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Strong Convergence Theorems for an Infinite Family of Demimetric Mappings in a Hilbert Space

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Using the idea of Halpern iteration, we prove a strong convergence theorem for finding a common fixed point of an infinite family of demimetric mappings in a Hilbert space. Using this result, we obtain well-known and new strong convergence theorems in a Hilbert space.

Keywords: Common fixed point, demimetric mapping, metric projection, inverse strongly monotone mapping, Halpern iteration.

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1. Introduction

Let H be a real Hilbert space, let C be a nonempty, closed and convex subset of H and let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T. Let k be a real number with $0 \leq k < 1$. A mapping $T : C \to H$ is called a *k-strict pseudo-contraction* [7] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2, \quad \forall x, y \in C. \tag{1}$$

If $k = 0$ in (1), T is nonexpansive. The following strong convergence theorem of Halpern iteration for nonexpansive mappings was proved by Wittmann [34]; see also [11, 25].

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Theorem 1.1 ([34]). *Let C be a nonempty, closed and convex subset of H and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by $x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n$ for $n = 1, 2, ...$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping $F : C \to H$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see, for instance, Browder [5] and Goebel and Kirk [10]. It is also known that a firmly nonexpansive mapping F is deduced from an equilibrium problem in a Hilbert space; see [9]. Kohsaka and Takahashi [19] introduced the following nonlinear mapping: A mapping $S : C \to H$ is called *nonspreading* if

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2$$

for all $x, y \in C$. A nonspreading mapping was first defined in a Banach space; see [18]. A nonspreading mapping in a Hilbert space is also deduced from a firmly nonexpansive mapping; see [13]. Kurokawa and Takahashi [20] proved the following strong convergence theorem of Halpern's type iteration for nonspreading mappings in a Hilbert space. They also posed a problem whether Theorem 1.1 holds or not for nonspreading mappings.

Theorem 1.2 ([20]). *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let T be a nonspreading mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, ...$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu, where P is the metric projection of H onto $F(T)$.

Motivated by nonexpansive mappings and nonspreading mappings, Aoyama, Iemoto, Kohsaka and Takahashi [3] introduced a class of mappings called λ-hybrid. Let $\lambda \in \mathbb{R}$ be given. A mapping $T : C \to H$ is called *λ-hybrid* [3] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1-\lambda)\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C.$$

Kocourek, Takahashi and Yao [16] introduced a wider class of nonlinear mappings containing λ-hybrid mappings. A mapping $U : C \to H$ is called *generalized hybrid* [16] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2 \qquad (2)$$

for all $x, y \in C$. Such a mapping U is called *(α, β)-generalized hybrid.* Aoyama [2] and Hojo and Takahashi [12] extended Theorem 1.2 to λ-hybrid mappings and (α, β)-generalized hybrid mappings, respectively. Kohsaka [17] proved the following nonlinear ergodic theorem for two λ-hybrid mappings in a Hilbert space. This result is a generalization of Aoyama [2].

Theorem 1.3 ([17]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S : C \to C$ be a λ-hybrid mapping for some $\lambda \in \mathbb{R}$ and let $T : C \to C$ be a μ-hybrid mapping for some $\mu \in \mathbb{R}$. Suppose that $ST = TS$ and $F(S) \cap F(T)$ is nonempty. Let $u, x \in C$ and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x, \quad \forall n \in 0 \cup \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence of $[0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a point of $F(S) \cap F(T)$.

Recently, Takahashi [28] introduced the notion of new nonlinear mappings. Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called *η-demimetric* [28] if, for any $x \in C$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2,$$

where J is the duality mapping on E. In a Hilbert space H, this notion is as follows: A mapping $U : C \to H$ with $F(U) \neq \emptyset$ is called *η-demimetric* if, for any $x \in C$ and $q \in F(U)$,

$$\langle x - q,\, x - Ux\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2.$$

We have that the class of η-demimetric mappings covers strict pseudo-contractions and generalized hybrid mappings.

In this paper, using the idea of Halpern iteration, we prove a strong convergence theorem for finding a common fixed point of an infinite family of demimetric mappings in a Hilbert space. This theorem is proved without assuming that demimetric mappings are commutative. Using this result, we obtain well-known and new strong convergence theorems in a Hilbert space. In particular, we solve a problem which was posed by Kurokawa and Takahashi [20].

