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Operator Systems and Convex Sets with Many Normal Cones

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The state space of an operator system of n-by-n matrices has, in a sense, many normal cones. This wealth of normal cones is a smoothness quality and it implies that each exposed face is an intersection of maximal exposed faces. A clustering property of exposed faces follows. We work on minimizing the assumptions which allow an arbitrary convex set of finite dimension to have analogous properties. One motivation for this work is an isomorphism from the exposed faces of the state space to the ground state spaces of the operator system.

Keywords: Normal cone, exposed ray, exposed face, coatom, operator system, state space, convex support, joint numerical range, ground state, coorbitope.

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1. Introduction

We are interested in the convex geometry of projections (linear images) of the set of density matrices $\mathcal{M}_n$, that is positive-semidefinite hermitian matrices of trace one. The convex set $\mathcal{M}_n$ represents the *state space* [1] of normalized positive linear forms on the C*-algebra M_n of complex n-by-n matrices, $n \in \mathbb{N}$, and is a statistical model of quantum mechanics [21, 7]. Let $S \subset M_n$ be an *operator system* [32], that is a complex vector space including the identity matrix $\mathbb{1}$ which is self-adjoint ($s \in S$ implies $s^* \in S$). The *state space* of normalized positive linear forms on S is represented by the projection $\mathcal{M}(S)$ of $\mathcal{M}_n$ onto S. The state space of S has a very broad use, for example in quantum state tomography [20], inference [22], or quantum chemistry [16], because it represents expected values of observables, measurement probabilities (POVM's), and reduced density matrices (quantum marginals).

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The convex hull of the *joint numerical range* [3, 26, 18, 25, 15, 14, 19, 40], also known as *joint algebraic numerical range* [29], is a coordinate representation of $\mathcal{M}(S)$, which we called *convex support* [42] in analogy with statistics [6]. The convex support is the *algebraic polar* of a spectrahedron [33], so convex algebraic geometry [30, 36, 39, 28] is useful to study $\mathcal{M}(S)$.

The present article focusses on the notion of *exposed face* of a convex set $C \subset \mathbb{R}^m$, which is defined as the set of maximizers of a linear form on C or as the empty set. We denote the set of exposed faces of C by $\mathcal{E}_C$. On $\mathcal{E}_C$ we consider the partial ordering by inclusion, whose general theory is well-understood [5, 27]. Properties of $\mathcal{M}(S)$ will be reflected in this paper only by assumptions on the normal cones of C.

Our results, discussed in Section 2, translate to the lattice of ground state spaces of S, which is order isomorphic to $\mathcal{E}_{\mathcal{M}(S)}$. The ground state energy of an energy operator, represented by a hermitian matrix s, is the least eigenvalue of s and the minimum is achieved for states supported on the corresponding eigenspace, the ground state space of s. Section 6 contains a detailed proof of this isomorphism.

This work contributes to a systematic analysis of signatures of quantum phase transitions in systems with a finite dimensional state space $\mathcal{M}_n$. These signatures are marked by abrupt changes of maximum-entropy states and ground state projections [2, 11], which depend on the convex geometry of state spaces [45, 46, 11, 35].

The convex geometry of the state space C of *k-local Hamiltonians* has a longer history [16], and is still a topic [31, 12, 13], because C represents k-party reduced density matrices. The partial ordering of $\mathcal{E}_C$ was studied from the point of view of minimal elements of $\mathcal{E}_C \setminus \{\emptyset\}$ and their suprema [10]. We continue the approach of [43] and study maximal elements of $\mathcal{E}_C \setminus \{C\}$ and their infima for arbitrary finite-dimensional convex sets C.

2. Discussion of the results

Let $C \subset \mathbb{R}^m$, $m \in \mathbb{N}$, be a convex subset. The *normal cone* of C at $x \in C$ is

$$N_C(x) := \{u \in \mathbb{R}^m \mid \forall y \in C : \langle u, y - x\rangle \leq 0\}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product. Elements of $N_C(x)$ are (outward pointing) *normal vectors* of C at x. The *relative interior* of a convex subset $C \subset \mathbb{R}^m$ is the interior of C with respect to the topology of the affine hull of C. The *normal cone* $N_C(F)$ of C at a non-empty convex subset F of C is well-defined as the normal cone $N_C(x)$ at any relative interior point x of F (see Section 2.2 of [38] or Section 4 of [43]). Let $\mathcal{N}_C$ denote the set of normal cones at points of C together with $N_C(\emptyset) := \mathbb{R}^m$. If C is not a singleton then $N_C : \mathcal{E}_C \to \mathcal{N}_C$ is an antitone lattice isomorphism, see equation (1) in Section 3.

This article focusses on exposed faces, but the simple Example 2.5(3) shows that state spaces do have a richer convex geometry. A *face* of C is a convex subset of C containing every closed segment in C whose relative interior it intersects. Exposed faces are faces, a *non-exposed face* is a face which is not an exposed face. If a face F is a singleton then its element is an *extreme point*, *exposed point*, or *non-exposed point* of C respectively, if F is a face, exposed face, or non-exposed face. We use the analogous definitions for faces which are rays.

To exclude trivialities we call an exposed face $F \in \mathcal{E}_C$ *proper exposed face* of C, if $F \notin \{\emptyset, C\}$. We define a *proper convex subset* of $\mathbb{R}^m$ to be a convex subset of $\mathbb{R}^m$ with interior points[1] which has a proper exposed face. A *proper normal cone* of C is any element of $\mathcal{N}_C$ other than $N_C(\emptyset)$ and $N_C(C)$.

Definition 2.1. Let $\mathcal{C}_m^0$, $\mathcal{C}_m$, $\mathcal{C}_m'$, and $\mathcal{C}_m''$ denote, respectively, the class of proper convex subsets of $\mathbb{R}^m$, $m \in \mathbb{N}$, such that for every proper normal cone N of C

$\mathcal{C}_m^0$	N has an exposed ray which is in $\mathcal{N}_C$,
$\mathcal{C}_m$	N has $\dim(N)$ linearly independent exposed rays which are in $\mathcal{N}_C$,
$\mathcal{C}_m'$	every extreme ray of N is in $\mathcal{N}_C$,
$\mathcal{C}_m''$	every non-empty face of N is in $\mathcal{N}_C$.

Let us critically discuss this definition. We call *convex body* a compact convex set.

Remark 2.2.

(1) Replacing *exposed ray* with *extreme ray* in the definition of $\mathcal{C}_m^0$ (resp. $\mathcal{C}_m$) does not change the set $\mathcal{C}_m^0$ (resp. $\mathcal{C}_m$), because normal cones of C included in N are exposed faces of N (Lemma 3.1).

(2) Replacing *extreme ray* with *exposed ray* in the definition of $\mathcal{C}_m'$ enlarges the set $\mathcal{C}_m'$ for $m > 2$ (Example 5.9). A convex body lies in $\mathcal{C}_m'$ if and only if its polar convex body has no non-exposed points (Theorem 5.4). Corollary 5.6, a sufficient condition for inclusion to $\mathcal{C}_m'$, shows that coorbitopes[2] lie in $\mathcal{C}_m'$.

[1] The simplifying assumption of interior points is fulfilled by any convex set after applying an affine embedding which removes codimensions. It guarantees that the normal cone at every point is a pointed closed convex cone. By definition, a *convex cone* is a non-empty convex subset K of $\mathbb{R}^m$ such that $\alpha x \in K$ for all $\alpha \geq 0$ and $x \in K$. A convex cone K is *pointed*, if $K \cap (-K) = \{0\}$.

[2] A *coorbitope* [37] is polar to an *orbitope*, defined as the convex hull of the orbit of a compact algebraic group acting linearly on a vector space. Proposition 2.2 of [37] shows that orbitopes have no non-exposed points because their extreme points lie on a sphere.

(3) Replacing *face* with *exposed face* in the definition of $\mathcal{C}''_m$ does not change the set $\mathcal{C}''_m$ (Lemma 4.8). A convex body lies in $\mathcal{C}''_m$ if and only if its polar convex body has no non-exposed faces (Theorem 5.4). In particular, state spaces of operator systems lie in $\mathcal{C}''_m$ (Corollary 6.3).
(4) Lemma 4.6 proves $\mathcal{C}'_m \subset \mathcal{C}_m$ and this implies $\mathcal{C}''_m \subset \mathcal{C}'_m \subset \mathcal{C}_m \subset \mathcal{C}^0_m$. See (5) for a refinement of this nesting of classes and for examples of strictness.

Section 3 studies smoothness of the class $\mathcal{C}^0_m$. We call *coatom*[3] of $\mathcal{E}_C$ an inclusion maximal element of $\mathcal{E}_C \setminus \{C\}$. Theorem 3.2 proves that a proper convex subset C of $\mathbb{R}^m$ lies in $\mathcal{C}^0_m$ if and only if every coatom of $\mathcal{E}_C$ is smooth, that is it has a unique unit normal vector. Theorem 3.3 proves that $C \in \mathcal{C}^0_m$ if and only if the boundary of C is covered by smooth coatoms of $\mathcal{E}_C$. See Example 2.5(1) for a convex set without this property.

