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Regularization via Sets Satisfying the Interior Sphere Condition

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For a given closed set $S \subset \mathbb{R}^n$, we provide inner approximation of S by sets satisfying the interior sphere condition. The fact that our approximation sets satisfy the interior sphere condition with variable radius, allows us to approach any corner and to use the Pompeiu-Hausdorff convergence even if the set S is unbounded.

Keywords: Regularization of sets, interior sphere condition, φ-convexity, Pompeiu-Hausdorff convergence, nonsmooth analysis.

1991 Mathematics Subject Classification: 35J62, 35J20, 35P30

1. Introduction

Let $S \subset \mathbb{R}^n$ be a nonempty and closed set. We recall that S is said to be satisfying the θ-*interior sphere condition* if there exists a positive and continuous function $\rho(\cdot)$ defined on the boundary of S such that for any boundary point s of S, one can find a closed ball in S of radius $\rho(s)$ containing s. If $\rho(\cdot) = \rho_0$ a positive constant, then we say that S satisfies a *uniform* interior sphere condition. The interior sphere condition is a well known condition in control theory, and is used especially in deriving regularity properties, more precisely the *semiconcavity* (see [4]), of the *minimal time function* associated

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to a given target set. Such results can be found in Cannarsa and Sinestrari [3] where the uniform interior sphere condition is assumed on the target set and in Cannarsa and Frankowska [2] and Sinestrari [21] where the uniform interior sphere condition is assumed on the dynamic. A generalization of these semiconcavity results to the non-Lipschitz case (that is, when the *Petrov condition* is not necessarily satisfied) can be found in Colombo, Marigonda and Wolenski [9] and Colombo and Nguyen [10].

Functions with hypographs satisfying the θ-interior sphere condition are studied in Nour, Stern and Takche [16] and Nguyen [14]. In the first paper, the authors proved that if a function f is locally Lipschitz, then there is an equivalence between the semiconcavity of f and the fact that the hypograph of f satisfies the uniform interior sphere condition. This led Nguyen to prove in [14] that a function with hypograph satisfying the uniform interior sphere condition enjoys some properties of a concave function, in particular a.e. twice differentiability. This result is applied in [14] to the minimal time function in the case of a nonlinear smooth dynamics and a target satisfying the uniform interior sphere condition. Another important result can be found in Nour, Stern and Takche [17] where the equivalence between the uniform interior sphere condition and the *union of closed balls property* is established. A generalisation of this result to the variable radius case can be found in Nour, Stern and Takche [18].

The goal of this paper is to provide, for a given nonempty and closed $S \subset \mathbb{R}^n$, conditions under which the set S can be approximated by sets $S_\varepsilon \subset S$ satisfying the θ-interior sphere condition. The fact that our approximation sets satisfy an interior sphere condition with variable radius allows us to approach any corner and to use the Pompeiu-Hausdorff convergence even if the set S is unbounded. Our results reached relate to results of Clarke, Ledyayev and Stern [5]. In this reference, the authors provided an inner approximation of a given *wedged* (or *epi-Lipschitz*) set S by φ_0-*convex* (or *proximally smooth*) sets. More precisely, Clarke, Ledyayev and Stern proved in [5] that if $S \subset \mathbb{R}^n$ is a nonempty, compact and wedged set then the r-*inner approximation* S_r is φ_0-convex, for r sufficiently small, with φ_0 depending on r (recall that S_r is the set of all points in S having a distance to S^c, the complement of S, greater than or equal to r). Some improvements of this result can be found in the works of Cornet and Czarnecki [11, 12]. Our main result differs from these results by several features. The main ones are that we do not make such wedgedness assumption on S and our approximation sets satisfy an interior sphere condition which is a different regularity property from the φ_0-convexity as is already shown in Nour, Stern and Takche [15].

After a section on our notations and preliminaries in nonsmooth analysis, our main results will be presented in Section 3.

2. Preliminaries

We begin this section by providing certain notations and definitions from nonsmooth analysis. Our general reference for these constructs is Clarke et al. [6]; see also Penot [19] and Rockafellar and Wets [20]. We denote by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, B and $\overline{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B(x;\rho) := x + \rho B$ and $\overline{B}(x;\rho) := x + \rho\overline{B}$. The segment joining two points x and y in $\mathbb{R}^n$ is denoted by $[x, y]$, that is,

$$[x, y] := \{(1-\lambda)x + \lambda y : \lambda \in [0, 1]\}.$$

We also denote, for $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, by $[A, B]$ the set of all segments joining a point in A to a point in B, that is,

$$[A, B] := \{[a, b] : a \in A \text{ and } b \in B\}.$$

Similarly we define:

$$[x, y[:= \{(1-\lambda)x + \lambda y : \lambda \in [0, 1[\} \quad \text{and} \quad [A, B[:= \{[a, b[: a \in A \text{ and } b \in B\}.$$

For a set $S \subset \mathbb{R}^n$, S^c, $\operatorname{int} S$, ∂S and $\operatorname{cl} S$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary and the closure of S, respectively. The set S^*, called *star of* S, is defined by:

$$S^* := \{s \in S : [s, x] \subset S \text{ for all } x \in S\}.$$

The distance from a point x to a set S is denoted by $d_S(x)$. We also denote by $\operatorname{proj}_S(x)$ the set of closest points in S to x, that is, the set of points $s \in S$ which satisfy $d_S(x) = \|s - x\|$. The *Pompeiu-Hausdorff distance* between two sets A and B will be denoted by $d_H(A, B)$. We terminate this part by introducing a new function, called *directional distance*, that will play an important role in our study. For $x \in \mathbb{R}^n$ a point and $\zeta \in \mathbb{R}^n$ a vector, the directional distance from x to a closed set $S \subset \mathbb{R}^n$ in the direction ζ, denoted by $d_S(x, \zeta)$, is defined by

$$d_S(x, \zeta) := \min\{t \geq 0 : x + t\zeta \in S\}.$$

Note that here the minimum of an empty set is taken to be ∞.

Now let S be a nonempty and closed subset of $\mathbb{R}^n$. For $x \in S$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to* S *at* x provided that there exists $\sigma = \sigma(x, \zeta) \geq 0$ such that

$$\langle \zeta, s - x\rangle \leq \sigma\|s - x\|^2 \quad \forall s \in S. \tag{1}$$

The relation (1) is commonly referred to as the *proximal normal inequality.* No nonzero ζ satisfying (1) exists if $x \in \operatorname{int} S$, but this may also occur for $x \in \partial S$, as is the case when S is the epigraph of the function $f(z) = -|z|$ and $x = (0, 0)$. For such a point, the only proximal normal is $\zeta = 0$. In view of (1), the set of all proximal normals to S at x is a convex cone, and we denote it by $N_S^P(x)$. Now let $x \in \partial S$, $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ be such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}\,;\, r\right) \cap S = \emptyset. \tag{2}$$

Then ζ is a proximal normal to S at x and we say that ζ is *realized by an r-sphere*. Note that ζ is then also realized by an r'-sphere for any $0 < r' < r$. One can show that ζ being realized by an r-sphere is equivalent to the proximal normal inequality holding with $\sigma = 1/2r$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}\,,\, s - x \right\rangle \leq \frac{1}{2r}\|s - x\|^2 \quad \forall s \in S. \tag{3}$$

We proceed to introduce two well-known regularity properties for a nonempty and closed subset S of $\mathbb{R}^n$: the θ-interior sphere condition and the φ-convexity. For more information about these two properties, we invite the reader to see the papers [1, 8, 16, 17, 18].