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. When $\{x_n\}$ is a sequence in H, we denote the *strong convergence* of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the *weak convergence* by $x_n \rightharpoonup x$. We have from [26] that, for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{3}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{4}$$

Furthermore, we have that, for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{5}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. A mapping $T : C \to H$ is called *quasi-nonexpansive* if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see [15, 26]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the *metric projection* of H onto D. We know that the metric projection P_D is firmly nonexpansive;

$$\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_D x, y - P_D x\rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [24, 26]. Using this inequality and (5), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \quad y \in D. \tag{6}$$

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $A : C \to H$ is called *inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Such a mapping A is called *α-inverse strongly monotone*. If $A : C \to H$ is α-inverse-strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned}
\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\
&= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\alpha\|Ax-Ay\|^2 + \lambda^2\|Ax-Ay\|^2 \\
&= \|x-y\|^2 + \lambda(\lambda - 2\alpha)\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Thus $I - \lambda A$ is nonexpansive; see [1, 23, 26] for more results of inverse strongly monotone mappings.

We give the following examples of demimetric mappings.

Examples

(1) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let k be a real number with $0 \leq k < 1$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is k-demimetric; see [28].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 0-demimetric. In fact, setting $x = u \in F(U)$ and $y = x \in C$ in (2), we have that

$$\alpha\|u - Ux\|^2 + (1-\alpha)\|u - Ux\|^2 \leq \beta\|u - x\|^2 + (1-\beta)\|u - x\|^2$$

and hence $\|Ux - u\|^2 \leq \|x - u\|^2$. From

$$\|Ux - x + x - u\|^2 = \|Ux - x\|^2 + 2\langle Ux - x, x - u\rangle + \|x - u\|^2\,,$$

we have that

$$2\langle x - u, x - Ux\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $u \in F(U)$. This means that U is 0-demimetric. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is nonspreading [18, 19] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [27] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [14]. We also know that the notion of generalized hybrid mappings is a generalization of the notion of λ-hybrid mappings. In fact, if T is λ-hybrid, then

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1-\lambda)\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

We have from (5) that

$$\begin{aligned}&\|Tx - Ty\|^2 \leq \\ &\quad \leq \|x - y\|^2 + (1-\lambda)(\|x - Ty\|^2 + \|Tx - y\|^2 - \|x - y\|^2 - \|Tx - Ty\|^2)\end{aligned}$$

and hence

$$(2-\lambda)\|Tx - Ty\|^2 - (1-\lambda)\|x - Ty\|^2 \leq (1-\lambda)\|Tx - y\|^2 + \lambda\|x - y\|^2.$$

This means that T is $(2-\lambda, 1-\lambda)$-generalized hybrid.

(3) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be an α-inverse strongly monotone mapping with $A^{-1}0 \neq \emptyset$. Then $1 - 2\alpha \in (-\infty, 0)$ and $I - A : C \to H$ is a $(1 - 2\alpha)$-demimetric mapping. In fact, since $A : C \to H$ is α-inverse strongly monotone, we have that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{7}$$

Setting $U = I - A$ and taking $y = z \in F(U) = A^{-1}0$ in (7), we have that

$$\langle x - z, x - Ux \rangle \geq \alpha \|x - Ux\|^2, \quad \forall x \in C, \ z \in F(U).$$

This implies that

$$2\langle x - z, x - Ux \rangle \geq (1 - (1 - 2\alpha))\|x - Ux\|^2, \quad \forall x \in C, \ z \in F(U)$$

and hence $U = I - A$ is $(1 - 2\alpha)$-demimetric.

(4) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric; see [28].

(5) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric; see [28].

The following lemma which was proved in [28] is important and crucial in the proof of our main result.

Lemma 2.1 ([28]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $k \in (-\infty, 1)$ and let U be a k-demimetric mapping of C into H. Then $F(U)$ is closed and convex.*

We also know the following lemma from [29]:

Lemma 2.2 ([29]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$ and let T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1 - k$ and define $S = (1 - \lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.*

We also know the following lemmas from Aoyama, Kimura, Takahashi and Toyoda [4], Xu [33] and Maingé [21].

Lemma 2.3 ([4], [33]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 ([21]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ satisfies $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \cdots$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0$.

