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A Bohr Mollerup Theorem for Interpolating the Triangular Numbers

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The Bohr-Mollerup Theorem (1922) provides an elegant criterion under which the gamma function is the unique function interpolating $n!$. We prove an analogous uniqueness theorem for interpolating the triangular numbers that, like the original, is grounded in the theory of convex functions. We then explore parallels with the class of quasi-gamma functions defined in a recent paper by Bermúdez, Martinón and Sadarangani (2014).

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1. The delta function

In 1729, Christian Goldbach engaged Leonard Euler in a series of letters discussing the existence of a function $f(x)$ defined on all of $\mathbb{R}$ with the property that $f(n) = n!$ for all positive integers n. One interesting aspect of this exchange is the unstated assumption between the correspondents that there was a unique right answer. From a modern viewpoint, interpolating $n!$ amounts to finding a curve in the plane passing through the points $(n, n!)$. There are obviously myriad possibilities to choose from, and insisting that the curve be continuous, differentiable, or even analytic does not change this fact. So how did Euler know when he had found the proper one? Part of the answer lies in the evolution of the definition of a function. In Euler's day, the term "function" meant something closer to "formula," and implicitly referred to mappings defined by analytic expressions comprised of the elementary functions and the tools of calculus. This is most certainly how Euler interpreted the original challenge from Goldbach. In fact, Euler found several equivalent analytic expressions for a function that successfully extended the domain of the factorial function to the real line. The one most familiar to current audiences is the

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gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x \in (0, \infty),$$

which is actually a modest variation of an Eulerian formula due to Legendre. A straightforward integration by parts calculation verifies the functional equation $\Gamma(x+1) = x\Gamma(x)$, and the condition $\Gamma(1) = 1$ makes $\Gamma(n+1) = n!$ immediate. But beyond its elegant formula, what makes $\Gamma(x)$ the "right" solution to Goldbach's interpolation problem?

For the mathematicians of Euler's time, the answer was tied up in the way $\Gamma(x)$ fits into the preexisting mathematical landscape. The discovery of identities such as $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ provides an overwhelming sense that the gamma function is an organic mathematical object whose significance extends far beyond its relationship to $n!$. There is, however, another piece of compelling evidence for the primacy of $\Gamma(x)$ that had to wait almost two hundred years to be discovered. In 1922, Harald Bohr and Johannes Mollerup proved the following:

Theorem 1.1. (Bohr-Mollerup) *If f is a positive function on $(0, \infty)$ satisfying*

(G1) $f(1) = 1$

(G2) $f(x+1) = xf(x)$

(G3) $\log f$ *is convex,*

then $f(x) = \Gamma(x)$.

Conditions (G1) and (G2) are prerequisites for any function that purports to be a generalization of $n!$, and, as previously stated, there are many examples of well-behaved functions that satisfy them. The Bohr-Mollerup result provides an unexpectedly simple third condition that completely characterizes $\Gamma(x)$ and guarantees uniqueness. Bohr and Mollerup's original proof was significantly simplified by Emil Artin, and this celebrated result is frequently referred to as the Bohr-Mollerup-Artin Theorem.

The geometric notion of convexity was not part of Euler's predominantly algebraic vocabulary, so it is not surprising that condition (G3) was a 20th century innovation. As the fundamental role of convexity in functional analysis (and beyond) was recognized, various abstractions and generalizations of it were explored. One of these – the concept of *quasi-convexity* – is the subject of a series of articles including [4], [5], [6], [7], and [8]. In a recent paper [2] from this journal, T. Bermúdez, A. Martinón, and K. Sadarangani replaced condition (G3) in Theorem 1.1 with the assumption that $\log f$ is quasi-convex. A consequence of using this weaker condition is that there is now a class Q of functions satisfying all three conditions. The authors refer to functions in Q as *quasi-gamma functions* and undertake a thorough study of the topology of Q as well as the geometric properties of its elements. Among their results is that the functions in Q are continuous and satisfy some smoothness conditions, that Q forms a closed subset of the Fréchet space $C(\mathbb{R}^+)$, but that Q is neither convex nor compact.