Definition 2.1. A nonempty and closed set $S \subset \mathbb{R}^n$ is said to satisfy the *θ-interior sphere condition* if and only if there exists a continuous function $\theta : \partial S \longrightarrow [0, +\infty[$ such that for all $x \in \partial S$ one can find a point $y_x \in S$ satisfying:

- $x \in \bar{B}\left(y_x; \frac{1}{2\theta(x)}\right) \subset S$, if $\theta(x) > 0$.
- $x \in \bar{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$, if $\theta(x) = 0$.

If $\theta(\cdot) = \theta_0$ a positive constant, then the θ_0-interior sphere condition coincides with the well known condition in control theory, the *uniform interior sphere condition*. This condition is important in deriving regularity properties of the minimal time function; see e.g. Cannarsa and Frankowska [2], Cannarsa and Sinestrari [2, 3] and Sinestrari [21].

Definition 2.2. A nonempty and closed set $S \subset \mathbb{R}^n$ is said to be *φ-convex* if and only if there exists a continuous function $\varphi : \partial S \longrightarrow [0, +\infty[$ such that for all $x \in \partial S$ and for all $0 \neq \zeta \in N_S^P(x)$ we have

- ζ is realized by a $\frac{1}{2\varphi(x)}$-sphere, if $\varphi(x) \neq 0$,
- ζ is realized by an r-sphere for all $r > 0$, if $\varphi(x) = 0$.

If $\varphi(\cdot) = \varphi_0$ a positive constant, then the φ_0-convexity coincides with the *positive reach* property and the *proximal smoothness* property defined in Federer [13] and Clarke et al. [7], respectively. More precisely, a set S is φ_0-convex if and only if it is $\frac{1}{2\varphi_0}$-proximal smooth (or has positive reach with radius $\frac{1}{2\varphi_0}$).

3. Main result

The goal of this section is to present our main result, that is, a sufficient condition for the existence of an inner approximation for a nonempty and closed set $S \subset \mathbb{R}^n$ by sets satisfying the interior sphere condition. This section will be divided in two subsections. In the first one, we give a strong sufficient condition for a special case, see Theorem 3.1. A weaker sufficient condition will be given in the second subsection for a more general case, see Theorem 3.5. Since the proof of Theorem 3.1 will be a part of the proof of Theorem 3.5, we will provide the details of the proof of Theorem 3.1 and then sketch its similar part in the proof of Theorem 3.5 with less details. This makes the paper more readable and easier to understand, especially with the complicated notations that will appear in the general case.

3.1. Special case

The following is our first main result in which we prove that if $S \subset \mathbb{R}^n$ is a nonempty and closed set with $\operatorname{int} S^* \neq \emptyset$, then S can be innerly approximated by sets satisfying the interior sphere condition.

Theorem 3.1. *Let $S \subset \mathbb{R}^n$ be a nonempty and closed set such that $\operatorname{int} S^* \neq \emptyset$. Then for all $\varepsilon > 0$ small, there exists a nonempty and closed set $S_\varepsilon \subset S$ such that*

- *S_ε satisfies the θ-interior sphere condition,*
- *$d_H(S_\varepsilon, S) \leq \varepsilon$.*

Proof. Let S be a nonempty and closed set and assume that $\operatorname{int} S^* \neq \emptyset$. Then there exist $s_0 \in S$ and $\rho_0 > 0$ such that $\bar{B}(s_0; 2\rho_0) \subset S^*$ and $B(s_0; 3\rho_0) \subset \operatorname{int} S$. Clearly for $s \in S$ we have $[\bar{B}(s_0; 2\rho_0), s] \subset S$ and $[B(s_0; 2\rho_0), s[\subset \operatorname{int} S$.

Claim 1: The function $d_{\partial S}(s_0, u)$ is continuous with respect to the unit vector u. Moreover $d_{\partial S}(s_0, u) \geq 3\rho_0$ for all u unit.

Since $B(s_0; 3\rho_0) \subset \operatorname{int} S$ we get directly that

$$d_{\partial S}(s_0, u) \geq 3\rho_0 \text{ for all } u \text{ unit.} \tag{4}$$

Now let u_i be a sequence of unit vectors that converges to a unit vector u_0. We need to prove that $d_{\partial S}(s_0, u_i) \longrightarrow d_{\partial S}(s_0, u_0)$.

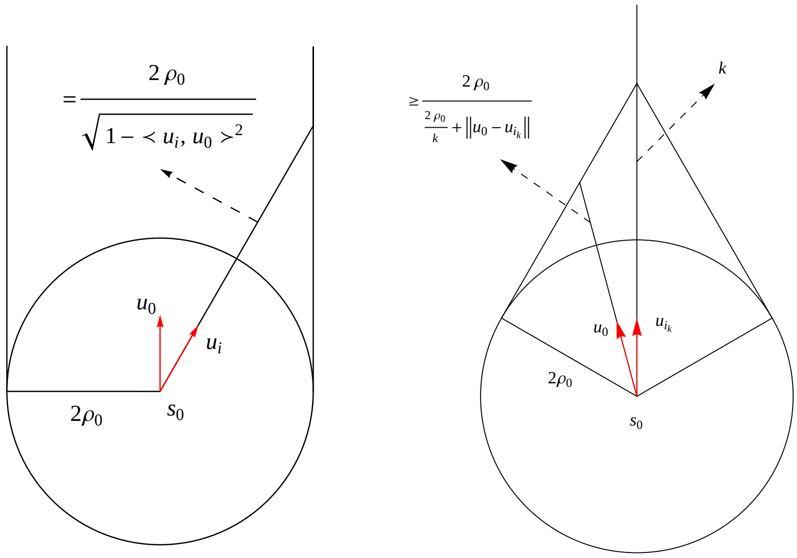
Case 1: $d_{\partial S}(s_0, u_0) = \infty$.

Then for all $t \geq 0$, we have that $[\bar{B}(s_0; 2\rho_0), s_0 + tu_0] \subset \operatorname{int} S$, and then

$$\bigcup_{t \geq 0} [\bar{B}(s_0; 2\rho_0), x_0 + tu_0] \subset \operatorname{int} S.$$

Therefore for i sufficiently large we get, see Figure 1(a), that

$$d_{\partial S}(s_0, u_i) \geq \frac{2\rho_0}{\sqrt{1 - \langle u_i, u_0 \rangle^2}}.$$



(a) $d_{\partial S}(s_0, u_0) = \infty$ (b) $d_{\partial S}(s_0, u_0) < \infty$

Figure 1

This gives, since $u_i \longrightarrow u_0$ and $\|u_0\| = 1$, that $d_{\partial S}(s_0, u_i) \longrightarrow \infty$.

Case 2: $d_{\partial S}(s_0, u_0) < \infty$.

