ for all t sufficiently small, then Y is weakly sequentially complete.*

Proof. By contradiction, assume that Y is not weakly sequentially complete. This means that there exists a nontrivial weak Cauchy sequence $\{y_m\}$ in Y, i.e.

$$\text{for all } y^* \in Y^* \text{ the sequence } y^*(y_m) \text{ converges}$$

and $\{y_m\}$ is not weakly convergent in Y. Let $0 \neq h \in X$ be a direction. Basing ourselves on the existence of the above sequence we construct a closed convex cone $K \subset Y$ and a K-convex mapping $F : A \to Y$ such that

$$F'(0; h) \text{ does not exist.}$$

Let us observe first that we can always find a functional $y^* \in Y^* \setminus \{0\}$ such that

$$y^*(y_m) \to z \in \mathbf{R} \setminus \{0\}, \tag{9}$$

since otherwise $\{y_m\}$ would weakly converge to zero. Let us note that without losing generality we can assume that $z > 0$. Consequently, by neglecting eventually a finite number of elements we can also assume that $y^*(y_m) > 0$, $m \in \mathbf{N}$. Let $K^* \subset Y^*$ be defined as

$$K^* := \cup_{\lambda \geq 0} \lambda y^*.$$

The cone K^* is a half-line emanating from 0 in the direction of y^*. This is a closed pointed convex cone in Y^*. Let

$$K := \{y \in Y \mid z^*(y) \geq 0 \text{ for all } z^* \in K^*\}. \tag{10}$$

By (9), $\{y_m\}_{m\in\mathbf{N}} \subset K$. The cone K is closed and convex. By Proposition 4.4, there exist a sequence $t_k \downarrow 0$, a subsequence $\{y_{m_k}\}_{k\in\mathbf{N}} \subset \{y_m\}$ and a convex function $g : \mathbf{R} \to \mathbf{R}$ such that $g(t_k) = a_k$, $k \in \mathbf{N}$. As in Proposition 4.4 the following cases should be discussed with z defined by (9).

Case 1. $t_k := (y^*(y_{m_k}))^{m_k}$, $a_k := t_k y^*(y_{m_k})$, if $\{y^*(y_{m_k})\}$ is nonincreasing and $0 < z < 1$,

Case 2. $t_k := (y^*(y_{m_k})\, q^{-1})^{m_k}$, $a_k := t_k y^*(y_{m_k})\frac{1}{q}$, where $q > z$, if $\{y^*(y_{m_k})\}$ is nonincreasing and $z > 1$,

Case 3. $t_k := (y^*(y_{m_k}))^{-m_k}$, $a_k := -t_k y^*(y_{m_k})$, if $\{y^*(y_{m_k})\}$ is increasing and $z > 1$,

Case 4. $t_k := (q\, y^*(y_{m_k})^{-1})^{m_k}$, $a_k := -t_k y^*(y_{m_k})\frac{1}{q}$ for some $q < z$, if $\{y^*(y_{m_k})\}$ is increasing and $0 < z < 1$.

Let $h \in X$, $h \neq 0$, and $A := \{x \in X : \; x = rh, \; r \geq 0\}$. In each of the above cases we construct a K-convex mapping $F : A \to Y$ satisfying the requirements of the theorem. We discuss Case 1 and Case 3 in details. Case 2 and Case 4 require only minor changes.

Case 1. We construct a K-convex mapping $F : A \to Y$ such that

$$F(t_k h) = a_k = t_k y_{m_k}, \quad k \in \mathbf{N},$$

where $\{t_k\}$ and $\{y_{m_k}\}$ are as in Case 1 of the theorem. Let $F_k : A \to Y$, $k \in \mathbf{N}$, be defined as follows. Let $F_1 : A \to Y$ be defined as

$$F_1(x) := \begin{cases} y_{m_1} t_1 + \frac{r-t_1}{t_2-t_1}\,(y_{m_2} t_2 - y_{m_1} t_1) & t_2 < r, \\ 0 & r < t_2. \end{cases}$$

For any $k \in \mathbf{N}$, $k \geq 2$ and any $x = rh$, $r > 0$ we define $F_k : A \to Y$ as

$$F_k(x) := \begin{cases} y_{m_k} t_k + \frac{r-t_k}{t_{k+1}-t_k}\,(y_{m_{k+1}} t_{k+1} - y_{m_k} t_k) & t_{k+1} < r \leq t_k, \\ 0 & r \notin (t_{k+1}, t_k]. \end{cases}$$

We start by showing that the mapping $F : A \to Y$ defined as $F(x) := \sum_{i=1}^{\infty} F_i(x)$ is well-defined. To this aim it is enough to observe that for any $x \in A$ with $x = rh$, where $t_{k+1} < r \leq t_k$, for any $N \in \mathbf{N}$ we have

$$\sum_{i=1}^{N} F_i(x) = \begin{cases} F_k(x) & N \geq k \\ 0 & N < k. \end{cases}$$