Section 4 improves the theorem in Section 1.2.4 of [43], which shows for $C \in \mathcal{C}''_m$ that every proper exposed face of C is an intersection of coatoms of $\mathcal{E}_C$. This property is well-known for polytopes [47], which are included in $\mathcal{C}''_m$ because they are convex support sets, see Remark 6.6 and Corollary 6.3. Theorem 4.1 weakens the assumptions from $C \in \mathcal{C}''_m$ to $C \in \mathcal{C}_m$ and adds a dimension dependent multiplicity:

Corollary 2.3 (Intersections). *Let $C \in \mathcal{C}_m$ and let $F \in \mathcal{E}_C$ be a proper exposed face. Then there exist $\dim(N_C(F))$ mutually distinct coatoms of $\mathcal{E}_C$ whose intersection is F and whose normal cones are linearly independent exposed rays of $N_C(F)$.*

Of course, if $\dim(N_C(F)) > 2$ then F can be the intersection of any number (at least two) of coatoms. An example is an octahedron where every vertex is the intersection of two, three, or four facets. Another example is a cone based on a disk whose apex is the intersection of two surface lines while lying on a continuum of them.

Corollary 2.3 contains a method to construct clusters of exposed faces of $C \in \mathcal{C}_m$. More precisely, we define a *cluster* as an equivalence class of coatoms of $\mathcal{E}_C$ where two coatoms F_1, F_2 are equivalent if there is a sequence of coatoms $G_0, \dots, G_k$, $k \in \mathbb{N}$, such that $G_0 = F_1$, $G_k = F_2$ and $G_{i-1} \cap G_i \neq \emptyset$ for $i = 1, \dots, k$.

Corollary 2.4 (Clustering). *Let $C \in \mathcal{C}_m$ and let $F, G \in \mathcal{E}_C$ be proper exposed faces. If F strictly contains G then there exists a proper exposed face $F' \neq F$ of C which strictly contains G. One can choose F' to be a coatom of $\mathcal{E}_C$.*

The described corollaries may be checked with 3D sets of Example 5.9, last paragraph, or of Example 6.7. 2D examples suffice to distinguish the classes $\mathcal{C}_m \subset \mathcal{C}^0_m$.

[3]Note that a coatom may not be a *facet*, that is a face of codimension one [38, 47]. For a polyhedral convex set the notions of coatom and facet are equivalent [17, 47].

Example 2.5. Let $\operatorname{conv}(X)$ denote the convex hull of a subset $X \subset \mathbb{R}^m$. We study

(1) the lens $C_{()} := \{(x,y) \in \mathbb{R}^2 \mid (x \pm \frac{3}{2})^2 + y^2 \leq (\frac{5}{2})^2\}$,
(2) the truncated disk $C_{(|} := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, y \leq \frac{1}{2}\}$,
(3) and the drop $C_{0>} := \operatorname{conv}(\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{a\}), \quad a := (0,2)$.

We have $C_{()} \notin \mathcal{C}_2^0$ because the coatom $\{a\}$ of $\mathcal{E}_{C_{()}}$ has a 2D normal cone. The exposed point $t_\pm := \frac{1}{2}(\pm\sqrt{3}, 1)$ of $C_{(|}$ has a 2D normal cone, too, but this normal cone has an exposed ray which is a normal cone of C, so $C_{(|} \in \mathcal{C}_2^0$ holds. Since this cone has only one exposed ray which is a normal cone of C we have $C_{(|} \in \mathcal{C}_2^0 \setminus \mathcal{C}_2$. Another argument for $C_{(|} \notin \mathcal{C}_2$ is that the construction of clusters of Corollary 2.4 fails because $t_\pm$ lies only on one coatom of $\mathcal{E}_C$, which is the segment $[t_-, t_+]$.

The drop $C_{0>}$ belongs to $\mathcal{C}_2$ because a is the only point with a 2D normal cone and because the exposed rays of this normal cone are normal cones. Alternatively, $C_{0>} \in \mathcal{C}_2$ by Theorem 5.4(4) since $C_{0>}$ is the polar of $C_{(|}$, which has no non-exposed faces. The inclusion $C_{0>} \in \mathcal{C}_2$ follows also from Remark 6.6(1) and Corollary 6.3 since $C_{0>}$ is the convex support of

$$F_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F_2 := \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Notice that the extreme point $t_\pm$ of $C_{0>}$ is a non-exposed point. Since $C_{0>} \in \mathcal{C}_2$ holds, this follows also by contradiction from Corollary 2.4: If $t_\pm$ was an exposed point of $C_{0>}$ then it had to lie on two coatoms of $\mathcal{E}_{C_{0>}}$.

Dimension three is needed to differentiate between the classes $\mathcal{C}''_m \subset \mathcal{C}'_m \subset \mathcal{C}_m$ (because $\mathcal{C}''_2 = \mathcal{C}_2$ holds) and to finish Remark 2.2. This discussion will be done with convex bodies for which Theorem 5.4 translates Definition 2.1 to polar convex bodies. Further, Section 5 recalls that the class $\mathcal{C}''_m$ is closed under projection to subspaces, which is wrong for $\mathcal{C}^0_m, \mathcal{C}_m$, and $\mathcal{C}'_m$.

Theorem 6.2 proves that the state space $\mathcal{M}(S)$ of an operator system $S \subset M_n$ is the projection of the state space of the algebra M_n onto S. Further topics of Section 6 are a proof of $\mathcal{M}(S) \in \mathcal{C}''_{\dim_{\mathbb{C}}(S)-1}$, coordinate representations of $\mathcal{M}(S)$ in terms of convex support sets and joint numerical ranges, the isomorphism between exposed faces of $\mathcal{M}(S)$ and ground state projections, and a discussion of state spaces of 3-by-3 matrices.

3. Smoothness

We characterize the class $\mathcal{C}^0_m$ in terms of smoothness properties. Further, we study some very special convex sets which are smooth, strictly convex, or both (ovals).

A *lattice* is a partially ordered set where the infimum and supremum of each two elements exists. An *atom* in a lattice $(\mathcal{L}, \leq, \wedge, \vee, 0)$ with smallest element 0 is an element $x \in \mathcal{L}$, $x \neq 0$, such that $y \leq x$, $y \neq x$ implies $y = 0$ for all $y \in \mathcal{L}$. Similarly, a *coatom* in a lattice $(\mathcal{L}, \leq, \wedge, \vee, 1)$ with greatest element 1 is an element $x \in \mathcal{L}$, $x \neq 1$, such that $y \geq x$, $y \neq x$ implies $y = 1$ for all $y \in \mathcal{L}$. The lattice $(\mathcal{L}, \leq, \wedge, \vee, 0)$ is *atomistic* if every element of $\mathcal{L}$ is the supremum of the atoms which it contains [27] (such a lattice is called *atomic* in [9, 47]). The lattice $(\mathcal{L}, \leq, \wedge, \vee, 1)$ is *coatomistic* if every element is the infimum of the coatoms in which it is contained. A lattice is *complete* if an arbitrary subset has an infimum and a supremum. The smallest and greatest elements of a lattice, when they exist, are called *improper* elements, all other elements are *proper* elements. If $x, y \in \mathcal{L}$ and $x \leq y$ then we define the *interval* $[x, y]_{\mathcal{L}} := \{z \in \mathcal{L} \mid x \leq z \leq y\}$.

Without reminder we will use the fact that the smallest exposed face containing a proper face is a proper exposed face (Lemma 4.6 of [43]). In particular, for a convex subset of $\mathbb{R}^m$ to have a proper exposed face is the same as to have a proper face.

Let $C \subset \mathbb{R}^m$, $m \in \mathbb{N}$, be a convex subset. Both $\mathcal{E}_C$ and $\mathcal{N}_C$ are, partially ordered by inclusion, complete lattices where the infimum is the intersection [43]. The improper elements of $\mathcal{E}_C$ are $\emptyset$ and C and the improper elements of $\mathcal{N}_C$ are $N_C(\emptyset) = \mathbb{E}^m$ and $N_C(C)$, the latter being the vector space which is the orthogonal complement of the affine hull of C. By Proposition 4.7 of [43], if C is not a singleton then

$$N_C : \mathcal{E}_C \to \mathcal{N}_C \tag{1}$$

is an antitone lattice isomorphism. Let C be a proper convex subset of $\mathbb{R}^m$. Then the coatoms of $\mathcal{E}_C$ are proper exposed faces and the atoms of $\mathcal{N}_C$ proper normal cones. Moreover, proper normal cones are pointed closed convex cones and the smallest element of $\mathcal{N}_C$ is $N_C(C) = \{0\}$.

We start with an observation about faces of normal cones. The *positive hull* of a non-empty subset $X \subset \mathbb{R}^m$ is $\operatorname{pos}(X) := \{\alpha x \mid \alpha \geq 0, x \in X\}$.