We claim that, for i sufficiently large, the sequence $d_{\partial S}(s_0, u_i)$ is bounded. Indeed, if not then there exists a subsequence $(i_k)_{k\geq 1}$ such that $i_k \geq k$ and $d_{\partial S}(s_0, u_{i_k}) \geq k$ for all $k \geq 1$. Since

$$u_{i_k} \longrightarrow u_0 \quad \text{and} \quad \left[\bar{B}(s_0; 2\rho_0), s_0 + k u_{i_k}\right[\subset \text{int}\, S,$$

we get, see Figure 1(b), that

$$d_{\partial S}(s_0, u_0) \geq \frac{2\rho_0}{\frac{2\rho_0}{k} + \|u_0 - u_{i_k}\|} \longrightarrow \infty, \quad \text{a contradiction.}$$

Now assume that $d_{\partial S}(s_0, u_i)$ is not converging to $d_{\partial S}(s_0, u_0)$, then there is a subsequence of u_i (we do not relabel) such that

$$d_{\partial S}(s_0, u_i) \longrightarrow t_0 \neq d_{\partial S}(s_0, u_0).$$

Clearly we have $s_0 + d_{\partial S}(s_0, u_i) u_i \longrightarrow s_0 + t_0 u_0 \in \partial S$. Hence $d_{\partial S}(s_0, u_0) < t_0$. Since $[B(s_0; 2\rho_0), s_0 + t_0 u_0[\subset \text{int}\, S$ we get that $s_0 + d_{\partial S}(s_0, u_0) u_0 \in \text{int}\, S$, which gives the desired contradiction and terminates the proof of the claim.

For $0 < \varepsilon \leq \rho_0$ and u a unit vector, we define the set C_ε^u, see Figure 2, by

$$C_\varepsilon^u := \left[\bar{B}(s_0 + \rho_0 u; \rho_0), \bar{B}\left(s_0 + \left(d_{\partial S}(s_0, u) - \varepsilon - \tfrac{\varepsilon\rho_0}{d_{\partial S}(s_0,u) - 2\rho_0}\right) u; \tfrac{\varepsilon\rho_0}{d_{\partial S}(s_0,u) - 2\rho_0}\right)\right].$$

Using the definition of $d_{\partial S}(s_0, u)$ and the fact that $\bar{B}(s_0; 2\rho_0) \subset S^*$, we have

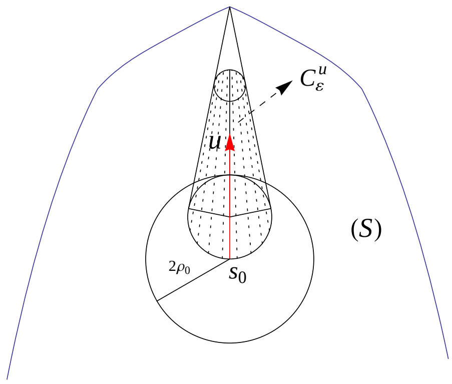


Figure 2: The set C_ε^u

that $C_\varepsilon^u \subset S$ for all u unit. On the other hand, one can easily verify that

$$C_\varepsilon^u = \bigcup_{t \in [2\rho_0, d_{\partial S}(s_0,u)-\varepsilon]} \bar{B}(s_0 + (t - \rho(u,t))u; \rho(u,t)), \tag{5}$$

where $\rho(\cdot,\cdot)$ is the function defined for u unit and $t \in [2\rho_0, d_{\partial S}(s_0,u)]$ by

$$\rho(u,t) := \begin{cases} \rho_0, & \text{if } d_{\partial S}(s_0,u) = \infty, \\ \dfrac{d_{\partial S}(s_0,u) - t}{d_{\partial S}(s_0,u) - 2\rho_0}\rho_0, & \text{if } d_{\partial S}(s_0,u_s) < \infty. \end{cases}$$

Note that if $d_{\partial S}(s_0,u) = \infty$ then

$$C_\varepsilon^u = \bigcup_{t \in [2\rho_0, \infty[} \bar{B}(s_0 + (t - \rho_0)u; \rho_0). \tag{6}$$

Using Claim 1 and the definition of $\rho(\cdot,\cdot)$ we have

- $\rho(u,t)$ is continuous with respect to (u,t).
- $0 \le \rho(u,t) \le \rho_0$ for all u unit and $t \in [2\rho_0, d_{\partial S}(s_0,u)]$.

Now we define the set S_ε by

$$S_\varepsilon := \bigcup_{u \,\text{unit}} C_\varepsilon^u \subset S.$$

Claim 2: S_ε is closed and $d_H(S, S_\varepsilon) \le \varepsilon$.

Let x_i be a sequence in S_ε that converges to a point x_0. We need to prove that $x_0 \in S_\varepsilon$. By the definition of S_ε and (5) there exist sequences of unit vectors u_i and $t_i \in [2\rho_0, d_{\partial S}(s_0,u_i) - \varepsilon]$ such that

$$x_i \in \bar{B}(s_0 + (t_i - \rho(u_i,t_i))u_i; \rho(u_i,t_i))\,.$$

Since the sequences u_i, x_i and $\rho(u_i, t_i)$ are bounded, we get that t_i is also bounded. Then we can assume the existence of unit vector u_0 and $t_0 \in [2\rho_0, d_{\partial S}(s_0, u_0) - \varepsilon]$ such that $u_i \longrightarrow u_0$ and $t_i \longrightarrow t_0$. Hence

$$x_i \longrightarrow x_0 \in \bar{B}(s_0 + (t_0 - \rho(u_0, t_0))u_0; \rho(u_0, t_0)) \subset C_\varepsilon^{u_0} \subset S_\varepsilon.$$

We proceed to prove that $d_H(S, S_\varepsilon) \leq \varepsilon$. But this follows directly from the facts that $S_\varepsilon \subset S$ and that for all $s \in S \cap S_\varepsilon^c$ we have

$$d_{S_\varepsilon}(x) \leq d_{C_\varepsilon^{u_x}}(x) \leq d(x, s_0 + (d_{\partial S}(s_0, u_x) - \varepsilon)u_x)) \leq \varepsilon,$$

where $u_x := \frac{x - s_0}{\|x - s_0\|}$. This terminates the proof of the claim.

Now for $s \notin B(s_0, 2\rho_0)$ and $u_s := \frac{s - s_0}{\|s - s_0\|}$ we define the function $r(s)$ by

$$r(s) := \frac{\rho(u_s, \|s - s_0\|)}{2}.$$

Claim 3: The function $r(\cdot)$ is well-defined and continuous on ∂S_ε.

Let $s \in \partial S_\varepsilon$. Since for all u unit, we have

$$\bar{B}(s_0 + \rho_0 u; \rho_0) \subset C_\varepsilon^u \subset S_\varepsilon,$$

we get that

$$\bar{B}(s_0; 2\rho_0) = \bigcup_{u \text{ unit}} \bar{B}(s_0 + \rho_0 u; \rho_0) \subset S_\varepsilon.$$

Hence $s \notin B(s_0, 2\rho_0)$ and this gives that $r(\cdot)$ is well-defined on ∂S_ε. The continuity of $r(\cdot)$ follows directly from the continuity of $\rho(\cdot, \cdot)$.

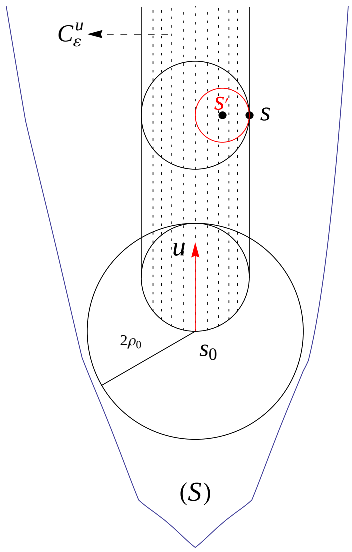


Figure 3: $d_{\partial S}(s_0, u) = \infty$

Claim 4: S_ε satisfies the θ-interior sphere condition with $\theta(\cdot) := \frac{1}{2r(\cdot)}$.