The starting point for our study is to replace $n!$ with the sequence of triangular numbers $t_n = 1+2+3+\cdots+n$. Job one is to seek out a function $\Delta(x)$, defined on all of $\mathbb{R}$, with the property that $\Delta(n) = t_n$. In this case we don't need Euler to help us find a suitable candidate. The familiar identity $t_n = n(n+1)/2$ immediately suggests setting

$$\Delta(x) = \frac{x(x+1)}{2},$$

but in what sense is this the "right" interpolation of t_n? As with $\Gamma(x)$, $\Delta(x)$ has the advantage of algebraic simplicity, but is there a Bohr-Mollerup type of result that characterizes $\Delta(x)$ as the unique function satisfying something akin to (G1), (G2), and (G3)? There are certainly natural analogs for properties (G1) and (G2); namely, $\Delta(0) = 0$ and $\Delta(x) = x + \Delta(x-1)$. The logarithmic convexity in (G3), however, is no longer a viable possibility because the growth rate of t_n is too slow. Dropping the logarithm and insisting that the interpolating function simply be convex fails for the same reason it fails with $n!$ – there are too many ways to satisfy the recursive functional equation with a convex function, even if we require that the convex function also be analytic.

In the next section we solve this problem by proving a uniqueness result that involves a convexity assumption on one-sided derivatives. Having found a reasonable analogue of the Bohr-Mollerup result for t_n, a natural question to explore is what happens when we follow Bermúdez et al. and replace convexity with the weaker requirement of quasi-convexity. As in [2], the resulting space Q of so-called *quasi-delta functions* is a closed subspace of $C(\mathbb{R})$, but in this case we shall see that Q is additionally compact and convex.

2. A Bohr-Mollerup theorem for interpolating t_n

Throughout this paper let $I \subseteq \mathbb{R}$ be an interval – possibly unbounded, possibly all of $\mathbb{R}$. A function $f\colon I \to \mathbb{R}$ is convex on I if for all $a < b$ in I and $t \in [0,1]$ it follows that

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b). \tag{1}$$

Alternately, we can look at the difference quotient

$$\phi(a,b) = \frac{f(b) - f(a)}{b-a}. \tag{2}$$

The function f is convex if and only if ϕ is increasing in both variables; i.e., for fixed b, $\phi(a,b)$ is increasing in a, and vice versa. One immediate consequence is that, for a convex function, the right and left derivatives f'_+ and f'_- are defined everywhere and increasing in the sense that

$$f'_-(a) \leq f'_+(a) \quad \text{for all } a \in I, \text{ and} \tag{3}$$
$$f'_+(a) \leq f'_-(b) \quad \text{for all } a < b \text{ in } I. \tag{4}$$

In fact, a function has one-sided derivatives satisfying (3) and (4) if and only if it is convex. (See [1], p. 4.)

The starting point for a Bohr-Mollerup type theorem for the triangular numbers is the functional equation $f(x) = x + f(x-1)$ along with the normalizing condition $f(0) = 0$. These two conditions ensure $f(n) = t_n$ and also define f on the negative integers. For example, $f(-1) = f(0) - 0 = 0$. One way to create a function defined on all of $\mathbb{R}$ satisfying this functional equation is to set $f(x) = 0$ for all $x \in [-1, 0]$, and use the functional equation to recursively define f on $[0, 1]$, $[1, 2]$, and so on. This results in a piecewise linear convex function which is most definitely not equal to $\Delta(x) = x(x+1)/2$ defined in the previous section. (See Figure 1.) This example illustrates that assuming f is convex is not sufficient for guaranteeing uniqueness. To find a satisfying analog for Theorem 1.1 we shall assume f has convex one-sided derivatives.

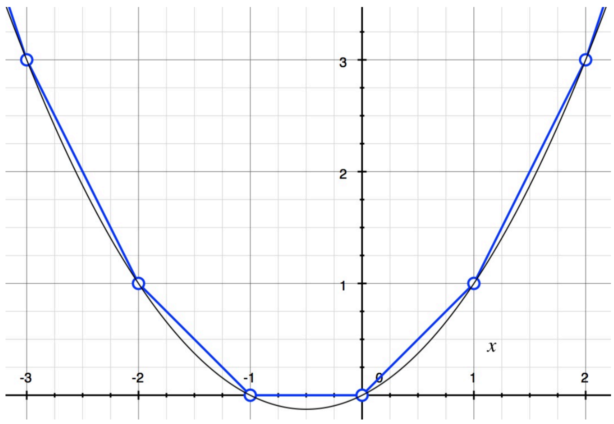

Figure 1: $\Delta(x)$ and a piecewise linear interpolation of t_n.