Lemma 3.1. *Let $C \subset \mathbb{R}^m$, $m \in \mathbb{N}$, be a convex set. If $K, L \in \mathcal{N}_C$, $K \subset L$, and L is a proper normal cone of C then K is an exposed face of L.*

Proof: Using the antitone lattice isomorphism (1) and applying a translation to C we can assume that $0 \in C$, and that $K = N_C(x)$ and $L = N_C(0)$ hold for some $x \in C$. The equalities

$$N_C(0) = N_{\operatorname{pos}(C)}(0), \qquad N_C(x) = N_{\operatorname{pos}(C)}(x)$$

are easy to prove, the second equality holds because all points y of the segment $[x, 0[$ have the same normal cone $N_C(y) = N_C(x)$. Secondly, it is easy to see that $N_{\operatorname{pos}(C)}(x)$ is the intersection of $N_{\operatorname{pos}(C)}(0)$ with the orthogonal complement of the face of $\operatorname{pos}(C)$ containing x in its relative interior. This intersection is an exposed face of $N_{\operatorname{pos}(C)}(0)$, known as *dual face* [41]. $\square$

Let $C \subset \mathbb{R}^m$ be a convex subset with interior points. A point $s \in C$ is a *smooth point* [38] of C if $N_C(s)$ is one-dimensional. In that case $N_C(s)$ is a ray and an atom of $\mathcal{N}_C$. Similarly we call $F \in \mathcal{E}_C$ a *smooth exposed face* of C if $N_C(F)$ is a ray and we call C a *smooth convex set* if all proper normal cones of C are rays.

Theorem 3.2. *Let C be a proper convex subset of $\mathbb{R}^m$, $m \in \mathbb{N}$. Then $C \in \mathcal{C}_m^0$ holds if and only if every atom of $\mathcal{N}_C$ is a ray. In that case the isomorphism (1) restricts to a bijection from the coatoms of $\mathcal{E}_C$ to the normal cones of C which are rays.*

Proof: Let $C \in \mathcal{C}_m^0$ and let N be an atom of $\mathcal{N}_C$. Then N is a proper normal cone and contains, by definition of $\mathcal{C}_m^0$, a ray which is a normal cone of C. Since N is minimal in $\mathcal{N}_C \setminus \{\{0\}\}$, it must be equal to that ray. Conversely, we recall that the face lattice of a finite-dimensional convex set has finite length. So does $\mathcal{E}_C$ and, by the isomorphism (1), $\mathcal{N}_C$. It follows that every proper normal cone N of C contains an atom K of $\mathcal{N}_C$. By assumption K is a ray and by Lemma 3.1 the ray K is an exposed ray of N. This proves the first statement. The second statement follows from the first one because normal cones which are rays are atoms of $\mathcal{N}_C$. $\square$

A *sublattice* [9] of a lattice $(\mathcal{L}, \leq, \wedge, \vee)$ is a subset $\mathcal{L}' \subset \mathcal{L}$ such that $a, b \in \mathcal{L}'$ implies that $a \wedge b$ and $a \vee b$ lie in $\mathcal{L}'$. Clearly, the intersection of any family of sublattices is another sublattice. Therefore the smallest sublattice of $\mathcal{L}$ containing a given subset $X \subset \mathcal{L}$ exists and is called the *sublattice generated* by X.

Let $C \subset \mathbb{R}^m$ be a convex subset with interior points. Every ray in $\mathcal{N}_C$ is an atom of $\mathcal{N}_C$ so every smooth exposed face of C is a coatom of $\mathcal{E}_C$ by (1). Hence, every smooth exposed point s of C is simultaneously a coatom and an atom of $\mathcal{E}_C$ (we identify a point with its singleton). Therefore $\{\emptyset, \{s\}, C\}$ is a sublattice of $\mathcal{E}_C$ and the remainder $\mathcal{E}_C \setminus \{s\}$ is a sublattice, too. Similarly, if we define

$$S_C := \{s \in C \mid s \text{ is a smooth exposed point of } C\}$$

then $S_C \cup \{\emptyset, C\}$ and $\mathcal{E}_C \setminus S_C$ are sublattices of $\mathcal{E}_C$. Let $\mathcal{L}_C$ be the sublattice of $\mathcal{E}_C$ generated by $\{\emptyset, C\}$ and by the smooth coatoms of $\mathcal{E}_C$ which are no singletons. We call $\mathcal{L}_C$ *lattice of large smooth coatoms* of $\mathcal{E}_C$. By what we have discussed in this paragraph, $\mathcal{L}_C \subset \mathcal{E}_C \setminus S_C$ holds.

Theorem 3.3. *Let C be a proper convex subset of $\mathbb{R}^m$, $m \in \mathbb{N}$. Then S_C and $\bigcup_{F \in \mathcal{L}_C \setminus C} F$ are disjoint. We have $C \in \mathcal{C}_m^0$ if and only if the boundary of C is the union of the smooth coatoms of $\mathcal{E}_C$.*

Proof: About the first statement, we have noticed in the preceding paragraph that $\mathcal{L}_C \subset \mathcal{E}_C \setminus S_C$ holds, so for every $s \in S_C$ it follow $\{s\} \notin \mathcal{L}_C$. Since $\{s\}$ is a coatom of $\mathcal{E}_C$ the point s cannot lie on any coatom of $\mathcal{E}_C$ other than $\{s\}$. This proves the disjointness.

We derive the second statement from the property that the union of coatoms of $\mathcal{E}_C$ is the boundary of C, which holds because the union of proper exposed faces of C is the boundary of C (see for example Theorem 18.2 of [34]). If $C \in \mathcal{C}_m^0$ then Theorem 3.2 shows that every coatom of $\mathcal{E}_C$ is smooth. The converse is true because every coatom of $\mathcal{E}_C$ is needed to fill up the boundary of C. Indeed, if a relative interior point x of a coatom F of C is covered by a coatom G of $\mathcal{E}_C$ then $G \subset F$ follows from Theorem 18.1 of [34]. This proves $F = G$. □

Example 6.7 discusses the partition of the boundary of C into S_C and its complement for convex support sets. Notice that the second assertion of Theorem 3.3 is weaker than $\mathcal{E}_C = S_C \cup \mathcal{L}_C$, which holds for $C \in \mathcal{C}_m$ by Theorem 4.1. The assumptions of these two theorems cannot be weakened arbitrarily: The boundary of the lens $C_{()} \notin \mathcal{C}_2^0$, see Example 2.5, is not covered by smooth coatoms of $\mathcal{E}_C$. We have a proper inclusion $S_C \cup \mathcal{L}_C \subsetneq \mathcal{E}_C$ for the truncated disk $C = C_{(|} \in \mathcal{C}_2^0 \setminus \mathcal{C}_2$.

We now collect properties of strictly convex bodies, which we meet again in Section 5 as the polars of smooth convex bodies. Let $C \subset \mathbb{R}^m$ be a convex subset with interior points. We call C *strictly convex* if the relative interior of every closed segment in C lies in the interior of C, *pseudo-oval* if C is smooth and strictly convex, and *oval* if C is a compact pseudo-oval.

Lemma 3.4. *Let C be a proper convex subset of $\mathbb{R}^m$, $m \in \mathbb{N}$. Then the following statements are equivalent.*

(1) *The set C is strictly convex,*
(2) *every coatom of $\mathcal{E}_C$ is a singleton,*
(3) *every boundary point of C is an exposed point of C.*

Proof: We show (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). The first implication is true because coatoms of $\mathcal{E}_C$ are non-empty convex subsets of the boundary of C, see the second paragraph of the proof of Theorem 3.3. The second implication is true because every boundary point of C lies in a proper exposed face of C. The third implication follows from the definition of a face. □

Lemma 3.4 and the following statement improve Theorem 3.1 of [40] about ovals. Notice that the lens $C_{()} \notin \mathcal{C}_2^0$ from Example 2.5 is strictly convex but no pseudo-oval.

Corollary 3.5. *Let $C \in \mathcal{C}_m^0$, $m \in \mathbb{N}$. If C is strictly convex then C is a pseudo-oval.*

Proof: Theorem 3.3 shows that the boundary of C is covered by smooth coatoms of $\mathcal{E}_C$, which are singletons by Lemma 3.4 because C is strictly convex. Therefore every proper normal cone of C is a ray which proves the claim. □

The next statement characterizes ovals under a stronger assumption than required in Corollary 3.5. The proof uses *Minkowski's theorem*, which states that every non-empty convex body is the convex hull of its extreme points (for a proof see for example Corollary 1.4.5 of [38]).

Lemma 3.6. *Let C be a proper smooth convex body in $\mathbb{R}^m$, $m \in \mathbb{N}$. Then C is an oval if and only if C has no non-exposed face.*

Proof: An oval has no non-exposed face because this would imply the existence of a proper exposed face of dimension one or larger. Conversely, by Lemma 3.4 it suffices to show that any coatom F of $\mathcal{E}_C$ is a singleton. Minkowski's theorem shows that F contains an extreme point x which, by hypothesis, is an exposed point of C. By contradiction, if $\{x\} \neq F$ then the isomorphism (1) shows that the normal cone $N_C(x)$ has at least dimension two, which contradicts the smoothness of C. □

Analogues of Lemma 3.6 about unbounded and non-closed sets are wrong because of possibly missing extreme points. Consider a closed half-space or an open square with an open segment attached to one of its sides.

4. Intersections of exposed faces

We show that every proper exposed face of a set of class $\mathcal{C}_m$ admits a representation as an intersection of coatoms of exposed faces, taking into account the dimension of normal cones. At the end of the section we continue the discussion of the classes $\mathcal{C}''_m \subset \mathcal{C}'_m \subset \mathcal{C}_m$ from Remark 2.2.

Corollary 2.3 is a consequence of the following statement.