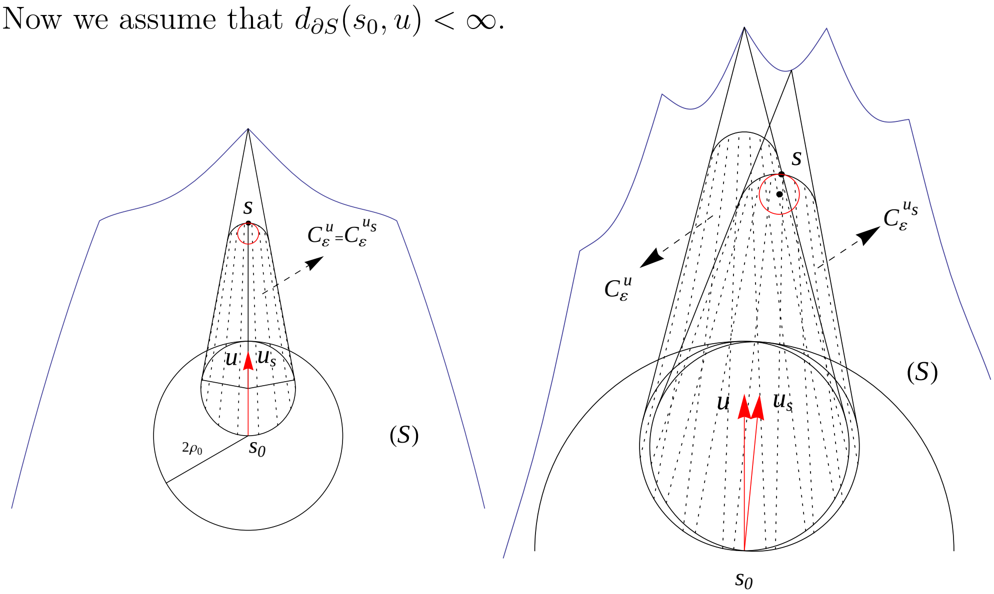
Let $s \in \partial S_\varepsilon$. Then there exists u unit such that $s \in \partial C_\varepsilon^u$. If $d_{\partial S}(s_0, u) = \infty$, see Figure 3, then by (6) there exists $t_0 \in [2\rho_0, +\infty[$ such that

$$s \in \bar{B}(s_0 + (t_0 - \rho_0)u; \rho_0) \subset C_\varepsilon^u \subset S_\varepsilon.$$

Since $r(s) = \frac{\rho(u_s, \|s - s_0\|)}{2} \leq \frac{\rho_0}{2}$, we get the existence of $s' \in \bar{B}(s_0 + (t_0 - \rho_0)u; \rho_0)$ such that

$$s \in \bar{B}\left(s'; r(s)\right) \subset \bar{B}(s_0 + (t_0 - \rho_0)u; \rho_0) \subset C_\varepsilon^u \subset S_\varepsilon.$$

Now we assume that $d_{\partial S}(s_0, u) < \infty$.



(a) $u_s = u$ (b) $u_s \neq u$ with $s \in \partial C_\varepsilon^{u_s}$

Figure 4

Case 1: $u_s = u$ (see Figure 4(a)). Then

$$\begin{aligned} s \in \bar{B}(s - r(s)u_s\,; r(s)) &= \bar{B}(s - r(s)u; r(s)) \\ &= \bar{B}\left(s_0 + \left(\|s - s_0\| - \frac{\rho(u, \|s - s_0\|)}{2}\right)u; \frac{\rho(u, \|s - s_0\|)}{2}\right) \\ &\subset \bar{B}(s_0 + (\|s - s_0\| - \rho(u, \|s - s_0\|))u; \rho(u, \|s - s_0\|)) \subset C_\varepsilon^u \subset S_\varepsilon\,, \end{aligned}$$

where the second inclusion above follows from (5) for $t = \|s - s_0\|$.

Case 2: $u_s \neq u$ but $s \in \partial C_\varepsilon^{u_s}$ (see Figure 4(b)). Then

$$\begin{aligned} s \in &\bar{B}(s - r(s)u_s\,; r(s)) = \\ &= \bar{B}\left(s_0 + \left(\|s - s_0\| - \frac{\rho(u_s, \|s - s_0\|)}{2}\right)u_s; \frac{\rho(u_s, \|s - s_0\|)}{2}\right) \\ &\subset \bar{B}(s_0 + (\|s - s_0\| - \rho(u_s, \|s - s_0\|))u_s; \rho(u_s, \|s - s_0\|)) \subset C_\varepsilon^{u_s} \subset S_\varepsilon\,, \end{aligned}$$

where the second inclusion above also follows from (5) for $t = \|s - s_0\|$.

Case 3: $u_s \neq u$ and $s \notin C_\varepsilon^{u_s}$ (see Figure (5)).

Then $d_{\partial S}(s_0, u_s) < \infty$. Since $s \in \partial S_\varepsilon$ we get that $s \in \partial C_\varepsilon^u$. Then there exists a unique $t_s \in [2\rho_0, d_{\partial S}(s_0, u) - \varepsilon]$ such that

$$s \in \bar{B}(s_0 + (t_s - \rho(u, t_s))u; \rho(u, t_s)).$$

We define

$$\alpha := d_{\partial S}(s_0, u) - \varepsilon - t_s \quad \text{and} \quad \beta := \|s - s_0\| - d_{\partial S}(s_0, u_s) + \varepsilon. \tag{7}$$

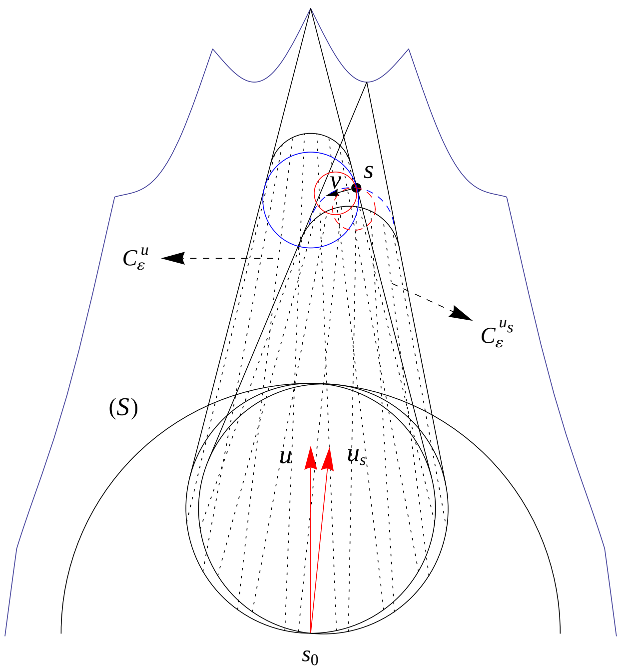


Figure 5: $u_s \neq u$ with $s \notin C_\varepsilon^{u_s}$

Clearly we have $\alpha \geq 0$. On the other hand, the fact that $s \notin C_\varepsilon^{u_s}$ gives that $\beta \geq 0$. Moreover, since s belongs to the upper part (with respect to u) of the boundary of the ball $\bar{B}(s_0 + (t_s - \rho(u, t_s))u; \rho(u, t_s))$, we have that

$$\|s - s_0\| \geq t_s - \rho(u, t_s). \tag{8}$$

Now will prove that $r(s) \leq \rho(u, t_s)$. Clearly we can assume that $\varepsilon \leq \rho_0$. Since $d_{\partial S}(s_0, u) < \infty$ and $d_{\partial S}(s_0, u_s) < \infty$ we have

$$\begin{aligned} r(s) \leq \rho(u, t_s) &\iff \rho(u_s, \|s - s_0\|) \leq 2\rho(u, t_s) \\ &\iff \frac{d_{\partial S}(s_0, u_s) - \|s - s_0\|}{d_{\partial S}(s_0, u_s) - 2\rho_0}\, \rho_0 \leq \frac{2(d_{\partial S}(s_0, u) - t_s)}{d_{\partial S}(s_0, u) - 2\rho_0}\, \rho_0. \end{aligned}$$