3. Strong Convergence Theorem

In this section, we prove a strong convergence theorem of Halpern's type iteration for finding a common fixed point of an infinite family of demimetric mappings in a Hilbert space. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called *demiclosed* if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - Ux_n \to 0$, $w = Uw$ holds. For example, if C is a nonempty, closed and convex subset of H and T is a nonexpansive mapping of C into H, then T is demiclosed; see [6] and [26, p. 114].

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{k_1, k_2, \dots\} \subset (-\infty, 1)$ and let $\{T_j\}_{j=1}^{\infty}$ be an infinite family of k_j-demimetric and demiclosed mappings of C into H. Assume that $F := \cap_{j=1}^{\infty} F(T_j) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. For $x_1 = x \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n) P_C(\alpha_n x_n + (1-\alpha_n)z_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\xi_1, \xi_2, \dots\} \subset (0, 1)$ and $\{\alpha_n\}, \{\delta_n\} \subset (0, 1)$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq \inf\{1 - k_1, 1 - k_2, \dots\}$;

(2) $\sum_{j=1}^{\infty} \xi_j = 1$;

(3) $0 < b \leq \alpha_n \leq c < 1$;

(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ converges strongly to $z_0 \in F$, where $z_0 = P_F u$.

Proof. For $\{\lambda_n\} \subset (0, \infty)$ and $\{\xi_1, \xi_2, \dots\} \subset (0, 1)$ such that

$$0 < \lambda_n \leq \inf\{1 - k_1, 1 - k_2, \dots\}$$

and $\sum_{j=1}^{\infty} \xi_j = 1$, respectly, define

$$S_n x = \sum_{j=1}^{\infty} \xi_j((1-\lambda_n)I + \lambda_n T_j)x$$

for all $n \in \mathbb{N}$ and $x \in C$; see Bruck [8]. Then we can prove that S_n is well defined. In fact, since $(1-\lambda_n)I + \lambda_n T_j$ is quasi-nonexpansive from Lemma 2.2, we have that, for $y_0 \in \cap_{j=1}^{\infty} F(T_j) = \cap_{j=1}^{\infty} F((1-\lambda_n)I + \lambda_n T_j)$,

$$\|((1-\lambda_n)I + \lambda_n T_j)x\| \leq \|((1-\lambda_n)I + \lambda_n T_j)x - y_0\| + \|y_0\| \leq \|x - y_0\| + \|y_0\|.$$

Thus $S_n x = \sum_{j=1}^{\infty} \xi_j((1-\lambda_n)I + \lambda_n T_j)x$ converges absolutely for each x in C. Furthermore, we know from Lemma 2.1 that $F(T_j)$ is closed and convex. Then we have that $\cap_{j=1}^{\infty} F(T_j)$ is nonempty, closed and convex. Thus we have that $P_{\cap_{j=1}^{\infty} F(T_j)}$ is well defined.

Since $(1-\lambda_n)I + \lambda_n T_j$ are quasi-nonexpansive from Lemma 2.2, we have that, for $z \in \cap_{j=1}^{\infty} F(T_j)$,

$$\begin{aligned} \|z_n - z\| &= \|\sum_{j=1}^{\infty} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n - z\| \\ &\leq \sum_{j=1}^{\infty} \xi_j \|((1-\lambda_n)I + \lambda_n T_j)x_n - z\| \\ &\leq \sum_{j=1}^{\infty} \xi_j \|x_n - z\| = \|x_n - z\|. \end{aligned} \tag{8}$$

Put $y_n = P_C(\alpha_n x_n + (1-\alpha_n)z_n)$. Then we have that

$$\begin{aligned} \|y_n - z\| &\leq \|\alpha_n x_n + (1-\alpha_n)z_n - z\| \\ &\leq \alpha_n \|x_n - z\| + (1-\alpha_n)\|z_n - z\| \\ &\leq \alpha_n \|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned} \tag{9}$$

Using this, we get that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\delta_n(u_n - z) + (1-\delta_n)(y_n - z)\| \\ &\leq \delta_n \|u_n - z\| + (1-\delta_n)\|y_n - z\| \\ &\leq \delta_n \|u_n - z\| + (1-\delta_n)\|x_n - z\|. \end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n \in \mathbb{N}} \|u_n - z\| \leq M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \leq K$. Suppose that $\|x_k - z\| \leq K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned} \|x_{k+1} - z\| &\leq \delta_k \|u_k - z\| + (1-\delta_k)\|x_k - z\| \\ &\leq \delta_k K + (1-\delta_k)K = K. \end{aligned}$$