Theorem 4.1. *Let C be a proper convex subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, and $F \in \mathcal{E}_C$ a proper exposed face, and set $N := N_C(F)$ and $d := \dim(N)$. Then the following statements are equivalent.*

(1) *N has d linearly independent exposed rays which are normal cones of C.*
(2) *There exist d mutually distinct coatoms of $\mathcal{E}_C$ whose intersection is F and whose normal cones are linearly independent exposed rays of N.*

Proof: Statement (2) is clearly stronger than (1); we prove that it follows from (1). Let N have d linearly independent exposed rays which lie in $\mathcal{N}_C$. Their supremum K in $\mathcal{N}_C$ is included in N, hence K is an exposed face of N by Lemma 3.1. This shows $K = N$ because $\dim(K) = d$ and because a proper face of any convex set has codimension at least one (Corollary 8.1.3 of [34]). The rays are atoms of $\mathcal{N}_C$, hence the isomorphism (1) shows that the corresponding exposed faces are coatoms of $\mathcal{E}_C$ and that their intersection is F. □

We discuss further corollaries of Theorem 4.1. Let $C \subset \mathbb{R}^m$, $m \in \mathbb{N}$, be any convex subset. We call a face $F \neq \emptyset$ of C a *corner*, if $\dim(N_C(F)) = m$. Every corner is an exposed face of C and a singleton, see for example Lemma 4.4 of [44]; we call its element a *corner point*.

Corollary 4.2. *Let C be a proper convex subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, and let $F \in \mathcal{E}_C$ be a proper exposed face for which the equivalent statements of Theorem 4.1 hold. Then the following is true.*

(1) *The normal cone $N_C(F)$ is a ray if and only if F is a coatom of $\mathcal{E}_C$,*
(2) $\dim(N_C(F)) = 2$ *holds if and only if F lies on a unique pair of coatoms of $\mathcal{E}_C$; in that case F is the intersection of the pair,*
(3) *F is a corner of C if and only if F is the intersection of m mutually distinct coatoms of $\mathcal{E}_C$ whose normal cones are exposed rays of $N_C(F)$ which span $\mathbb{R}^m$.*
(4) *The lattice $[F, C]_{\mathcal{E}_C}$ is coatomistic. The coatoms of $[F, C]_{\mathcal{E}_C}$ are the coatoms of $\mathcal{E}_C$ which are included in $[F, C]_{\mathcal{E}_C}$.*

Proof: This corollary follows mainly from Theorem 4.1(2). A case selection of the dimension of $N_C(F)$ suffices to complete the proofs of the Items (1)–(3). Item (4) is completed by using basic properties of intervals and coatoms. □

Corollary 4.2(4) has an interpretation in terms of normal cones.

Remark 4.3. If F is a proper exposed face of a proper convex subset $C \subset \mathbb{R}^m$ then the isomorphism (1) restricts to an antitone lattice isomorphism

$$[F, C]_{\mathcal{E}_C} \to [\{0\}, N_C(F)]_{\mathcal{N}_C}$$

to the interval of normal cones $[\{0\}, N_C(F)]_{\mathcal{N}_C}$, which is therefore atomistic. This isomorphism is simplest possible for $C \in \mathcal{C}''_m$: All non-empty faces of $N_C(F)$ lie in $\mathcal{N}_C$ by Lemma 3.1 and they all belong to the interval $[\{0\}, N_C(F)]_{\mathcal{N}_C}$.

We have to clarify a subtlety when stating that $\mathcal{E}_C$ is coatomistic. Namely, the intersection of proper exposed faces may be non-empty for some non-closed sets and is indeed non-empty for all closed convex cones[4]. This is not so for convex bodies.

Lemma 4.4. *Let C be a proper convex body of $\mathbb{R}^m$, $m \in \mathbb{N}$. Then the intersection of coatoms of $\mathcal{E}_C$ is empty.*

Proof: If the intersection of coatoms of $\mathcal{E}_C$ is non-empty then by (1) there is a proper normal cone $N \in \mathcal{N}_C$ which contains all proper normal cones. Since N is pointed, some vectors of $\mathbb{E}^m$ are no normal vectors at points of C. This can only happen when C is unbounded or not closed. □

Lemma 4.4 shows that Corollary 4.5(1) applies to convex bodies.

Corollary 4.5. *Let $C \in \mathcal{C}_m$, $m \in \mathbb{N}$.*

(1) *If the intersection of coatoms of $\mathcal{E}_C$ is empty then $\mathcal{E}_C$ is coatomistic.*
(2) *If the intersection of coatoms of $\mathcal{E}_C$ is $A \neq \emptyset$ then $\mathcal{E}_C \setminus \{\emptyset\} = [A, C]_{\mathcal{E}_C}$ is a coatomistic lattice where the infimum is the intersection.*

[4]If the intersection of coatoms of $\mathcal{E}_C$ is non-empty then $\mathcal{E}_C \setminus \{\emptyset\}$ would be a more natural definition of a lattice of exposed faces of C.

Proof: This follows immediately from Theorem 4.1(2). □

We show $\mathcal{C}'_m \subset \mathcal{C}_m$ using Straszewicz's theorem, which affirms that every extreme point of a convex body is a limit of exposed points (for a proof see for example Theorem 1.4.7 of [38]).

Lemma 4.6. *Let $N \subset \mathbb{R}^m$ be a pointed closed convex cone. Then N has* $\dim(N)$ *linearly independent exposed extreme rays.*

Proof: Since N is a closed pointed cone, it admits a compact hyperplane intersection through its relative interior. Straszewicz's theorem shows that this intersection is the closed convex hull of its exposed points, so N is the closed convex hull of its exposed rays. Hence N has $\dim(N)$ exposed rays which are linearly independent. □

Actually, Lemma 4.6 proves a bit more than $\mathcal{C}'_m \subset \mathcal{C}_m$, we will return to it in (5). We prove the claim of Remark 2.2(3) that the class $\mathcal{C}''_m$ does not increase when *face* is replaced with *exposed face* in the Definition 2.1 of $\mathcal{C}''_m$.

Definition 4.7. Let $\mathcal{C}''^{*}_m$ denote the class of proper convex subsets of $\mathbb{R}^m$, $m \in \mathbb{N}$, such that every non-empty exposed face of every proper normal cone of C is a normal cone of C.

Let C be a convex subset of $\mathbb{R}^m$, $m \in \mathbb{N}$. A subset $F \subset C$ is a *poonem* [17] of C if there are sets $F_0 \subset \cdots \subset F_k$, $k \in \mathbb{N}$, such that $F_0 = F$ and $F_k = C$ and such that F_{i-1} is an exposed face of F_i for $i = 1, \ldots, k$. It is not hard to prove that $F \subset C$ is a poonem of C if and only if F is a face of C, see Section 1.2.1 of [43].

Lemma 4.8. *For all $m \in \mathbb{N}$ we have $\mathcal{C}''_m = \mathcal{C}''^{*}_m$.*

Proof: Clearly $\mathcal{C}''_m \subset \mathcal{C}''^{*}_m$ holds. Conversely let $C \in \mathcal{C}''^{*}_m$. It suffices to prove for every non-empty face T of every proper normal cone N of C that T is a normal cone of C. Since T is a poonem of N there are sets $F_0 \subset \cdots \subset F_k$, $k \in \mathbb{N}$, such that $F_0 = T$ and $F_k = N$ and such that F_{i-1} is an exposed face of F_i for $i = 1, \ldots, k$. By the assumption of $C \in \mathcal{C}''^{*}_m$ the exposed face F_{k-1} of N is a normal cone of C. By induction T is a normal cone of C. □

5. Convex bodies

Theorem 5.4 is an equivalent statement of Definition 2.1 for convex bodies in terms of their polars. It allows to present the Examples 5.8, 5.9, 5.10 which finish Remark 2.2. Remark 5.3 recalls that $\mathcal{C}''_m$ is closed under projection to subspaces and Example 5.8 shows that 2D projections of sets from $\mathcal{C}'_3$ may not lie in $\mathcal{C}^0_2$.

Throughout this section let $K \subset \mathbb{R}^m$, $m \in \mathbb{N}$, be a convex body including the origin $0 \in \mathbb{R}^m$ in its interior. The *polar* of K is

$$K^\circ := \{u \in \mathbb{R}^m \mid \langle u, x\rangle \leq 1, x \in K\}. \tag{2}$$

It is well-known [34, 38] that K° is a convex body with the origin in its interior and that $(K^\circ)^\circ = K$ holds.

Definition 5.1. Let $C \subset \mathbb{R}^m$ be a convex subset and assume $u \in \mathbb{R}^m$ is such that $x \mapsto \langle x, u\rangle$ has a maximum on C. Then the exposed face $F := \operatorname{argmax}_{x\in C}\langle x, u\rangle$ is non-empty and the *touching cone* of C at u is defined as the face of the normal cone $N_C(F)$ including u in its relative interior [38]. The normal cones $N_C(\emptyset) = \mathbb{R}^m$ and $N_C(C)$ are touching cone by definition. The set of touching cones of C is denoted by $\mathcal{T}_C$, the set of faces of C by $\mathcal{F}_C$.