Consequently, for $t_{k+1} < r \leq t_k$, we have

$$F(x) = \sum_{i=1}^{\infty} F_i(x) = \lim_{N\to\infty} \sum_{i=1}^{N} F_i(x) = F_k(x)\,. \tag{11}$$

It is easy to see that for $r > t_2$ we have $F(x) = F_1(x)$. Moreover, $F(0) = 0$. We show that the mapping F is K-convex on A with respect to the cone K defined by (10). To this aim we use Lemma 3.2,

More precisely, we start by showing that for $x \in A$, $x = rh$, $r > 0$ we have

$$y^*(F(x)) = g(r),$$

where the function g is as in Proposition 4.4. Indeed, if $r \in (t_{k+1}, t_k]$ for some $k \geq 2$, then by (7) and (11),

$$\begin{aligned} y^*(F(x)) = y^*(F_k(rh)) &= y^* \left(y_{m_k} t_k + \frac{r - t_k}{t_{k+1} - t_k} \, (y_{m_{k+1}} t_{k+1} - y_{m_k} t_k) \right) \\ &= f_k(r) = g(r). \end{aligned}$$

If $r > t_2$, then

$$y^*(F(x)) = y^*(F_1(rh)) = y^* \left(y_{m_1} t_1 + \frac{r - t_1}{t_2 - t_1} (y_{m_2} t_2 - y_{m_1} t_1) \right) = f_1(r) = g(r).$$

Now, take any $z^* \in K^*$. By the definition of K^*, $z^* = \beta y^*$ for some $\beta \geq 0$ and by convexity of g we get

$$\begin{aligned} z^*(F((\lambda a_1 + (1-\lambda) a_2) h)) &= \beta y^*(F\,((\lambda a_1 + (1-\lambda) a_2) h)) \\ &= \beta g(\lambda a_1 + (1-\lambda) a_2) \leq \beta \lambda g(a_1) + \beta (1-\lambda) g(a_2) \\ &= \lambda z^*(F(a_1 h)) + (1-\lambda) z^*(F(a_2 h)). \end{aligned} \tag{12}$$

By Lemma 3.2, this proves that F is K-convex on A.

Case 2. We have $t_k := (y^*(y_{m_k})\, q^{-1})^{m_k}$ and $a_k := t_k y^*(y_{m_k}) \frac{1}{q}$, where $q > z$, $k \in \mathbf{N}$. With this choice of $\{t_k\}$, $\{a_k\}$, by repeating the reasoning from Case 1, we get the mapping F with the required properties.

Case 3. We construct a K-convex mapping $F : A \to Y$ such that

$$F(t_k h) = a_k = -t_k y_{m_k}, \quad k \in \mathbf{N},$$

where $\{t_k\}$ and $\{y_{m_k}\}$ are as in Case 3 of the theorem.

Let $F_k : A \to Y$, $k \in \mathbf{N}$, be defined as follows. Let $F_1 : A \to Y$ be defined as

$$F_1(x) := \begin{cases} -y_{m_1}t_1 - \frac{r-t_1}{t_2-t_1}(y_{m_2}t_2 - y_{m_1}t_1) & t_2 < r, \\ 0 & r < t_2. \end{cases}$$

For any $k \in \mathbf{N}$, $k \geq 2$ and any $x \in A$, $x = rh$, $r > 0$ we define $F_k : A \to Y$ as

$$F_k(x) := \begin{cases} -y_{m_k}t_k - \frac{r-t_k}{t_{k+1}-t_k}(y_{m_{k+1}}t_{k+1} - y_{m_k}t_k) & t_{k+1} < r \leq t_k, \\ 0 & r \notin (t_{k+1}, t_k]. \end{cases}$$

Now the mapping $F : A \to Y$ is defined by formula (11) as in Case 1. As previously, for $r > t_2$ we have $F(x) = F_1(x)$ and $F(0) = 0$. For any $z^* \in K^*$, the convexity of function $z^*(F)$ is proved by the same arguments as in (12).

Case 4. We have $t_k := (q\, y^*(y_{m_k})^{-1})^{m_k}$ and $a_k := -t_k y^*(y_{m_k})\frac{1}{q}$, where $q < z$, $k \in \mathbf{N}$. With this choice of $\{t_k\}$, $\{a_k\}$, by repeating the reasoning from Case 3, we get the mapping F with the required properties.