By induction, we obtain that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Take $z_0 = P_{\cap_{j=1}^{\infty} F(T_j)} u$. Using (4), we have that

$$\begin{aligned}\|y_n - z_0\|^2 &\leq \|\alpha_n x_n + (1-\alpha_n) z_n - z_0\|^2 \\ &= \alpha_n \|x_n - z_0\|^2 + (1-\alpha_n)\|z_n - z_0\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \\ &\leq \alpha_n \|x_n - z_0\|^2 + (1-\alpha_n)\|x_n - z_0\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \\ &= \|x_n - z_0\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\delta_n(u_n - z_0) + (1-\delta_n)(y_n - z_0)\|^2 \\ &\leq \delta_n\|u_n - z_0\|^2 + (1-\delta_n)\|y_n - z_0\|^2 \\ &\leq \delta_n\|u_n - z_0\|^2 + \|y_n - z_0\|^2 \\ &\leq \delta_n\|u_n - z_0\|^2 + \|x_n - z_0\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 .\end{aligned}$$

Using $0 < b \leq \alpha_n \leq c < 1$, we have that

$$\begin{aligned}b(1-c)\|x_n - z_n\|^2 &\leq \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \\ &\leq \delta_n\|u_n - z_0\|^2 + \|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2.\end{aligned} \tag{10}$$

We also have that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \delta_n\|u_n - x_n\| + (1-\delta_n)\|y_n - x_n\| \\ &\leq \delta_n\|u_n - x_n\| + \|y_n - x_n\| \\ &\leq \delta_n\|u_n - x_n\| + \|\alpha_n x_n + (1-\alpha_n)z_n - x_n\| \\ &\leq \delta_n\|u_n - x_n\| + \|(1-\alpha_n)(z_n - x_n)\| \\ &\leq \delta_n\|u_n - x_n\| + \|z_n - x_n\|.\end{aligned} \tag{11}$$

We will divide the proof into two cases.

Case 1: Put $\Gamma_n = \|x_n - z_0\|^2$ for all $n \in \mathbb{N}$. Suppose that there exists a natural number N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\lim_{n\to\infty} \Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using $\delta_n \to 0$, we have from (10) that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{12}$$

From (11) and $\delta_n \to 0$, we also have that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{13}$$

For $z_0 = P_{\cap_{j=1}^{\infty} F(T_j)} u$, we show that

$$\limsup_{n\to\infty} \langle u - z_0, x_n - z_0 \rangle \leq 0.$$

Put $s = \limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle$. Without loss of generality, there exists a subsequence $\{x_l\}$ of $\{x_n\}$ such that $s = \lim_{l\to\infty}\langle u - z_0, x_l - z_0\rangle$ and $\{x_l\}$ converges weakly to some point $w \in C$. On the other hand, since T_j is k_j-demimetric for all $j \in \{1, 2, \dots\}$, we have that, for $z \in \cap_{j=1}^{\infty} F(T_j)$,

$$\begin{aligned}
\langle x_n - z, x_n - z_n\rangle &= \langle x_n - z, x_n - \sum_{j=1}^{\infty}\xi_j((1-\lambda_n)I + \lambda_n T_j)x_n\rangle \\
&= \sum_{j=1}^{\infty}\xi_j\langle x_n - z, x_n - ((1-\lambda_n)I + \lambda_n T_j)x_n\rangle \\
&= \sum_{j=1}^{\infty}\xi_j\lambda_n\langle x_n - z, x_n - T_j x_n\rangle \\
&\geq \sum_{j=1}^{\infty}\xi_j\lambda_n\frac{1-k_j}{2}\|x_n - T_j x_n\|^2 \\
&\geq \sum_{j=1}^{\infty}\xi_j a\frac{1-k_j}{2}\|x_n - T_j x_n\|^2.
\end{aligned}$$

We have from $\lim_{n\to\infty}\|z_n - x_n\| = 0$ that

$$\lim_{n\to\infty}\|x_n - T_j x_n\| = 0, \quad \forall j \in \{1, 2, \dots\}.$$

Since T_j are demiclosed for all $j \in \{1, 2, \dots\}$, we have $w \in \cap_{j=1}^{\infty} F(T_j)$.