Partially ordered by inclusion, $\mathcal{T}_C$ and $\mathcal{F}_C$ are complete lattices where the infimum is the intersection [43]. Recall that every normal cone is a touching cone and every exposed face is a face. We shall use Theorem 7.4 of [43]:

Fact 5.2. *Let $C \subset \mathbb{R}^m$ be a convex subset. Then $\mathcal{T}_C$ is the set of non-empty faces of normal cones of C. If C has an interior point and if $T_0 \in \mathcal{T}_C$ is not equal to $\mathbb{R}^m$ then the interval $[\{0\}, T_0]_{\mathcal{T}_C}$ is equal to $\{T \subset \mathbb{R}^m \mid T$ is a non-empty face of $T_0\}$.*

The class $\mathcal{C}''_m$ is closed under projection.

Remark 5.3. Fact 5.2 shows that a convex subset $C \subset \mathbb{R}^m$, $m \in \mathbb{N}$, with interior point lies in $\mathcal{C}''_m$ if and only if $\mathcal{N}_C = \mathcal{T}_C$. This can be used to show that $\mathcal{C}''_m$ is closed under projection. More precisely, for $k = 1, \ldots, m$ any k-dimensional image of any element of $\mathcal{C}''_m$ under a linear map $\mathbb{R}^m \to \mathbb{R}^k$ belongs to $\mathcal{C}''_k$. Indeed, the property that all non-empty faces of normal cones are normal cones is passed from C to its linear images by Corollary 7.7 of [43].

The lattices of K and K° are related by a commutative diagram [43, 44]:

$$\begin{tikzcd}
\mathcal{F}_K \arrow[d, "\mathrm{pos}"'] & \supset & \mathcal{E}_K \arrow[d, "\mathrm{pos}"'] & \cong & \mathcal{N}_K & \subset & \mathcal{T}_K \\
\mathcal{T}_{K^\circ} & \supset & \mathcal{N}_{K^\circ} & \cong & \mathcal{E}_{K^\circ} \arrow[u, "\mathrm{pos}"'] & \subset & \mathcal{F}_{K^\circ} \arrow[u, "\mathrm{pos}"']
\end{tikzcd} \tag{3}$$

The antitone lattice isomorphism (1) appears once as $\mathcal{E}_K \to \mathcal{N}_K$ and once as $\mathcal{E}_{K^\circ} \to \mathcal{N}_{K^\circ}$. The positive hull operator pos defines isotone lattice isomorphisms $\mathcal{F}_K \to \mathcal{T}_{K^\circ}$, $\mathcal{E}_K \to \mathcal{N}_{K^\circ}$, $\mathcal{E}_{K^\circ} \to \mathcal{N}_K$, and $\mathcal{F}_{K^\circ} \to \mathcal{T}_K$ (we set $\mathrm{pos}(\emptyset) = \{0\}$). The map $\mathcal{E}_K \to \mathcal{E}_{K^\circ}$ which makes the diagram commute maps $F \in \mathcal{E}_K$ to its *conjugate face* $\{u \in K^\circ \mid \langle u, x\rangle = 1$ for all $x \in F\}$.

The diagram (3) allows Definition 2.1 to be reformulated in terms of the polar.

Theorem 5.4.
(1) $K \in \mathcal{C}_m^0 \iff$ *every proper exposed face of* K° *contains an exposed point of* K°,
(2) $K \in \mathcal{C}_m \iff$ *every proper exposed face* F *of* K° *contains* $\dim(F)+1$ *affinely independent exposed points of* K°,
(3) $K \in \mathcal{C}_m' \iff K^\circ$ *has no non-exposed points,*
(4) $K \in \mathcal{C}_m'' \iff K^\circ$ *has no non-exposed faces* $\iff \mathcal{N}_K = \mathcal{T}_K$.

Proof: Let N be a proper normal cone of K. Then N is a pointed closed cone. Let F be the unique proper exposed face of K° such that $N = \mathrm{pos}(F)$, which exists by the isomorphism $\mathcal{E}_{K^\circ} \to \mathcal{N}_K$ of diagram (3). Then pos restricts to a lattice isomorphism $\mathcal{F}_F \to \mathcal{F}_N \setminus \{\emptyset\}$ as was observed in Lemma 3.4 of [43]. It is easy to see that pos restricts further to a lattice isomorphism $\mathcal{E}_F \to \mathcal{E}_N \setminus \{\emptyset\}$.

The isomorphism $\mathcal{E}_{K^\circ} \to \mathcal{N}_K$ of diagram (3), which identifies proper exposed faces F of K° with proper normal cones N of K combined with the isomorphisms $\mathcal{F}_F \to \mathcal{F}_N \setminus \{\emptyset\}$ and $\mathcal{E}_F \to \mathcal{E}_N \setminus \{\emptyset\}$ proves (1), (2), and proves further that $K \in \mathcal{C}_m''$ holds if and only if every face of every proper exposed face of K° lies in $\mathcal{E}_{K^\circ}$.

To prove (4) we assume the last statement to be true and show that any proper face G of K° is exposed. The proper face G lies in a proper exposed face F of K°, hence G is a face of F and by assumption G is an exposed face of K°. This shows $\mathcal{E}_{K^\circ} = \mathcal{F}_{K^\circ}$. The converse is clear because every face of every face of K° is a face of K° [34]. The second condition $\mathcal{N}_K = \mathcal{T}_K$ of (4) follows from the isomorphisms $\mathcal{E}_{K^\circ} \to \mathcal{N}_K$ and $\mathcal{F}_{K^\circ} \to \mathcal{T}_K$ of diagram (3) or from Fact 5.2 (already observed in Remark 5.3).

The proof of (3) is analogous to the proof of the first condition of (4), now with G an extreme point rather than a face. $\square$

We observe that Theorem 5.4(4) simplifies (3) to the following commutative diagram, valid for $K \in \mathcal{C}_m''$.

$$\begin{tikzcd}
\mathcal{F}_K \arrow[d, "\mathrm{pos}"'] & \supset & \mathcal{E}_K \arrow[d, "\mathrm{pos}"'] & \cong & \mathcal{N}_K & = & \mathcal{T}_K \\
\mathcal{T}_{K^\circ} & \supset & \mathcal{N}_{K^\circ} & \cong & \mathcal{E}_{K^\circ} \arrow[u, "\mathrm{pos}"'] & = & \mathcal{F}_{K^\circ} \arrow[u, "\mathrm{pos}"']
\end{tikzcd} \tag{4}$$

Notably, if $K \in \mathcal{C}_m''$ then Minkowski's theorem shows that $\mathcal{E}_{K^\circ} = \mathcal{F}_{K^\circ}$ is atomistic, so $\mathcal{E}_K$ is coatomistic (compare to Corollary 4.5 for general convex sets).

Another consequence of the isomorphism $\mathcal{E}_{K^\circ} \to \mathcal{N}_K$ of diagram (3) and of Lemma 3.4(2) is the following.

Corollary 5.5. *The convex body* K *is smooth if and only if* K° *is strictly convex.*

In a sense, the next statement generalizes Corollary 5.5 from $\mathcal{C}''_m$ to the class $\mathcal{C}'_m$. The proof is inspired by Proposition 2.2 of [37].

Corollary 5.6. *If the extreme points of K° lie on the boundary of a strictly convex set with interior points then $K \in \mathcal{C}'_m$.*

Proof: Let C be the strictly convex set containing the extreme points of K°. Every boundary point of C is an exposed point of C by Lemma 3.4 so the extreme points of K° are exposed points of C and *a fortiori* of K°. This proves that K° has no non-exposed points so Theorem 5.4(3) shows $K \in \mathcal{C}'_m$. $\square$

We finish the discussion of Remark 2.2.

Definition 5.7. Let $\mathcal{C}'^*_m$ denote the class of proper convex subsets of $\mathbb{R}^m$, $m \in \mathbb{N}$, such that every exposed ray of every proper normal cone of C lies in $\mathcal{N}_C$.

Lemma 4.6 proves $\mathcal{C}'^*_m \subset \mathcal{C}_m$ and this implies

$$\mathcal{C}''_m \subset \mathcal{C}'_m \subset \mathcal{C}'^*_m \subset \mathcal{C}_m \subset \mathcal{C}^0_m. \tag{5}$$

For $m = 3$ the first three inclusions of (5) are strict by the Examples 5.8, 5.9, and 5.10. The last inclusion is strict already for $m = 2$ by Example 2.5.

Example 5.8 (Convex hull of ball and lens, $\mathcal{C}''_3 \subsetneq \mathcal{C}'_3$). We use Corollary 5.6 and construct a convex body K the extreme points of whose polar K° lie on a sphere. Consider the lens $C_{()} = D_- \cap D_+$ from Example 2.5, where

$$D_\pm := \{(x, y) \in \mathbb{R}^2 \mid (x \pm \tfrac{3}{2})^2 + y^2 \leq (\tfrac{5}{2})^2\},$$

and its embedding into $\mathbb{R}^3$ defined by

$$\widetilde{C_{()}} := C_{()} \oplus \{0\} = \{(x, y, 0) \in \mathbb{R}^3 \mid (x, y) \in C_{()}\}.$$

Let $K := \operatorname{conv}(\widetilde{C_{()}} \cup B)$ where B is the closed Euclidean unit ball of $\mathbb{R}^3$. Corollary 16.5.2 of [34] proves

$$K^\circ = \operatorname{conv}(D^\circ_- \cup D^\circ_+) \oplus \mathbb{R} \cap B \tag{6}$$

which shows clearly that the extreme points of K° lie on the unit sphere of $\mathbb{R}^3$. So Corollary 5.6 proves $K \in \mathcal{C}'_3$.