Theorem 2.1. *If f is defined on $\mathbb{R}$ and satisfies*

(D1) $f(0) = 0$

(D2) $f(x) = x + f(x-1)$

(D3) f'_+ *and* f'_- *exist and are convex,*

then $f(x) = \Delta(x)$.

Proof. The convexity of the one-sided derivatives means that $f'_+(x)$ and $f'_-(x)$ are continuous. This tells us that all four Dini derivatives are continuous which is more than sufficient to conclude $f'_-(x) = f'_+(x)$. (See, for example, [3], p. 40.) Thus, f' exists and is a convex function.

Now set $g(x) = f'(x) - f'(0)$. Differentiating across (D2) yields the functional equation $g(x) = 1 + g(x-1)$ from which it follows that $g(m) = m$ for all $m \in \mathbb{Z}$.

Given $x > 0$, choose $m \in \mathbb{Z}$ so that $x \leq m < x + 1$. Applying the inequality in (1) to the convex function g with $a = 0$, $b = m$ and $t = 1 - x/m$ yields $g(x) \leq x$, but in fact we have $g(x) = x$. To see precisely why observe that if $g(x) < x$ then

$$\phi(m, x+1) = \frac{g(x+1) - m}{(x+1) - m} = \frac{g(x) + 1 - m}{x + 1 - m} < \frac{x + 1 - m}{x + 1 - m} = 1,$$

while $\phi(x, x+1) = 1$, contradicting the characterization of convexity following equation (2). The case $x < 0$ can be handled in a similar fashion to conclude $g(x) = x$ on all of $\mathbb{R}$.

Finally, $g(x) = x$ implies $f'(x) = x + f'(0)$. Thus $f(x) = x^2/2 + f'(0)x + f(0)$, and the result follows. □

3. Properties of quasi-delta functions

The next definition broadens the definition of convexity by relaxing the condition in (1).

Definition 3.1. A function $f : I \to \mathbb{R}$ is *quasi-convex* if for all $a < b$ in I and $t \in [0, 1]$, it follows that

$$f(ta + (1-t)b) \leq \max\{f(a), f(b)\}. \tag{5}$$

Quasi-convexity is studied at length in papers by Crouzeix [4], Greenberg and Pierskalla [6], López Cerdá and Valls Vallejo [7], Martinez-Legaz and Martinón [8], Frenk and Kassey [5], and Bermúdez, Martinón and Sadarangani [2]. To get an initial sense of the difference between convexity and quasi-convexity, observe that a convex function must be continuous, and a twice differentiable function is convex if and only if it has a non-negative second derivative. Quasi-convex function need not satisfy either of these conditions. The most useful characterization of quasi-convexity, employed throughout the literature, is the following:

Lemma 3.2. *A function $f : I \to \mathbb{R}$ is quasi-convex if and only if (i) f is monotone, or (ii) there exists a point $c \in I$ such that f is decreasing on $(-\infty, c] \cap I$ and increasing on $[c, \infty) \cap I$.*

In [2], Bermúdez et al. replace condition (G3) in Theorem 1.1 with the requirement that $\log f$ only be quasi-convex, and in so doing create an interesting class of what they refer to as "quasi-gamma functions." Applying this same idea to Theorem 2.1 leads to a similarly interesting class of functions.

Definition 3.3. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *quasi-delta* if it satisfies

(D1) $f(0) = 0$

(D2) $f(x) = x + f(x-1)$

(QD) f'_+ and f'_- exist and are quasi-convex.

We have already seen two examples of quasi-delta functions – $\Delta(x)$ and the piecewise linear interpolation of t_n in Figure 1. In fact, these two examples are somewhat representative of the scope of possibilities, as the next two results illustrate.

Lemma 3.4. *If f is quasi-delta then* (i) *f is continuous, and* (ii) *f is differentiable except possibly at some countable number of points.*

Proof. Because both one-sided derivatives exist, a quasi-delta function is continuous from the right and from the left. This proves (i).