Since $\{x_l\}$ converges weakly to $w \in \cap_{j=1}^{\infty} F(T_j)$, we have that

$$s = \lim_{l\to\infty}\langle u - z_0, x_l - z_0\rangle = \langle u - z_0, w - z_0\rangle \leq 0.$$

Since $x_{n+1} - z_0 = \delta_n(u_n - z_0) + (1-\delta_n)(y_n - z_0)$, we have from (3) that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq (1-\delta_n)^2\|y_n - z_0\|^2 + 2\delta_n\langle u_n - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\delta_n)\|x_n - z_0\|^2 + 2\delta_n\langle u_n - u, x_{n+1} - z_0\rangle \\
&\qquad + 2\delta_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&= (1-\delta_n)\|x_n - z_0\|^2 + 2\delta_n\langle u_n - u, x_{n+1} - z_0\rangle \\
&\qquad + 2\delta_n\langle u - z_0, x_{n+1} - x_n\rangle + 2\delta_n\langle u - z_0, x_n - z_0\rangle.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\delta_n = \infty$, we obtain from Lemma 2.3 that $x_n \to z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.4 that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. Thus we have from (10) that for all $n \in \mathbb{N}$,

$$\begin{aligned} b(1-c)\|x_{\tau(n)} - z_{\tau(n)}\|^2 &\leq \\ &\leq \delta_{\tau(n)}\|u_{\tau(n)} - z_0\|^2 + \left\|x_{\tau(n)} - z_0\right\|^2 - \|x_{\tau(n)+1} - z_0\|^2 \\ &\leq \delta_{\tau(n)}\|u_{\tau(n)} - z_0\|^2. \end{aligned} \tag{14}$$

Using $\delta_{\tau(n)} \to 0$, we have from (14) that

$$\lim_{n\to\infty} \|z_{\tau(n)} - x_{\tau(n)}\| = 0.$$

As in the proof of Case 1, we have from $\lim_{n\to\infty} \|z_{\tau(n)} - x_{\tau(n)}\| = 0$ that

$$\lim_{n\to\infty} \|x_{\tau(n)} - T_j x_{\tau(n)}\| = 0, \quad \forall j \in \{1, 2, \dots\}. \tag{15}$$

As in the proof of Case 1, we also have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{16}$$

For $z_0 = P_{\cap_{j=1}^\infty F(T_j)} u$, let us show that

$$\limsup_{n\to\infty} \langle z_0 - u, x_{\tau(n)} - z_0\rangle \geq 0.$$

Put $s = \limsup_{n\to\infty}\langle z_0 - u, x_{\tau(n)} - z_0\rangle$. Without loss of generality, there exists a subsequence $\{x_{\tau(l)}\}$ of $\{x_{\tau(n)}\}$ such that $s = \lim_{l\to\infty}\langle z_0 - u, x_{\tau(l)} - z_0\rangle$ and $\{x_{\tau(l)}\}$ converges weakly to some point $w \in C$. Since T_j are demiclosed for all $j \in \{1, 2, \dots\}$, we have from (15) that $w \in \cap_{j=1}^\infty F(T_j)$. Since $\{x_{\tau(l)}\}$ converges weakly to $w \in \cap_{j=1}^\infty F(T_j)$, we have

$$s = \lim_{l\to\infty}\langle z_0 - u, x_{\tau(l)} - z_0\rangle = \langle z_0 - u, w - z_0\rangle \geq 0.$$

As in the proof of Case 1, we also have that

$$\begin{aligned} \|x_{\tau(n)+1} - z_0\|^2 \leq (1 - \delta_{\tau(n)})\|x_{\tau(n)} - z_0\|^2 &+ 2\delta_{\tau(n)}\langle u_{\tau(n)} - u, x_{\tau(n)+1} - z_0\rangle \\ + 2\delta_{\tau(n)}\langle u - z_0, x_{\tau(n)+1} - x_{\tau(n)}\rangle &+ 2\delta_{\tau(n)}\langle u - z_0, x_{\tau(n)} - z_0\rangle. \end{aligned}$$