We use Theorem 5.23 and Algorithm 5.1 of [36] to compute the polar of $D_\pm$,

$$D^\circ_\pm = \{(x, y) \in \mathbb{R}^2 \mid \tfrac{1}{25}(8x \mp 3)^2 + (2y)^2 \leq 1\}.$$

Clearly $\operatorname{conv}(D^\circ_- \cup D^\circ_+)$ has the non-exposed points $(\sigma\frac{3}{8}, \tau\frac{1}{2})$ for signs $\sigma, \tau \in \{+, -\}$. The two ellipses $D^\circ_\pm$ lie in the unit disk, so the segments

$$[(\sigma\tfrac{3}{8}, \tau\tfrac{1}{2}, -\tfrac{\sqrt{39}}{8}), (\sigma\tfrac{3}{8}, \tau\tfrac{1}{2}, +\tfrac{\sqrt{39}}{8})], \quad \sigma, \tau \in \{+, -\}$$

are non-exposed faces of K° by equation (6). Now Theorem 5.4(4) shows $K \notin \mathcal{C}''_3$. The projection of K onto the x-y-plane is $C_{()}$, which does not belong to $\mathcal{C}^0_2$. Therefore, the classes $\mathcal{C}^0_m$, $\mathcal{C}_m$, and $\mathcal{C}'_m$ are not closed under linear maps.

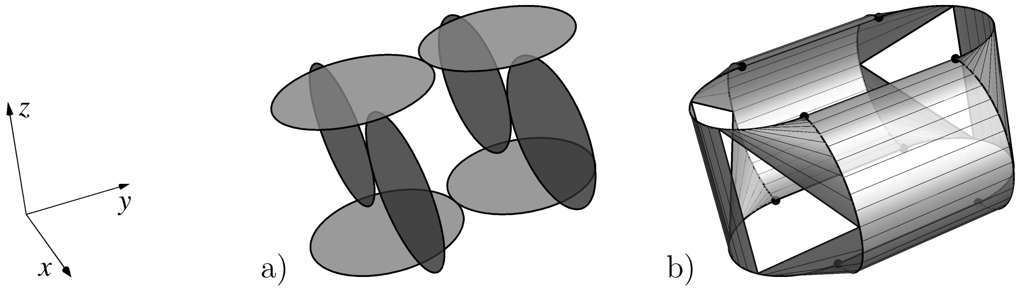

Figure 5.1: **a)** Unit disks with normal vectors in z-direction centered at $(0, \pm 1, \pm 1)$ (respectively, y-direction centered at $(\pm 1, \pm 1, 0)$). **b)** Convex hull of the eight disks from a), only the boundary is depicted, two-dimensional faces are deleted, ruled surfaces are striped, and non-exposed points are marked at $(\pm 1, \pm 1, \pm 1)$.

Example 5.9 (Polar of four stadia, $\mathcal{C}'_3 \subsetneq \mathcal{C}'^*_3$). In analogy to statement (3) in Theorem 5.4 one proves that $K \in \mathcal{C}'^*_m$ holds if and only if every exposed point of every proper exposed face of K° lies in $\mathcal{E}_{K^\circ}$. Therefore, if L is a convex body with a non-exposed point every exposed point of whose proper exposed faces is an exposed point of L, then the polar $K^\circ = L$ satisfies $K \in \mathcal{C}'^*_3 \setminus \mathcal{C}'_3$.

Let L be the convex set defined in Figure 5.1, which is the convex hull of a cube with four half-cylinders attached, or the convex hull of four stadia. Two of the stadia lie in $\mathcal{E}_L$ and include eight non-exposed points of L. Their four straight sides are non-exposed faces of L. Therefore the eight non-exposed points of L are no exposed points of any proper exposed face of L. This completes the example.

Although unrelated to the inclusion $\mathcal{C}'_3 \subsetneq \mathcal{C}'^*_3$, it is nevertheless interesting that L is the convex support (10) of three 16-by-16 matrices and that therefore the results of the Sections 3 and 4 apply to L. Indeed, the unit disk with normal vector in z-direction centered at $(0, 1, 1)$ is the convex support of

$$F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly the other seven disks of Figure 5.1a) are convex support sets and their convex hull L is the convex support of the direct sum of those matrices.

The lattice of exposed faces $\mathcal{E}_L$ has six coatoms of dimension two (two stadia and four triangles). The reminder of the boundary of L is covered by four cylindrical and eight curved ruled surfaces. The straight segments on the curved ruled surface of the positive octant have end-points $(\cos(\varphi), 1 + \sin(\varphi), 1)$ and

$(1+\sin(\psi), 1, \cos(\psi))$, where

$$\cos(\psi) = \frac{2\cos(\varphi) - 1}{2 - 2\cos(\varphi) + \cos(2\varphi)}\ , \quad \sin(\psi) = \frac{4\cos(\varphi)\sin(\frac{\varphi}{2})^2}{2 - 2\cos(\varphi) + \cos(2\varphi)}\ ,$$

and $\varphi \in (0, \frac{\pi}{3})$. All boundary points of L are smooth except for the exposed points. They cover a half-circle of each of the eight disks used to define L in Figure 5.1. Having a two-dimensional normal cone, each of them is the intersection of a unique pair of coatoms of $\mathcal{E}_L$ (segment, triangle, or stadium) by Corollary 4.2(2).

Example 5.10 (Polar of truncated torus, $\mathcal{C}_3'^* \subsetneq \mathcal{C}_3$). We construct a truncated torus L and show that its polar L° lies in $\mathcal{C}_3 \setminus \mathcal{C}_3'^*$. Consider the convex hull K of the torus $\{(x, y, z) \in \mathbb{R}^3 \mid p(x, y, z) = 0\}$ defined by

$$p := (x^2 + y^2 + z^2)^2 - 4(x^2 + y^2)$$

which is a torus with rotation symmetry about the z-axis and self-intersection at the origin $0 \in \mathbb{R}^3$. The radius from 0 to the center of the torus tube and the radius of the tube are both equal one. The convex body K is smooth, so $K \in \mathcal{C}_3''$, and has the following proper exposed faces:

$$D_\pm := \{(x, y, \pm 1) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\} \qquad \text{(two disks)}$$

and exposed points (x, y, z) for $x, y, z \in \mathbb{R}$ such that $p(x, y, z) = 0$ and $x^2+y^2 > 1$. The non-exposed points of K are

$$\xi_\pm(\varphi) := (\cos(\varphi), \sin(\varphi), \pm 1), \quad \varphi \in [0, 2\pi).$$

By intersecting K with three half-spaces ($\varphi_0 = 0, \varphi_1 = \frac{2}{3}\pi, \varphi_2 = \frac{4}{3}\pi$) we define

$$L := \{(x, y, z) \in K \mid x\cos(\varphi_i) + y\sin(\varphi_i) \leq 1, i = 0, 1, 2\}.$$

The convex body L has five 2D exposed faces, namely $D_\pm$ and the exposed faces in the boundaries of the three half-spaces. The latter three exposed faces of L contain no non-exposed points of L. The disk $D_\pm$ contains three exposed points $\xi_\pm(\varphi_1)$, $\xi_\pm(\varphi_2)$, $\xi_\pm(\varphi_3)$ of L while the remaining exposed points of $D_\pm$ are non-exposed points of L. The convex body L has no 1D exposed faces. This discussion of faces of L and the polar definitions of $\mathcal{C}_3$, see Theorem 5.4(2), and of $\mathcal{C}_3'^*$, see Example 5.9, show $L^\circ \in \mathcal{C}_3 \setminus \mathcal{C}_3'^*$.

6. State spaces of operator systems

Theorem 6.2 proves that the state space $\mathcal{M}(S)$ of an operator system S of M_n is the projection of the state space $\mathcal{M}_n$ of the matrix algebra M_n onto the real space $S_\mathrm{h} \subset S$ of hermitian matrices. We point out that

- $\mathcal{M}(S) \in \mathcal{C}''_m$ for $m = \dim_{\mathbb{C}}(S) - 1$ (remove one codimension of $\mathcal{M}(S)$ from $S_{\rm h}$),
- there is a lattice isomorphism from the ground state projections of $S_{\rm h}$ to the exposed faces of $\mathcal{M}(S)$,
- the ground state projections of $S_{\rm h}$ form a coatomistic lattice.

Some of these properties may be well-known, but we are unaware of proofs in the literature. We finish the article with representations of $\mathcal{M}(S)$ as a convex support set and as the convex hull of a joint numerical range. Finally we discuss convex support sets of 3-by-3 matrices.