For (ii), we observe that the quasi-convexity of f'_- and f'_+ means that each derivative is either monotone or consists of two monotone components. The set of discontinuous points of a monotone function is at most countable, so there exists a countable set D such that both f'_- and f'_+ are continuous at every $x \notin D$. Just as we saw in the proof of Theorem 2.1, this implies $f'_-(x) = f'_+(x)$ for all $x \notin D$ and f is therefore differentiable off of D. $\square$

Theorem 3.5. *If f is quasi-delta, then f is convex.*

Proof. The strategy of the proof is to show that f'_+ and f'_- satisfy the inequalities in (3) and (4). We begin by showing they are each monotonically increasing.

Fix $m \in \mathbb{Z}$. From (D2) we get $f(m) - f(m-1) = m$. Applying the mean value theorem for one-sided derivatives on the interval $[m-1, m]$ (see [1], p. 3) it follows that there is a point $x_m \in (m-1, m)$ where m falls between $f'_-(x_m)$ and $f'_+(x_m)$. In particular, this tells us that at least one of the derivatives is not bounded below as $x \to -\infty$.

Now Lemma 3.2 offers two options for the geometric structure of f'_- and f'_+, and it is clear from the previous observation that at least one of these two derivatives must be monotonically increasing on $\mathbb{R}$. But Lemma 3.4 implies $f'_-(x) = f'_+(x)$ off of a countable set, from which we conclude that both derivatives are monotonically increasing functions on $\mathbb{R}$.

Having established the monotonicity of each derivative, let's fix $x_0 \in \mathbb{R}$ and show $f'_-(x_0) \leq f'_+(x_0)$. We claim that

$$\frac{f(x_0) - f(y)}{x_0 - y} \leq f'_+(x_0) \quad \text{for all } y < x_0. \tag{6}$$

To verify the claim assume, for contradiction, that there exists $y < x_0$ where the inequality fails. Once again applying the mean value theorem we find that there exists $c \in (y, x_0)$ where the difference quotient $(f(x_0) - f(y))/(x_0 - y)$ is between $f'_-(c)$ and $f'_+(c)$. Because f'_+ is increasing,

$$\frac{f(x_0) - f(y)}{x - y} > f'_+(x_0) \geq f'_+(c)$$

which means $f'_-(c) > f'_+(x_0)$.

Now f'_- is also increasing and so $f'_-(t) \geq f'_-(c) > f'_+(x_0)$ for all t in the interval $[c, x_0]$. By Lemma 3.4 there must be a t_0 in $[c, x_0]$ where $f'_+(t_0) = f'_-(t_0)$, and for such a point we then get $f'_+(t_0) > f'_+(x_0)$. This contradicts the fact that f'_+ is increasing and proves the claim in (6).

Finally, taking the limit as $y \to x_0$ from the left in (6) yields $f'_-(x_0) \leq f'_+(x_0)$, as desired. This is sufficient to establish inequalities (3) and (4), from which it follows that f is a convex function. □

4. The class Q of quasi-delta functions

In this section we take a step back and look at the structure of the class of all quasi-delta functions, which we denote by Q. From Lemma 3.4 it is clear that Q is a subset of $C(\mathbb{R})$, the vector space of continuous functions on $\mathbb{R}$.

Theorem 4.1. *Q is a convex subset of $C(\mathbb{R})$.*

Proof. Let $f, g \in Q$ be arbitrary, and fix $t \in [0, 1]$. We must show $tf + (1-t)g$ is in Q. It is straightforward to verify that $tf + (1-t)g$ satisfies (D1) and (D2) in Definition 3.3. For (QD), observe that f and g are convex by Theorem 3.5, and the linear combination of convex functions is convex. As we have discussed, convex functions have monotone one-sided derivatives, and so by the criterion in Lemma 3.2, $tf + (1-t)g$ satisfies (QD) and is in Q, as desired. □

The standard way to make $C(\mathbb{R})$ into a topological vector space is to define the family of semi-norms p_n on $C(\mathbb{R})$ for each positive integer n by $p_n(g) = \sup\{|g(x)| : -n \leq x \leq n\}$. We then define the metric

$$d(h, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(g-h)}{(1 + p_n(g-h))} \qquad \text{for } h, g \in C(\mathbb{R}). \tag{7}$$

The metric in (7) induces a complete, locally convex topology on $C(\mathbb{R})$ (giving us a Fréchet space) where a sequence $f_n \to f$ in $C(\mathbb{R})$ (i.e., with respect to d) if and only if $f_n \to f$ uniformly on compact sets. (See [9], p. 31.)