In all four cases, the directional derivative of F at zero in the direction $h \in X$ equals

$$F'(0;h) = \lim_{t \downarrow 0} \frac{F(th)}{t} \ .$$

Summing up the considerations above we get the following formulas in the respective cases:

Case 1. $F'(0;h) = \lim\limits_{k\to+\infty} \frac{F(t_k h)}{t_k} = \lim\limits_{k\to+\infty} y_{m_k}$,

Case 2. $F'(0;h) = \lim\limits_{k\to+\infty} \frac{F(t_k h)}{t_k} = \lim\limits_{k\to+\infty} \frac{1}{q} y_{m_k}$, where $q > z$.

Case 3. $F'(0;h) = \lim\limits_{k\to+\infty} \frac{F(t_k h)}{t_k} = \lim\limits_{k\to+\infty} -y_{m_k}$.

Case 4. $F'(0;h) = \lim\limits_{k\to+\infty} \frac{F(t_k h)}{t_k} = \lim\limits_{k\to+\infty} -\frac{1}{q} y_{m_k}$, where $q < z$,

for the respective sequences $t_k \downarrow 0$ and $\{y_{m_k}\}$.

On the other hand, by Remark 2.5, the subsequences $\{y_{m_k}\}$ appearing in the above formulas are not weakly convergent.

Hence, we constructed a cone K and a K-convex mapping F on A which is not directionally differentiable at 0 in the direction h, which completes the proof. □

6. Extension of mapping F

When X is a Hilbert space we can extend the mapping F from Theorem 5.2 to the half-space $H := \{x \in X : \langle x, h\rangle \geq 0\}$, where h is as in Theorem 5.2.

Proposition 6.1. *Let X be a Hilbert space and $h \in X \setminus \{0\}$. Let set A, mapping $F : A \to Y$, and cone K be defined as in the proof of Theorem 5.2. Then there exists a convex extension $\bar{F}\colon H \to Y$ of F to the half-space $H = \{x \in X : \langle x, h\rangle \geq 0\}$.*

Proof. Let direction $h \neq 0$, set A, cone K, and mapping F be defined as in the proof of Theorem 5.2. Let $\bar{F} : H \to Y$ be defined as follows

$$\bar{F}(x) = F(\langle x, h\rangle h).$$

Let $x_1, x_2 \in H$ and $\lambda \in [0, 1]$. The K-convexity of the mapping $\bar{F}$ on H follows from the K-convexity of the mapping F on A,

$$\begin{aligned}
\bar{F}(\lambda x_1 + (1-\lambda)x_2) &= F(\langle \lambda x_1 + (1-\lambda)x_2, h\rangle h) \\
&= F(\lambda\langle x_1, h\rangle h + (1-\lambda)\langle x_2, h\rangle h) \leq_K \lambda F(\langle x_1, h\rangle) + (1-\lambda)F(\langle x_2, h\rangle h) \\
&= \lambda\bar{F}(x_1) + (1-\lambda)\bar{F}(x_2).
\end{aligned}$$

□

In view of Proposition 6.1 we formulate the following theorem.

Theorem 6.2. *Let X be a Hilbert space and let Y be a Banach space. If for every closed convex cone $K \subset Y$, and every K-convex mapping $F : H \to Y$ defined on $H = \{x \in X\colon \langle x, h\rangle \geq 0\}$, where $h \in X$, $\|h\| = 1$ the directional derivative $F'(0; h)$ exists, then Y is weakly sequentially complete.*

Proof. Without loss of generality, in Theorem 5.2, we can define K-convex mapping F on $A = \{x \in X : x = rh,\ r \geq 0\}$, where direction $h \in X$ is such that $\|h\| = 1$. By Proposition 6.1, we can extend the K-convex mapping F defined on A to the K-convex mapping $\bar{F}$ defined on H. Since $\bar{F}(0) = 0$, the directional derivative of $\bar{F}$ at 0 in the direction h is equal to

$$\lim_{t\downarrow 0} \frac{\bar{F}(\langle th, h\rangle h)}{t} = \lim_{t\downarrow 0} \frac{\bar{F}(t\|h\|^2 h)}{t} = \lim_{t\downarrow 0} \frac{F(th)}{t}.$$

As shown in the proof of Theorem 5.2 the directional derivative of F at zero in the direction $h \in X$ does not exist which completes the proof. □

7. Comments

In [23] directional derivatives of cone convex mappings are defined via the concept of infimum of the set $\left\{\frac{F(x_0+th)-F(x_0)}{t} : \ t > 0\right\}$. Let K be a pointed cone. An element a is the infimum of A if a is a lower bound, i.e. $a \leq_K x\ \forall x \in A$ and, for any lower bound a' of A, we have $a' \leq_K a$ (c.f. [4] Ch.7, par.1, nr 8, [15] Ch.2, p.18). Without additional assumptions on cone K one cannot expect the equality

$$\lim_{t\downarrow 0} \frac{F(x_0+th)-F(x_0)}{t} = \inf\left\{\frac{F(x_0+th)-F(x_0)}{t} : \ t > 0\right\}. \tag{13}$$

In ([15]) we can find the following proposition.