From $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have that

$$\begin{aligned} \delta_{\tau(n)}\|x_{\tau(n)} - z_0\|^2 \leq\ & 2\delta_{\tau(n)}\langle u_{\tau(n)} - u, x_{\tau(n)+1} - z_0\rangle \\ &+ 2\delta_{\tau(n)}\langle u - z_0, x_{\tau(n)+1} - x_{\tau(n)}\rangle + 2\delta_{\tau(n)}\langle u - z_0, x_{\tau(n)} - z_0\rangle. \end{aligned}$$

Since $\delta_{\tau(n)} > 0$, we have that

$$\begin{aligned} \|x_{\tau(n)} - z_0\|^2 \leq\ & 2\langle u_{\tau(n)} - u, x_{\tau(n)+1} - z_0\rangle \\ &+ 2\langle u - z_0, x_{\tau(n)+1} - x_{\tau(n)}\rangle + 2\langle u - z_0, x_{\tau(n)} - z_0\rangle. \end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty} \left\|x_{\tau(n)} - z_0\right\|^2 \leq 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$. From (16), we have also that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.4 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

4. Applications

In this section, we apply Theorem 3.1 to obtain well-known and new strong convergence theorems in Hilbert spaces. We know the following lemmas obtained by Marino and Xu [22] and Kocourek, Takahashi and Yao [16]; see also [30, 32].

Lemma 4.1 ([22, 30]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Lemma 4.2 ([16, 32]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

The following is a strong convergence theorem for an infinite family of strict pseudo-contractions in a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{k_1, k_2, \dots\} \subset [0, 1)$ and let $\{T_j\}_{j=1}^{\infty}$ be an infinite family of k_j-strict pseudo-contractions of C into H. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Assume that $F := \cap_{j=1}^{\infty} F(T_j) \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)P_C(\alpha_n x_n + \beta_n z_n), \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\xi_1, \xi_2, \dots\} \subset (0, 1)$ and $\{\alpha_n\}, \{\delta_n\} \subset (0, 1)$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq \inf\{1 - k_1, 1 - k_2, \dots\}$;
(2) $\sum_{j=1}^{\infty} \xi_j = 1$;
(3) $0 < b \leq \alpha_n \leq c < 1$;
(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ converges strongly to $z_0 \in F$, where $z_0 = P_F u$.

Proof. Since T_j is a k_j-strict pseudo-contraction of C into H such that $F(T_j) \neq \emptyset$, from (1) in Examples, T_j is k_j-demimetric. Furthermore, from Lemma 4.1, T_j is demiclosed. Thus, we have the desired result from Theorem 3.1. $\square$

We also know the following result from [31].

Lemma 4.4 ([31]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = 2\alpha\, U + (1 - 2\alpha)I$. Then, the following are equivalent:*

(1) *A is an α-inverse strongly monotone mapping, i.e.,*

$$\alpha\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle, \quad \forall x, y \in C;$$

(2) *U is a widely $(1 - 2\alpha)$-strict pseudo-contraction, i.e.,*

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + (1 - 2\alpha)\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C;$$

(3) *T is a nonexpansive mapping.*

Using Lemma 4.4, we can prove the following weak convergence theorem for an infinite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 4.5. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{\mu_1, \mu_2, \dots\} \subset (0, \infty)$ and let $\{B_j\}_{j=1}^{\infty}$ be an infinite family of μ_j-inverse strongly monotone mappings of C into H. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Assume that $B := \cap_{j=1}^{\infty} B_j^{-1}0 \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j (I - \lambda_n B_j) x_n, \\ x_{n+1} = \delta_n u_n + (1 - \delta_n) P_C(\alpha_n x_n + (1 - \alpha_n) z_n), \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\xi_1, \xi_2, \dots\} \subset (0, 1)$ and $\{\alpha_n\}, \{\delta_n\} \subset (0, 1)$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq 2 \inf\{\mu_1, \mu_2, \dots\}$;
(2) $\sum_{j=1}^{\infty} \xi_j = 1$;
(3) $0 < b \leq \alpha_n \leq c < 1$;
(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ converges strongly to $z_0 \in B$, where $z_0 = \lim_{n\to\infty} P_B x_n$.