Therefore it is sufficient to prove that

$$2(d_{\partial S}(s_0, u) - t_s)(d_{\partial S}(s_0, u_s) - 2\rho_0) - (d_{\partial S}(s_0, u_s) - \|s - s_0\|)(d_{\partial S}(s_0, u) - 2\rho_0) \geq 0.$$

But

$$
\begin{aligned}
&2(d_{\partial S}(s_0,u)-t_s)(d_{\partial S}(s_0,u_s)-2\rho_0)-(d_{\partial S}(s_0,u_s)-\|s-s_0\|)(d_{\partial S}(s_0,u)-2\rho_0)\\
&=2(\alpha+\varepsilon)(d_{\partial S}(s_0,u_s)-2\rho_0)-(\varepsilon-\beta)(d_{\partial S}(s_0,u)-2\rho_0) && \text{[by (7)]}\\
&=\varepsilon(2d_{\partial S}(s_0,u_s)-d_{\partial S}(s_0,u)-2\rho_0)+2\alpha(d_{\partial S}(s_0,u_s)-2\rho_0)+\beta(d_{\partial S}(s_0,u)-2\rho_0)\\
&\geq \varepsilon(2d_{\partial S}(s_0,u_s)-d_{\partial S}(s_0,u)-2\rho_0)+2\alpha\varepsilon+\beta\varepsilon && \text{[by (4) and since } \varepsilon\leq\rho_0]\\
&\geq \varepsilon(d_{\partial S}(s_0,u_s)-d_{\partial S}(s_0,u)+\rho_0)+2\alpha\varepsilon+\beta\varepsilon && \text{[by (4)]}\\
&=\varepsilon(d_{\partial S}(s_0,u_s)-d_{\partial S}(s_0,u)+\rho_0+2\alpha+\beta)\\
&\geq \varepsilon(\|s-s_0\|-t_s+\rho_0+\alpha)\geq\varepsilon(\|s-s_0\|-t_s+\rho_0) && \text{[by (7) and since } \alpha\geq 0]\\
&\geq \varepsilon(\rho_0-\rho(u,t_s))\geq 0. && \text{[by (8)]}
\end{aligned}
$$

Therefore, $r(s)\leq\rho(u,t_s)$. Now, to terminate the proof of Claim 4, it is sufficient to remark that for

$$v:=\frac{s_0+(t_s-\rho(u,t_s)u-s}{\|s_0+(t_s-\rho(u,t_s)u-s\|},$$

we have

$$s\in\bar{B}\left(s+r(s)v;r(s)\right)\subset\bar{B}(s_0+(t_s-\rho(u,t_s))u;\rho(u,t_s))\subset C_\varepsilon^u\subset S_\varepsilon.$$

This also terminates the proof of the theorem. □

3.2. General case

Before presenting our general main result, we will introduce the following new geometric property that will play an important role in our study.

Definition 3.2. Let A and S be two nonempty sets in $\mathbb{R}^n$ such that A is closed and $A\subset S$. We say that A is S*-convex* if and only for all $s\in S\cap A^c$ and $a\neq a'\in A$ we have

$$s-a\notin N_A^P(a)\ \text{ or }\ s-a'\notin N_A^P(a').$$

Wee illustrate Definition 3.2 with some basic properties.

Proposition 3.3. *Let A and S be two nonempty sets in $\mathbb{R}^n$ such that A is closed and $A\subset S$. Then we have the following:*

(i) *A is convex if and only if A is $\mathbb{R}^n$-convex.*
(ii) *If A is S-convex and $A\subset S'\subset S$ then A is S'-convex.*
(iii) *If A is S-convex then each point in S has a unique projection on A.*

Proof. Follows directly from the definition of S-convexity. The details are left to the reader. □

Remark 3.4. The converse of (*iii*) of Proposition 3.3 is not necessarily true. Indeed, in $\mathbb{R}^2$, let $S:=\{(x,x):x>0\}^c$ and $A:=\{(x,y):x>0\text{ and }y>0\}^c$. Clearly each point in S has a unique projection on A but A is not S-convex.

The following is our general main result in which we give a more general sufficient condition for the existence of an inner approximation for a nonempty and closed set $S \subset \mathbb{R}^n$ by sets satisfying the interior sphere condition. Theorem 3.1 is a direct consequence of this general result as we will see in Remark 3.7.

Theorem 3.5. *Let $S \subset \mathbb{R}^n$ be a nonempty and closed set satisfying the following: there exists a closed set $A \subset \operatorname{int} S$ such that*

- *A is S-convex,*
- *A is φ_0-convex,*
- *For $s \in A^c \cap S$, $a \in \operatorname{proj}_A(s)$ and $\zeta := \frac{s-a}{\|s-a\|}$, we have*
    - $s \in \left[\bar{B}\left(a + \frac{1}{2\varphi_0}\zeta, \frac{1}{2\varphi_0}\right), a + d_{\partial S}(a,\zeta)\,\zeta\right] \subset S$, *if* $d_{\partial S}(a,\zeta) < \infty$.
    - $s \in \left[\bar{B}\left(a + \frac{1}{2\varphi_0}\zeta, \frac{1}{2\varphi_0}\right), a + t\zeta\right] \subset S$ *for all* $t \geq 0$, *if* $d_{\partial S}(a,\zeta) = \infty$.

Then for all $\varepsilon > 0$ small, there exists a nonempty and closed set $S_\varepsilon \subset S$ such that

- *S_ε satisfies the θ-interior sphere condition,*
- *$d_H(S_\varepsilon, S) \leq \varepsilon$.*

Proof. Let S be a nonempty and closed set and assume the existence of a closed set $A \subset \operatorname{int} S$ satisfying the three conditions above. Then there exists $\varphi_0 > 0$ such that for all $a \in \partial A$ and for all $0 \neq \zeta \in N_A^P(a)$ we have that ζ is realized by a $\frac{1}{2\varphi_0}$-sphere, that is,

$$B\left(a + \frac{1}{2\varphi_0}\frac{\zeta}{\|\zeta\|}\,;\frac{1}{2\varphi_0}\right) \cap A = \emptyset.$$

We define the set $\mathcal{N}_A$ by

$$\mathcal{N}_A := \{(a,\zeta) : a \in \partial A \text{ and } \zeta \in N_A^P(a) \text{ is unit}\}.$$

Since A is φ_0-convex, we have that $\mathcal{N}_A$ is closed. Moreover, we can assume that for all $(a,\zeta) \in \mathcal{N}_A$, we have

$$B\left(a + \frac{3}{4\varphi_0}\zeta\,;\frac{3}{4\varphi_0}\right) \subset \operatorname{int} S.$$

Claim 1: The function $d_{\partial S}(\cdot,\cdot)$ is continuous on $\mathcal{N}_A$. Moreover, $d_{\partial S}(a,\zeta) \geq \frac{3}{2\varphi_0}$ for all $(a,\zeta) \in \mathcal{N}_A$.