A subset E of a topological vector space is bounded if for every neighborhood V of 0, $E \subseteq \alpha V$ for some $\alpha > 0$. In the case of $C(\mathbb{R})$, this is equivalent to showing that each p_n is bounded on E. For example, the sets $V_n = \{f \in$

$C(\mathbb{R}) : p_n(f) < 1/n\}$ form a basis for the neighborhoods of 0. Because each V_n contains functions that are arbitrarily large outside of $[-n, n]$, no V_n is bounded (and hence there is no norm compatible with this topology.) On the other hand, Q is a bounded subset of $C(\mathbb{R})$. In fact, we show that it is also closed and compact.

Lemma 4.2. *Q is closed and bounded in $C(\mathbb{R})$.*

Proof. Let $g \in Q$. Since g is convex with $g(-1) = g(0) = 0$ there exists a point $c \in [-1, 0]$ such that g is decreasing on $(-\infty, c]$ and increasing on $[c, \infty)$. By incorporating condition (D2) it is possible to show $f(c) \geq -1/4$. Thus, for each positive integer n we have

$$p_n(g) = \sup\{|g(x)| : -n \leq x \leq n\} = g(n) = \frac{n(n+1)}{2}.$$

Since p_n is bounded on Q for each n, Q is bounded.

To show Q is closed let $f_n \to f$ in $C(\mathbb{R})$ where the sequence (f_n) is contained in Q. Then $f_n \to f$ uniformly on compact sets and thus pointwise on all of $\mathbb{R}$. Each f_n is convex by Theorem 3.5 and this convexity is inherited by f ([1], p. 3). As we have previously argued, f convex implies f satisfies (QD). Properties (D1) and (D2) are immediate, and we conclude $f \in Q$ and Q is closed. □

Theorem 4.3. *Q is compact in $C(\mathbb{R})$.*

Proof. With an eye toward using the Arzela-Ascoli Theorem, we temporarily focus on the compact domain $K = [-1, 0]$ and the space $C(K)$ of continuous functions defined on K. Let Q_K be the set of functions in Q restricted to K. From Lemma 4.2 we can deduce that Q_K is a closed and bounded subset of $C(K)$. We claim that Q_K is also equicontinuous.

Let $\epsilon > 0$, and choose $\delta = \epsilon/2$. Given $g_K \in Q_K$, it follows from property (D2) that g_K is the restriction to K of a unique $g \in Q$. As discussed in the previous proof, there is a point $c \in [-1, 0]$ such that g is decreasing on $(-\infty, c]$ and increasing on $[c, \infty)$. Let $x, y \in [-1, 0]$ satisfy $0 < y - x < \delta$, and consider each of the following cases:

If $x < y \leq c$ then $g(x) \geq g(y)$ while $g(x+1) \leq g(y+1)$. Invoking (D2) once again, it follows that $0 \leq g(x) - g(y) \leq y - x$.

If $c \leq x < y$ a similar argument shows $0 \leq g(y) - g(x) \leq y - x$.

If $x < c < y$ then

$$|g(x) - g(y)| \leq |g(x) - g(c)| + |g(c) - g(y)| \leq 2(y - x).$$

Taken together, we have that $|g(x) - g(y)| < \epsilon$ whenever $|x - y| < \delta$, and it follows that Q_K is equicontinuous. By the Arzela-Ascoli Theorem, Q_K is a compact subset of $C(K)$.

To see that Q is compact in $C(\mathbb{R})$, let (f_n) be a sequence in Q. Because Q_K is compact, (f_n) admits a subsequence (f_{n_k}) that converges uniformly when restricted to $[-1, 0]$. But this, in fact, implies that (f_{n_k}) converges uniformly on all of $\mathbb{R}$. This is because, for a given $x \in \mathbb{R}$, we can choose an integer m such that $x - m \in [-1, 0]$. Property (D2) then allows us to write

$$|f_{n_k}(x) - f_{n_l}(x)| = |f_{n_k}(x - m) - f_{n_l}(x - m)|,$$

and the result follows. Uniform convergence of (f_{n_k}) on $\mathbb{R}$ is more than enough to conclude convergence in $C(\mathbb{R})$, and the limit of this subsequence is most definitely in Q. Thus Q is compact. □

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