Proof. Since B_j is μ_j-inverse strongly monotone, from (3) in Examples, $U_j = I - B_j$ is $(1 - 2\mu_j)$-demimetric. Furthermore, we have that

$$I - \lambda_n B_j = I - \lambda_n(I - (I - B_j)) = (1 - \lambda_n)I + \lambda_n(I - B_j) = (1 - \lambda_n)I + \lambda_n U_j$$

and $0 < a \leq \lambda_n \leq \inf\{1-(1-2\mu_1), 1-(1-2\mu_2), \ \ldots\} = 2\inf\{\mu_1, \mu_2, \ \ldots\}$. Furthermore, $U_j = I - B_j$ is demiclosed. In fact, let $x_n \rightharpoonup w$ and $x_n - U_j x_n \to 0$. Putting $T_j = 2\mu_j U_j + (1-2\mu_j)I$, we have that

$$x_n - U_j x_n = (I - U_j)x_n = \frac{1}{2\mu_j}(I - T_j)x_n \to 0.$$

Since T_j is nonexpansive from Lemma 4.4 and hence T_j is demiclosed, we have $w \in F(T_j) = F(U_j)$. Thus $U_j = I - B_j$ is demiclosed. Then we have the desired result from Theorem 3.1. $\square$

The following is a strong convergence theorem for an infinite family of generalized hybrid mappings in a Hilbert space.

Theorem 4.6. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{T_j\}_{j=1}^{\infty}$ be an infinite family of generalized hybrid mappings of C into H. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Assume that $F := \cap_{j=1}^{\infty} F(T_j) \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j T_j x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n) P_C(\alpha_n x_n + (1-\alpha_n) z_n), \end{cases}$$

where $b, c \in \mathbb{R}$, $\{\xi_1, \xi_2, \ldots\} \subset (0,1)$ and $\{\alpha_n\}, \{\delta_n\} \subset (0,1)$ satisfy the following conditions:

(1) $\sum_{j=1}^{\infty} \xi_j = 1$;
(2) $0 < b \leq \alpha_n \leq c < 1$;
(3) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ converges strongly to $z_0 \in F$, where $z_0 = P_F u$.

Proof. Since T_j is a generalized hybrid mapping of C into H such that $F(T_j) \neq \emptyset$, from (2) in Examples, T_j is 0-demimetric. Furthermore, from Lemma 4.2, T_j is demiclosed. Therefore, we have the desired result from Theorem 3.1. $\square$

We have not known whether Theorem 1.1 of the Halpern iteration holds or not for nonspreading mappings. Such a problem was also posed by Hojo and Takahashi [12] in the setting of generalized hybrid mappings. As a direct consequence of Theorem 4.6, we solve this problem of Hojo and Takahashi.

Theorem 4.7. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a generalized hybrid mapping of C into H. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Assume that $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$x_{n+1} = \delta_n u_n + (1-\delta_n) P_C(\alpha_n x_n + (1-\alpha_n) T x_n),$$

where $b, c \in \mathbb{R}$ and $\{\alpha_n\}, \{\delta_n\} \subset (0,1)$ satisfy the following conditions:

(1) $0 < b \leq \alpha_n \leq c < 1$;

(2) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ *converges strongly to* $z_0 \in F(T)$, *where* $z_0 = P_{F(T)}u$.

The following is a strong convergence theorem for an infinite family of nonexpansive mappings in a Hilbert space.

Theorem 4.8. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $\{T_j\}_{j=1}^{\infty}$ *be an infinite family of nonexpansive mappings of* C *into* H. *Let* $\{u_n\}$ *be a sequence in* C *such that* $u_n \to u$. *Assume that* $F := \cap_{j=1}^{\infty} F(T_j) \neq \emptyset$. *For any* $x_1 = x \in C$, *define* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j T_j x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n) P_C(\alpha_n x_n + (1-\alpha_n) z_n), \end{cases}$$

where $b, c \in \mathbb{R}$, $\{\xi_1, \xi_2, \dots\} \subset (0,1)$ *and* $\{\alpha_n\}, \{\delta_n\} \subset (0,1)$ *satisfy the following conditions:*

(1) $\sum_{j=1}^{\infty} \xi_j = 1$;

(2) $0 < b \leq \alpha_n \leq c < 1$;

(3) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ *converges strongly to* $z_0 \in F$, *where* $z_0 = P_F u$.

Proof. As in the proof of Theorem 4.6, T_j is 0-demimetric and demiclosed. We get the desired result from Theorem 3.1. □

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