The proof of this claim follows using the same techniques as in the proof of the Claim 1 of Theorem 3.1. The details are left to the reader.

Now let $0 < \varepsilon \leq \frac{1}{2\varphi_0}$. Since $d_{\partial S}(a,\zeta) \geq \frac{3}{2\varphi_0}$ for all $(a,\zeta) \in \mathcal{N}_A$, we have

$$\varepsilon \leq \frac{1}{2\varphi_0} \leq d_{\partial S}(a,\zeta) - \frac{1}{\varphi_0} \quad \text{for all} \quad (a,\zeta) \in \mathcal{N}_A. \tag{9}$$

For $(a,\zeta)\in\mathcal{N}_A$, we denote by $C_\varepsilon^{a,\zeta}$ the set

$$\left[\bar{B}\left(a+\tfrac{1}{2\varphi_0}\zeta;\tfrac{1}{2\varphi_0}\right),\bar{B}\left(a+\left(d_{\partial S}(a,\zeta)-\varepsilon-\tfrac{\varepsilon}{2\varphi_0 d_{\partial S}(a,\zeta)-2}\right)\zeta;\tfrac{\varepsilon}{2\varphi_0 d_{\partial S}(a,\zeta)-2}\right)\right].$$

Clearly we have $C_\varepsilon^{a,\zeta}\subset S$ for $(a,\zeta)\in\mathcal{N}_A$. On the other hand and as in the special case above, we have

$$C_\varepsilon^{a,\zeta}=\bigcup_{t\in\left[\frac{1}{\varphi_0},d_{\partial S}(a,\zeta)-\varepsilon\right]}\bar{B}\left(a+\left(t-\rho(a,\zeta,t)\right)\zeta;\rho(a,\zeta,t)\right),\tag{10}$$

where $\rho(\cdot,\cdot,\cdot)$ is the function defined for $(a,\zeta)\in\mathcal{N}_A$ and $t\in\left[\frac{1}{\varphi_0},d_{\partial S}(a,\zeta)\right]$ by

$$\rho(a,\zeta,t):=\begin{cases}\dfrac{1}{2\varphi_0}, & \text{if } d_{\partial S}(a,\zeta)=\infty,\\ \dfrac{d_{\partial S}(a,\zeta)-t}{2\varphi_0 d_{\partial S}(a,\zeta)-2}, & \text{if } d_{\partial S}(a,\zeta)<\infty.\end{cases}$$

Note that if $d_{\partial S}(a,\zeta)=\infty$ then

$$C_\varepsilon^{a,\zeta}=\bigcup_{t\in\left[\frac{1}{\varphi_0},\infty\right[}\bar{B}\left(a+\left(t-\frac{1}{2\varphi_0}\right)\zeta;\frac{1}{2\varphi_0}\right).\tag{11}$$

By Claim 1 and from the definition of $\rho(\cdot,\cdot,\cdot)$ we have

- $\rho(a,\zeta,t)$ is continuous with respect to (a,ζ,t),
- $0\le\rho(a,\zeta,t)\le\frac{1}{2\varphi_0}$ for all $(a,\zeta)\in\mathcal{N}_A$ and $t\in\left[\frac{1}{\varphi_0},d_{\partial S}(a,\zeta)\right]$.

We set

$$S_\varepsilon:=\bigcup_{(a,\zeta)\in\mathcal{N}_A}C_\varepsilon^{a,\zeta}\subset S.$$

Claim 2: S_ε is closed and $d_H(S,S_\varepsilon)\le\varepsilon$.

This assertion follows using the same techniques as in the proof of Claim 2 of Theorem 3.1.

For $s\in A^c\cap S$, we denote by a_s the unique projection of s on A (the uniqueness of the projection follows from the S-convexity of A). We also denote by ζ_s the vector

$$\zeta_s:=\frac{s-a_s}{\|s-a_s\|}.$$

For any $s\in A^c\cap S$ satisfying $s\notin B\left(a_s+\frac{1}{2\varphi_0}\zeta_s;\frac{1}{2\varphi_0}\right)$, we define $r(s)$ by

$$r(s):=\frac{\rho(a_s,\zeta_s,\|s-a_s\|)}{2}.$$

Claim 3: The function $r(\cdot)$ is well-defined and continuous on ∂S_ε.

The function is well-defined since

$$B\left(a_s + \frac{1}{2\varphi_0}\zeta_s; \frac{1}{2\varphi_0}\right) \subset \operatorname{int} S_\varepsilon \text{ for all } \ s \in \partial S_\varepsilon.$$

The continuity of $r(\cdot)$ follows from the continuity of $\rho(\cdot,\cdot,\cdot)$ and the continuity the function $s \mapsto a_s$ on $A^c \cap S$.

Claim 4: S_ε satisfies the θ-interior sphere condition with $\theta(\cdot) = \frac{1}{2r(\cdot)}$.

Let $s \in \partial S_\varepsilon$. Then there exists $(a,\zeta) \in \mathcal{N}_A$ such that $s \in \partial C_\varepsilon^{a,\zeta}$. If $d_{\partial S}(a,\zeta) = \infty$ then by (11) there exists $t_0 \in [\frac{1}{\varphi_0}, \infty[$ such that

$$s \in \bar{B}\left(a + \left(t_0 - \frac{1}{2\varphi_0}\right)\zeta; \frac{1}{2\varphi_0}\right) \subset C_\varepsilon^{a,\zeta} \subset S_\varepsilon.$$

Since $r(s) = \frac{\rho(a_s,\zeta_s,\|s-a_s\|)}{2} \leq \frac{1}{4\varphi_0}$ there exists $s' \in \bar{B}\left(a + \left(t_0 - \frac{1}{2\varphi_0}\right)\zeta; \frac{1}{2\varphi_0}\right)$ such that

$$s \in \bar{B}(s'; r(s)) \subset \bar{B}\left(a + \left(t_0 - \frac{1}{2\varphi_0}\right)\zeta; \frac{1}{2\varphi_0}\right) \subset C_\varepsilon^{a,\zeta} \subset S_\varepsilon.$$

Now we assume that $d_{\partial S}(a,\zeta) < \infty$.

Case 1: $a_s = a$.

This case is similar to the special case (Theorem 3.1) with

$$s_0 := a = a_s, \quad u := \zeta \quad \text{and} \quad u_s := \zeta_s.$$

Therefore, there exists $s' \in S$ such that $s \in \bar{B}(s'; r(s)) \subset S_\varepsilon$.

Case 2: $a_s \neq a$ (see Figure 6).