Proposition 7.1 ([15], Proposition 2.2.4). *Let (X, τ) be a Hausdorff topological vector space ordered by the closed convex pointed cone K. If the net $(x_i)_{i \in I} \subset X$ is nonincreasing and convergent to $x \in X$, then $\{x_i : \; i \in I\}$ is bounded below and $x = \inf\{x_i : i \in I\}$.*

As a corollary we get

Proposition 7.2. *If the directional derivative $F'(x_0; h)$ exists, there exists also the infimum,*

$$\inf\left\{\frac{F(x_0 + th) - F(x_0)}{t} \; : \; t > 0\right\}.$$

On the other hand, in [21] we can find the following example. Let $X = l^\infty$ and let us consider partial order generated by the cone $l^\infty_+ := \{x \in l^\infty : x^k \geq 0 \; \forall k \geq 1\}$, l^∞_+ is a pointed closed convex cone. The sequence $\{x_n\} \subset l^\infty$ defined (for n fixed) $x^k_n := \begin{cases} -1 & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n, \end{cases}$ is nonincreasing, and $\inf\{x_n : n \geq 1\} = (-1, -1, -1, \ldots)$. But $\{x_n\}$ does not converge to its infimum.

Observe that we proved the cone convexity of the mapping F constructed in the proof of Theorem 5.2 (and in the proof of Theorem 6.2) for the cone K

$$K = \{y \in Y | \;\; z^*(y) \geq 0 \;\text{ for all } z^* \in K^*\}.$$

This cone is not pointed, henceforth the concept of the infimum is not well-defined. Let us note that the concept of directional derivative introduced in Definition 5.1 is well-defined independently on whether the cone K is pointed or not.

References

[1] F. Albiac, N. J. Kalton: Topics in Banach Spaces, Springer-Verlag, Berlin (2000).

[2] S. Banach: Théorie des Opérations Linéaires, Monografie Matematyczne Vol. I, Warszawa (1932).

[3] C. Bilek, C. Jermann, A. Neumaier: Global Optimization and Constraint Satisfaction, Springer-Verlag, Berlin (2003).

[4] N. Bourbaki: Algebra II, Chapters 4-7, Springer-Verlag, Berlin-Heidelberg-Tokyo (1980).

[5] E. Casini, E. Miglierina: Cones with bounded and unbounded bases and reflexivity, Nonlinear Analysis 72 (2010) 2356–2366.

[6] J. B. Conway: A Course in Functional Analysis, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York (1990).

[7] C. Costara, D. Popa: Exercises in Functional Analysis, Springer Science and Business Media, New York (2003).

[8] L. M. Graña Drummond, F. M. P. Raupp, B. F. Svaiter: A quadratically convergent Newton method for vector optimization, Optimization 63(5) (2014) 661–677.

[9] J. Diestel: Sequences and Series in Banach Spaces, Springer-Verlag, Berlin-Heidelberg-New York (1984).

[10] R. Engelking: General Topology, Heldermann Verlag, Berlin (1989).

[11] W. A. Eberlein: Weak compactness in Banach spaces, Proc. Nat. Acad. Sci. USA. 33 (1947) 317–328.

[12] J. Jahn: Mathematical Vector Optimization in Partially Ordered Linear Spaces, Verlag Peter Lang, Frankfurt am Main (1986).

[13] K. D. Joshi: Introduction to General Topology, Wiley Eastern (1984).

[14] M. I. Kadets, A. Pełczyński: Bases sequences, biorthogonal systems and norming sets in Banach and Fréchet spaces, Studia Math. 25 (1965) 297–323.

[15] A. A. Khan, C. Tammer, C. Zălinescu: Set-Valued Optimization. An Introduction with Applications, Springer-Verlag, Berlin (2015).

[16] T. Konderla: A construction of convex functions, Math. Notes 91(1) (2012) 74–78.

[17] Pei-Kee Lin: Kothe-Bochner Function Spaces, Springer Science and Business Media, New York (2014).

[18] J. Lindenstrauss, L. Tzafriri: Classical Banach Spaces I and II, Springer-Verlag, Berlin (1996).

[19] A. Pełczyński: A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces", Studia Mathematica XXI (1962) 371–374.

[20] T. Pennanen, J. Eckstein: Generalized Jacobians of Vector-Valued Convex Functions, RUTCOR Research Report (1997).

[21] A. L. Peressini: Ordered Topological Vector Spaces, Harper & Row, New York-Evanston-London (1967).

[22] H. P. Rosenthal: A Characterization of Banach Spaces Containing l^1, Proc. Nat. Acad. Sci. USA 71 (1974) 2411–2413.

[23] M. Valadier: Sous-différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné, Math. Sand. 30 (1972) 65–74.

[24] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, New Jersey-London-Singapore-Hong Kong (2002).