Let $S \subset M_n$ be an operator system. Since S is a finite-dimensional complex vector space, every linear form $S \to \mathbb{C}$ is represented in terms of a unique $a \in S$ by

$$f_a(s) = \langle a, s \rangle, \quad s \in S.$$

Here $\langle a, b \rangle = \operatorname{tr}(a^* b)$ is the Hilbert-Schmidt inner product of $a, b \in S$. The real part of this inner product is a Euclidean scalar product under which $S = S_{\rm h} + \mathrm{i}\, S_{\rm h}$ is an orthogonal direct sum. The form f_a has real values on $S_{\rm h}$ if and only if $a \in S_{\rm h}$ so the Hilbert-Schmidt inner product restricts to a Euclidean scalar product on $S_{\rm h}$.

Definition 6.1. A *state* on an operator system $S \subset M_n$ is a linear form $f : S \to \mathbb{C}$ which is *positive*, that is $f(s) \geq 0$ holds for all positive-semidefinite $s \in S$, and *normalized*, that is $f(\mathbb{1}) = 1$. Let

$$\mathcal{M}(S) := \{ s \in S \mid f_s \text{ is a state on } S \}.$$

In abuse of notation we call elements of $\mathcal{M}(S)$ *states* on S and say that $\mathcal{M}(S)$ is the *state space* of S.

Let $K \subset \mathbb{R}^m$ be a convex body including the origin $0 \in \mathbb{R}^m$ in its interior. The *dual* of K is defined as the point reflection of the polar K°, introduced in (2),

$$K^* := -K^\circ = \{ u \in \mathbb{R}^m \mid 1 + \langle u, x \rangle \geq 0,\ x \in K \}.$$

It follows from the properties of K° that K^* is a convex body with the origin in its interior and that $(K^*)^* = K$ holds. Similarly, the *dual* of a convex cone $K \subset \mathbb{R}^m$ is

$$K^* = \{ u \in \mathbb{R}^m \mid \langle u, x \rangle \geq 0,\ x \in K \}$$

and K is *self-dual*, if $K = K^*$ holds. For every subspace $X \subset (M_n)_{\rm h}$ the mapping $\pi_X : (M_n)_{\rm h} \to X$ denotes the orthogonal projection from $(M_n)_{\rm h}$ onto X.

The following proof uses a property of self-dual convex cones: Projections of certain cone bases are dual to bounded affine sections through their interior.

Theorem 6.2. *If $S \subset M_n$ is an operator system then $\mathcal{M}(S) = \pi_{S_{\rm h}}(\mathcal{M}_n)$.*

Proof: It is well-known, see for instance [8], that the cone K of positive semi-definite matrices of M_n is self-dual with respect to $(M_n)_\mathrm{h}$. Without loss of generality, let $\dim_\mathbb{C}(S) > 1$. Then the space $S_0 \subset S_\mathrm{h}$ of traceless hermitian matrices of S has dimension $\dim_\mathbb{R}(S_0) = \dim_\mathbb{C}(S) - 1 > 0$. The affine section

$$\mathbb{A} := K \cap (\tfrac{1}{n}\mathbb{1} + S_0) = \mathcal{M}_n \cap (\tfrac{1}{n}\mathbb{1} + S_0)$$

of K is bounded and contains the trace state $\frac{1}{n}\mathbb{1}$, which is an interior point with respect to $(M_n)_\mathrm{h}$ of K. Since $\mathcal{M}_n = (\frac{1}{n}\mathbb{1} + \mathbb{1}^\perp) \cap K$ holds, where $\mathbb{1}^\perp$ denotes orthogonal complement, Theorem 2.16 of [42] proves (duals with respect to S_0)

(i) $\mathbb{A} - \frac{1}{n}\mathbb{1} = \frac{1}{n}\pi_{S_0}(\mathcal{M}_n)^*$
(ii) $\pi_{S_0}(\mathcal{M}_n) = \frac{1}{n}(\mathbb{A} - \frac{1}{n}\mathbb{1})^*$.

On the other hand, an easy computation shows

$$\begin{aligned}\mathcal{M}(S) &= \{s \in S_\mathrm{h} \mid \langle s, \mathbb{1}\rangle = 1, \langle s, a\rangle \geq 0 \,\forall a \in \mathbb{A}\}\\ &= \tfrac{1}{n}\mathbb{1} + \{s_0 \in S_0 \mid \langle \tfrac{1}{n}\mathbb{1} + s_0, a\rangle \geq 0 \,\forall a \in \mathbb{A}\}\\ &= \tfrac{1}{n}\mathbb{1} + \{s_0 \in S_0 \mid \tfrac{1}{n} + \langle s_0, a_0\rangle \geq 0 \,\forall a_0 \in \mathbb{A} - \tfrac{1}{n}\mathbb{1}\},\end{aligned}$$

that is,

(iii) $\mathcal{M}(S) = \frac{1}{n}(\mathbb{1} + (\mathbb{A} - \frac{1}{n}\mathbb{1})^*)$.

Substituting (ii) into (iii) gives $\mathcal{M}(S) = \frac{1}{n}\mathbb{1} + \pi_{S_0}(\mathcal{M}_n)$, as claimed. □

State spaces lie in $\mathcal{C}''_m$. More precisely the following holds, if we identify the space of traceless hermitian matrices of a d-dimensional operator system with $\mathbb{R}^{d-1}$.

Corollary 6.3. *If $S \subset M_n$ is an operator system of dimension $d := \dim_\mathbb{C}(S) > 1$ then $\mathcal{M}(S) - \frac{1}{n}\mathbb{1} \in \mathcal{C}''_{d-1}$.*

Proof: One can see from the proof of Theorem 6.2 that zero is an interior point with respect to S_0 of $\mathcal{M}(S) - \frac{1}{n}\mathbb{1} = \pi_{S_0}(\mathcal{M}_n)$. Further *i)* of Theorem 6.2 shows that the polar of $\pi_{S_0}(\mathcal{M}_n)$ is an affine section of $\mathcal{M}_n$. Since $\mathcal{M}_n$ has no non-exposed faces [33, 1] it follows that this affine section has no non-exposed faces either. Then Theorem 5.4(4) proves the claim. □

Analogues of Corollary 6.3 were proved in Fact 3.2 of [40] for convex support sets, defined below. One of the proofs uses that $\mathcal{M}_n - \frac{1}{n}\mathbb{1}$ lies in $\mathcal{C}''_{n^2-1}$ so the projection $\pi_{S_0}(\mathcal{M}_n)$ lies in $\mathcal{C}''_{d-1}$ by Remark 5.3.

We explain the lattice isomorphism from ground state spaces of an operator system to the exposed faces of the state space. We hope that inward pointing normal vectors will be agreeably natural in the setting of ground state spaces.

Remark 6.4. Let us begin with the algebra M_n where the isomorphism is probably better known, see Section 6 of [4], Chapter 3 of [1], and the references therein. We denote the set of projections $p = p^* = p^2 \in M_n$ by $\mathcal{P}$. Endowed with the partial ordering on $(M_n)_{\mathrm{h}}$ defined by

$$a \preceq b \quad\iff\quad b \succeq a \quad\iff\quad b-a \text{ is positive semi-definite} \tag{7}$$

the set $\mathcal{P}$ is a complete lattice. One can represent $\mathcal{P}$ in terms of the images of its elements, ordered by inclusion, see for example Chapter 2 of [1]. The infimum in this representation is the intersection. Let $u \in (M_n)_{\mathrm{h}}$ be a hermitian matrix and

$$u = \sum_\lambda \lambda p_{u,\lambda}, \quad \mathbb{1} = \sum_\lambda p_{u,\lambda}, \quad p_{u,\lambda} \in \mathcal{P}$$

its spectral decomposition (λ extends over the eigenvalues of u). The *ground state projection* of u is the spectral projection $p_-(u) := p_{u,\lambda_-(u)}$ of the least eigenvalue $\lambda_-(u)$ of u. The *ground state space* of u is the image of $p_-(u)$. The exposed face of $\mathcal{M}_n$ with inward pointing normal vector $u \in (M_n)_{\mathrm{h}}$

$$F_{\mathcal{M}_n}(u) := \operatorname{argmin}\{\langle \rho, u\rangle \mid \rho \in \mathcal{M}_n\}$$

is well-known to be the set of states which are supported on the ground state space of u, that is

$$F_{\mathcal{M}_n}(u) = \{\rho \in \mathcal{M}_n \mid \mathrm{Image}(\rho) \subset \mathrm{Image}(p_-(u))\}.$$

Further, the map $\mathcal{P} \to \mathcal{E}_{\mathcal{M}_n}$, with $p \mapsto F_{\mathcal{M}_n}(-p)$, if $p \neq 0$, and with $0 \mapsto \emptyset$, is a lattice isomorphism to the exposed faces of $\mathcal{M}_n$.