Then $d_{\partial S}(a_s,\zeta_s) < \infty$. Since $s \in \partial S_\varepsilon$ we have $s \in \partial C_\varepsilon^{a,\zeta}$. Then there exists a unique $t_s \in \left[\frac{1}{\varphi_0}, d_{\partial S}(a,\zeta) - \varepsilon\right]$ such that $s \in \bar{B}(a+(t_s-\rho(a,\zeta,t_s))\zeta; \rho(a,\zeta,t_s))$. We will prove that $r(s) \leq \rho(a,\zeta,t_s)$. If $\rho(a,\zeta,t_s) > \frac{1}{4\varphi_0}$ then

$$r(s) = \frac{\rho(a_s,\zeta_s,\|s-a_s\|)}{2} \leq \frac{1}{4\varphi_0} < \rho(a,\zeta,t_s).$$

Assume now that $\rho(a,\zeta,t_s) \leq \frac{1}{4\varphi_0}$. Since $s \in \bar{B}(a+(t_s-\rho(a,\zeta,t_s))\zeta; \rho(a,\zeta,t_s))$ we get

$$\begin{aligned}
&\|a+d_{\partial S}(a,\zeta)\zeta - s\| = \\
&\qquad = \|(a+(t_s-\rho(a,\zeta,t_s))\zeta - s) + (d_{\partial S}(a,\zeta) - t_s + \rho(a,\zeta,t_s))\zeta\| \\
&\qquad \leq \rho(a,\zeta,t_s) + d_{\partial S}(a,\zeta) - t_s + \rho(a,\zeta,t_s) \\
&\qquad = d_{\partial S}(a,\zeta) - t_s + 2\rho(a,\zeta,t_s) \ \leq \ d_{\partial S}(a,\zeta) - t_s + \frac{1}{2\varphi_0}.
\end{aligned} \tag{12}$$

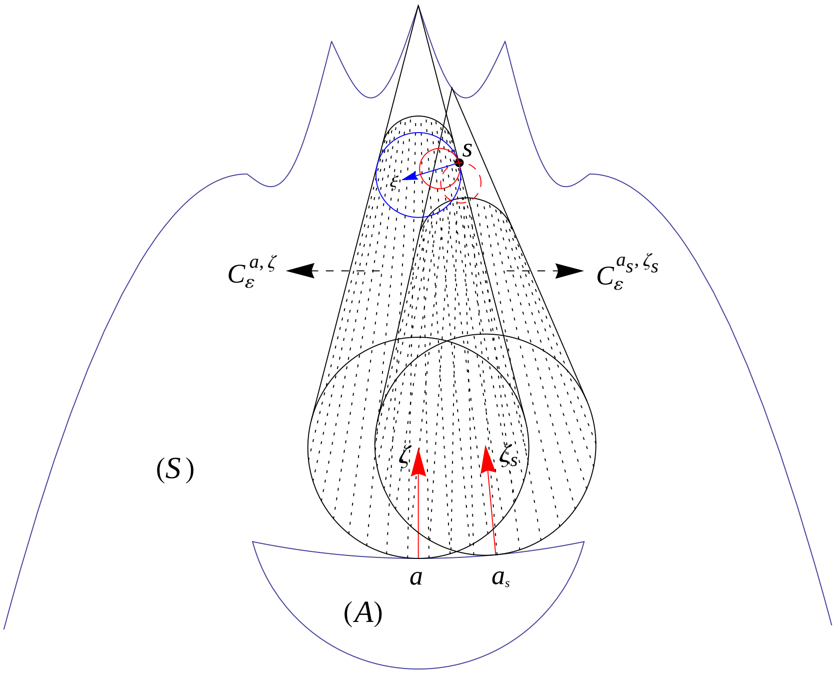


Figure 6: $a_s \neq a$

Due to the S-convexity of A and using the fact that $((a + d_{\partial S}(a,\zeta)\zeta) - a) \in N_A^P(a)$, we get

$$a \in \operatorname{proj}_A(a + d_{\partial S}(a,\zeta,\zeta)). \tag{13}$$

Then

$$\begin{aligned} d_{\partial S}(a,\zeta) &\leq \|a + d_{\partial S}(a,\zeta)\zeta - a_s\| \\ &\leq \|a + d_{\partial S}(a,\zeta)\zeta - s\| + \|s - a_s\| \\ &\leq d_{\partial S}(a,\zeta) - t_s + \frac{1}{2\varphi_0} + \|s - a_s\|, \end{aligned}$$

where the last inequality follows from (12). Therefore

$$\|s - a_s\| - t_s + \frac{1}{2\varphi_0} \geq 0. \tag{14}$$

Now we define

$$\alpha := d_{\partial S}(a,\zeta) - \varepsilon - t_s \ \text{ and } \ \beta := \|s - a_s\| - d_{\partial S}(a_s,\zeta_s) + \varepsilon. \tag{15}$$

Clearly we have $\alpha \geq 0$. On the other hand, the fact that $s \notin C_\varepsilon^{a_s,\zeta_s}$ gives that $\beta \geq 0$. We have

$$\begin{aligned} r(s) \leq \rho(a,\zeta,t_s) &\iff \rho(a_s,\zeta_s,\|s - a_s\|) \leq 2\rho(a,\zeta,t_s) \\ &\iff \frac{d_{\partial S}(a_s,\zeta_s) - \|s - a_s\|}{2\varphi_0 d_{\partial S}(a_s,\zeta_s) - 2} \leq \frac{d_{\partial S}(a,\zeta) - t_s}{\varphi_0 d_{\partial S}(a,\zeta) - 1}. \end{aligned}$$

Therefore it is sufficient to prove

$$2(d_{\partial S}(a,\zeta)-t_s)(\varphi_0 d_{\partial S}(a_s,\zeta_s)-1)-(d_{\partial S}(a_s,\zeta_s)-\|s-a_s\|)(\varphi_0 d_{\partial S}(a,\zeta)-1)\geq 0.$$

But

$$\begin{aligned}
&2(d_{\partial S}(a,\zeta)-t_s)(\varphi_0 d_{\partial S}(a_s,\zeta_s)-1)-(d_{\partial S}(a_s,\zeta_s)-\|s-a_s\|)(\varphi_0 d_{\partial S}(a,\zeta)-1)\\
&=2(\alpha+\varepsilon)(\varphi_0 d_{\partial S}(a_s,\zeta_s)-1)-(\varepsilon-\beta)(\varphi_0 d_{\partial S}(a,\zeta)-1) && \text{[by (15)]}\\
&=\varepsilon(2\varphi_0 d_{\partial S}(a_s,\zeta_s)-\varphi_0 d_{\partial S}(a,\zeta)-1)+2\alpha(\varphi_0 d_{\partial S}(a_s,\zeta_s)-1)+\beta(\varphi_0 d_{\partial S}(a,\zeta)-1)\\
&\geq \varepsilon\varphi_0\left(2d_{\partial S}(a_s,\zeta_s)-d_{\partial S}(a,\zeta)-\tfrac{1}{\varphi_0}\right)+\alpha+\tfrac{\beta}{2} && \text{[by (9)]}\\
&\geq \varepsilon\varphi_0\left(d_{\partial S}(a_s,\zeta_s)-d_{\partial S}(a,\zeta)+\tfrac{1}{2\varphi_0}\right)+\alpha+\tfrac{\beta}{2} && \text{[by (9)]}\\
&=\varepsilon\varphi_0\left(\|s-a_s\|-\beta-\alpha-t_s+\tfrac{1}{2\varphi_0}\right)+\alpha+\tfrac{\beta}{2} && \text{[by (15)]}\\
&\geq \varepsilon\varphi_0(-\alpha-\beta)+\alpha+\tfrac{\beta}{2}\geq \tfrac{1}{2}(-\alpha-\beta)+\alpha+\tfrac{\beta}{2}=\tfrac{\alpha}{2}\geq 0. && \text{[by (14) and (9)]}
\end{aligned}$$

Therefore, $r(s)\leq \rho(a,\zeta,t_s)$. Now, to terminate the proof of Claim 4, it is sufficient to remark that for

$$\xi := \frac{a+(t_s-\rho(a,\zeta,t_s)\zeta-s}{\|a+(t_s-\rho(a,\zeta,t_s)\zeta-s\|}\,,$$

we have

$$s\in \bar{B}\left(s+r(s)\xi;r(s)\right)\subset \bar{B}(a+(t_s-\rho(a,\zeta,t_s))\zeta;\rho(a,\zeta,t_s))\subset C_\varepsilon^{a,\zeta}\subset S_\varepsilon.$$

This also terminates the proof of the theorem. □

Remark 3.6. In this remark we will prove that in Theorem 3.5, the S-convexity assumption of A can be replaced by the following weaker condition:

(C) For all $s\in S\cap A^c$ and $a\neq a'\in A$ satisfying

$$\|s-a\|\leq d_{\partial S}\left(a,\frac{s-a}{\|s-a\|}\right)\ \text{ and }\ \|s-a'\|\leq d_{\partial S}\left(a',\frac{s-a'}{\|s-a'\|}\right),$$

we have

$$s-a\notin N_A^P(a)\ \text{ or }\ s-a'\notin N_A^P(a').$$

It is sufficient to prove that any point $s\in A^c\cap S$ has a unique projection a_s on A and that (13) holds. For the uniqueness of the projection, it is sufficient to remark that if $s\in A^c\cap S$ has two projections $a_s\neq a_s'$ on A, then by **(C)** we get that

$$\|s-a_s\|>d_{\partial S}\left(a_s,\frac{s-a_s}{\|s-a_s\|}\right)\ \text{ or }\ \|s-a_s'\|>d_{\partial S}\left(a_s',\frac{s-a_s'}{\|s-a_s'\|}\right),$$

which contradicts the third assumption of Theorem 3.5. Now we will prove that (13) holds. Let $a'\in \mathrm{proj}_A(a+d_{\partial_S}(a,\zeta)\zeta)$ and assume that $a'\neq a$. Clearly we have

- $\|(a+d_{\partial_S}(a,\zeta)\zeta)-a\| = d_{\partial_S}(a,\zeta) = d_{\partial S}\left(a, \frac{(a+d_{\partial_S}(a,\zeta)\zeta)-a}{\|(a+d_{\partial_S}(a,\zeta)\zeta)-a\|}\right)$,
- $\|(a+d_{\partial_S}(a,\zeta)\zeta)-a'\| \leq d_{\partial S}\left(a', \frac{(a+d_{\partial_S}(a,\zeta)\zeta)-a'}{\|(a+d_{\partial_S}(a,\zeta)\zeta)-a'\|}\right)$
(by the third assumption of Theorem 3.5),
- $(a+d_{\partial_S}(a,\zeta)\zeta)-a = d_{\partial_S}(a,\zeta)\zeta \in N_A^P(a)$,
- $(a+d_{\partial_S}(a,\zeta)\zeta)-a' \in N_A^P(a')$.

But this contradicts the condition **(C)**. Therefore $a \in \mathrm{proj}_A(a+d_{\partial_S}(a,\zeta,\zeta))$.

Remark 3.7. Theorem 3.1 is a corollary of Theorem 3.5. Indeed, it is sufficient to take $A=\{s_0\}$ and $\varphi_0 = \frac{1}{2\rho_0}$, where $\bar{B}(s_0;2\rho_0) \subset S^*$.

Another important corollary of Theorem 3.5 is the following result.

Corollary 3.8. *Let $S \subset \mathbb{R}^n$ be a nonempty and closed set satisfying the following: there exist a closed and convex set $A \subset \operatorname{int} S$ and $\rho_0 > 0$ such that for $s \in A^c \cap S$, $a \in \mathrm{proj}_A(s)$ and $\zeta := \frac{s-a}{\|s-a\|}$, we have*

- $s \in \left[\bar{B}\left(a+\rho_0\zeta,\rho_0\right), a+d_{\partial S}(a,\zeta)\,\zeta\right] \subset S$, *if* $d_{\partial S}(a,\zeta) < \infty$.
- $s \in \left[\bar{B}\left(a+\rho_0\zeta,\rho_0\right), a+t\zeta\right] \subset S$ *for all* $t \geq 0$, *if* $d_{\partial S}(a,\zeta) = \infty$.

Then for all $\varepsilon > 0$ small, there exists a nonempty and closed set $S_\varepsilon \subset S$ such that

- *S_ε satisfies the θ-interior sphere condition,*
- $d_H(S_\varepsilon, S) \leq \varepsilon$.

Proof. Since A is convex, we get that A is S-convex and $\frac{1}{2\rho_0}$-convex. Now it is sufficient to apply Theorem 3.5. □

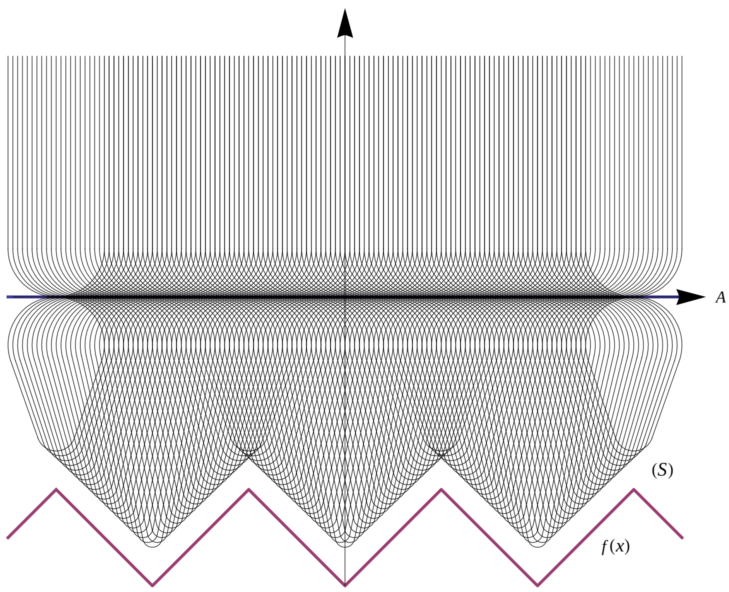


Figure 7: Example 3.9

The following is an example of a set satisfying the conditions of Theorem 3.5 but it fails to satisfy the condition of Theorem 3.1.

Example 3.9. Let S be the epigraph of the periodic function $f(\cdot)$, see Figure 7, defined on $\mathbb{R}$ by

- $f(x) = |x| - 3$ for all $x \in [-1, 1]$,
- $f(x+2) = f(x)$ for all $x \in \mathbb{R}$.

Cleary we have $\operatorname{int} S^* = \emptyset$. On the other hand, one can easily verify that the convex set $A := \{(x, 0) : x \in \mathbb{R}\}$ satisfies the conditions of Theorem 3.5 and then S can be innerly approximated by sets S_ε satisfying the interior sphere condition (the shaded region in Figure 7 is an example of an S_ε).

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