Consider now the state space $\mathcal{M}(S)$ of an operator system $S \subset M_n$. It is easy to see that for $s \in S_{\mathrm{h}}$ the exposed face

$$F_{\mathcal{M}(S)}(s) := \operatorname{argmin}\{\langle \rho, s\rangle \mid \rho \in \mathcal{M}(S)\}$$

lifts to the corresponding exposed face of $\mathcal{M}_n$, i.e. $F_{\mathcal{M}_n}(s) = \pi_{S_{\mathrm{h}}}|_{\mathcal{M}_n}^{-1}(F_{\mathcal{M}(S)}(s))$. Indeed, the map

$$\mathcal{E}_{\mathcal{M}(S)} \to \mathcal{E}'_{\mathcal{M}(S)} := \{\pi_{S_{\mathrm{h}}}|_{\mathcal{M}_n}^{-1}(F) \mid F \in \mathcal{E}_{\mathcal{M}(S)}\} \tag{8}$$

is a lattice isomorphism whose range $\mathcal{E}'_{\mathcal{M}(S)}$ is ordered by inclusion. Moreover, the infimum of $\mathcal{E}'_{\mathcal{M}(S)}$ is just the restriction of the infimum of $\mathcal{E}_{\mathcal{M}_n}$ to $\mathcal{E}'_{\mathcal{M}(S)}$, which is the intersection (see Proposition 5.6 of [43]). Let

$$\mathcal{P}(S) := \{p_-(s) \mid s \in S_{\mathrm{h}}\} \cup \{0\}$$

denote the set of ground state projections of S_{h} endowed with the partial ordering (7). Notice that $p_-(s)$ may not lie in S for some $s \in S_{\mathrm{h}}$. Since (8) is a lattice isomorphism, the lattice isomorphism $\mathcal{P} \to \mathcal{E}_{\mathcal{M}_n}$ restricts to a lattice isomorphism $\mathcal{P}(S) \to \mathcal{E}'_{\mathcal{M}(S)}$ and combines with (8) to a lattice isomorphism

$$\mathcal{P}(S) \to \mathcal{E}_{\mathcal{M}(S)}, \quad p \mapsto \begin{cases} \pi_{S_h}(F_{\mathcal{M}_n}(-p)), & \text{if } p \neq 0, \\ \emptyset, & \text{if } p = 0. \end{cases} \tag{9}$$

The isomorphism (9) is described in Section 3.1 of [42]. We stress that the infimum of $\mathcal{P}(S)$ is the restriction of the infimum of $\mathcal{P}$ because the infimum of $\mathcal{E}'_{\mathcal{M}(S)}$ is the restriction of the infimum of $\mathcal{E}_{\mathcal{M}_n}$.

If the projections of $\mathcal{P}(S)$ are represented in terms of their images then the infima of $\mathcal{E}_{\mathcal{M}(S)}$ and of $\mathcal{P}(S)$ are both given by the intersection. This is especially nice in the following corollary.

Corollary 6.5. *If $S \subset M_n$ is an operator system then the lattice of ground state projections $\mathcal{P}(S)$ is coatomistic.*

Proof: This follows from Corollary 6.3, Lemma 4.4, Corollary 4.5(1), and equation (9). □

We now recall two coordinate representations of state spaces (and show that polytopes belong to them). We define the *convex support* of $F_i \in (M_n)_{\mathrm{h}}$, $i = 1, \ldots, k$, $k \in \mathbb{N}$, by

$$\mathrm{cs}(F_1, \ldots, F_k) := \{\langle \rho, F_i \rangle_{i=1}^k \mid \rho \in \mathcal{M}_n\} \tag{10}$$

and the *joint numerical range* (using the inner product of $\mathbb{C}^n$) by

$$W(F_1, \ldots, F_k) := \{\langle x, F_i(x) \rangle_{i=1}^k \mid \langle x, x \rangle = 1, x \in \mathbb{C}^n\}.$$

Remark 6.6 (Convex support sets).

(1) *State spaces and convex support sets.* Let $S \subset M_n$ be an operator system and let $F_i \in (M_n)_{\mathrm{h}}$ for $i = 1, \ldots, k$, such that $\{\mathbb{1}, F_1, \ldots, F_k\}$ spans S_{h}. Then the state space $\mathcal{M}(S)$ and the convex support $\mathrm{cs}(\mathbb{1}, F_1, \ldots, F_k)$ are affinely isomorphic. Indeed, $\mathcal{M}(S) = \pi_{S_{\mathrm{h}}}(\mathcal{M}_n)$ holds by Theorem 6.2 and the linear map

$$\alpha : S_{\mathrm{h}} \to \mathbb{R}^{k+1}, s \mapsto (\langle s, \mathbb{1} \rangle, \langle s, F_1 \rangle, \ldots, \langle s, F_k \rangle)$$

is injective, see Remark 1.1(1) of [42] for a proof. As all matrices of $\mathcal{M}(S)$ have unit trace, the state space $\mathcal{M}(S)$ is affinely isomorphic to the convex support $\mathrm{cs}(F_1, \ldots, F_k)$.

(2) *Convex support sets and joint numerical ranges.* If $F_i \in (M_n)_{\mathrm{h}}$, $i = 1, \ldots, k$, then

$$\mathrm{cs}(F_1, \ldots, F_k) = \mathrm{conv}\left(W(F_1, \ldots, F_k)\right).$$

For a proof see [29] or Section 2 of [40]. For $k = 2$ the joint numerical range $W(F_1, F_2)$ is the numerical range of $F_1 + \mathrm{i}\, F_2$ which is convex for all $n \in \mathbb{N}$ by the Toeplitz-Hausdorff theorem. The joint numerical range is convex for $k = 3$ and $n \geq 3$, but fails to be convex in general for $k \geq 4$ [3, 26, 18].

(3) *Polytopes.* Consider a collection of n points $p_j \in \mathbb{R}^m$, $n, m \in \mathbb{N}$. Define an m-by-n-matrix M whose j's column is p_j, $j = 1, \ldots, n$, and define diagonal matrices $F_i \in M_n$ where $\mathrm{diag}(F_i)$ is the i's row of M, $i = 1, \ldots, m$. Then the convex support $\mathrm{cs}(F_1, \ldots, F_m)$ is the convex hull of $\{p_1, \ldots, p_n\} \subset \mathbb{R}^m$.

We discuss convex support sets of 3-by-3 matrices.

Example 6.7. Consider the convex support $\mathrm{cs}(F)$ of $k \in \mathbb{N}$ hermitian 3-by-3 matrices $F = (F_1, \ldots, F_k)$ and assume without loss of generality that $\mathrm{cs}(F)$ has an interior point. Then $\mathrm{cs}(F) \in \mathcal{C}_k''$ holds by Remark 6.6(1) and Corollary 6.3.

The subset of non-smooth points $\partial_\mathrm{n} \mathrm{cs}(F)$ of the boundary $\partial \mathrm{cs}(F)$ consists of the non-smooth exposed points. Indeed, the form of a proper exposed face G of $\mathrm{cs}(F)$ is very restricted for 3-by-3 matrices. It may be a singleton (exposed point), either smooth or not. Otherwise G is a segment, a filled ellipse, or filled ellipsoid [40]. We see that proper exposed faces of $\mathrm{cs}(F)$ have no segments on their relative boundary. Therefore, if G is not a singleton then it is necessarily a coatom of $\mathcal{E}_{\mathrm{cs}(F)}$ and is therefore smooth (has a unique unit normal vector) by Theorem 3.2. For the same reason, every non-exposed face of $\mathrm{cs}(F)$ is a singleton. Moreover, every non-exposed point is smooth because it has the same normal cone as the smallest exposed face in which it is contained (see Lemma 4.6 of [43]). This proves the claim. Equivalently by Corollary 2.3, the set of non-smooth points $\partial_\mathrm{n} \mathrm{cs}(F)$ consists of intersection points of pairs of mutually distinct coatoms of $\mathcal{E}_{\mathrm{cs}(F)}$.

A notable property of 3-by-3 matrices is that each two coatoms of $\mathcal{E}_{\mathrm{cs}(F)}$ which are both no singletons must intersect. This follows from the analogue property of $k = 2$ by projecting $\mathrm{cs}(F)$ onto the span of the normal vectors of these coatoms (for a proof when $k = 3$ see Lemma 5.1 of [40], $k > 3$ is analogous). The case $k = 2$ is solved by the classification of the numerical range of 3-by-3 matrices [24, 23]. So, $\mathcal{E}_{\mathrm{cs}(F)}$ has only one cluster of coatoms of positive dimension, in the sense of Corollary 2.4.

The classification [40] proves that $\partial_\mathrm{n} \mathrm{cs}(F)$ is closed for $k = 3$. If $\mathrm{cs}(F)$ has a corner then $\mathrm{cs}(F)$ is either the convex hull of an ellipsoid and a point outside the ellipsoid or the convex hull of an ellipse and a point outside the affine hull of the ellipse. In the first case $\partial_\mathrm{n} \mathrm{cs}(F)$ is a singleton, in the second case $\partial_\mathrm{n} \mathrm{cs}(F)$ is the union of an ellipse and a singleton. If $\mathrm{cs}(F)$ has no corner then the cluster of $\mathcal{E}_{\mathrm{cs}(F)}$ with coatoms of positive dimension contains s segments and e ellipses such that

$$(s, e) \in \{(0,0), (0,1), (0,2), (0,3), (0,4), (1,0), (1,1), (1,2)\}.$$

The coatoms of this cluster intersect in pairs but not in triples [40], so $\partial_\mathrm{n} \mathrm{cs}(F)$ has cardinality $\binom{s+e}{2} \in \{0, 1, 3, 6\}$. Since $\partial_\mathrm{n} \mathrm{cs}(F)$ is closed it follows that its complement $\partial \mathrm{cs}(F) \setminus \partial_\mathrm{n} \mathrm{cs}(F)$ is a C^1-submanifold of $\mathbb{R}^3$ (Theorem 2.2.4 of [38] can be used locally to prove this).

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